mathematics

MDPI

Article

Axiomatic Results for Weighted Allocation Rules under Multiattribute Situations

Yu-Hsien Liao

Department of Applied Mathematics, National Pingtung University, Pingtung County 900391, Taiwan; twincos@ms25.hinet.net or twincos@mail.nptu.edu.tw; Tel.: +886-958-631-010

Abstract: In many interactive environments, operators may have to deal with different work objectives at the same time. In a realistic context, such as differences in the target type to be addressed, or changes in the behavior of other operators, operators may therefore have to cope with by adopting different work levels (strategies) at any given time. On the other hand, the importance or influence brought by operators may vary depending on many subjective and objective factors, such as the size of the constituency represented by a congressman, and the bargaining power of a business personnel which may vary. Therefore, it is reasonable that weights are apportioned to operators and arbitrary usability should be distributed according to these weights under various working levels and multiattribute situations. In pre-existing results for allocation rules, weights might be always apportioned to the "operators" or the "levels" to modify the differences among the operators or its working levels respectively. By applying weights to the operators and its working levels (strategies) simultaneously, we adopt the maximal marginal variations among working level (strategy) vectors to propose an allocation rule under multiattribute situations. Furthermore, we introduce some axiomatic outcomes to display the rationality for this weighted allocation rule. By replacing weights to be maximal marginal variations, a generalized index is also introduced.

Keywords: allocation rule; weight; the maximal marginal variation; axiomatic result; multiattribute situation

Citation: Liao, Y.-H. Axiomatic Results for Weighted Allocation Rules under Multiattribute Situations. *Mathematics* **2021**, *9*, 617. https://doi.org/10.3390/math9060617

Academic Editor: Andrés Jiménez Losada

Received: 8 February 2021
Accepted: 12 March 2021
Published: 15 March 2021

1. Introduction

In different topics, from biomedical engineering, sciences to environment, and the management sciences, operators confront a raising demand to concentrate on multiple objectives effectively in its working procedures. Related conditions involve dissecting distribution tradeoffs, selecting optimal decisions or planning course, or arbitrary situations where one requires an effective rule with tradeoffs among several objectives. In many cases these real-world effective conditions might be modeled as a mathematical multiattribute optimization game. The rules of such conditions lacks appropriate notions to present optimal outcomes that—unlike traditional notions or viewpoints—apply various functions of the objectives into account. Several pre-existing results considered multiattribute situations. For example, Bednarczuk et al. [1] converted the multiple-choice knapsack issue into a bi-purpose optimal issue whose solution collection covers solutions of the original multiple-choice knapsack issue. Goli et al. [2] addressed the optimization of the multivariate manufacture portfolio issue under return uncertainty, which is addressed here. The main outcome is based on the utilization of a hybrid ameliorated artificial intelligence and robust optimization, providing a new notion for determining the risk of a manufacture portfolio. A two-objective (maximizing return and minimizing risk) mathematical shape is also introduced. By focusing on multiattribute analysis techniques under several complex conditions (e.g., with various viewpoints to be pondered and multi-grade operators involved), the goal of the outcomes due to Guarini et al. [3] is to outline a procedure with which to pick the rule best matched to the particular demands of evaluation, which commonly appear while addressing strategy-making issues. A resilient combinatorial optimization modeling process due to Mustakerov et al. [4] is progressed for multi-choice yielding with various strategy maker prerequisites. The characterized process is built on formulation of multiattribute linear mixed integer optimization assignments. Tirkolaee et al. [5] pointed out the multiattribute multi-mode utility-constrained manufacture scheduling issue with remuneration planning where the energies could be completed through one of the possible modes and the goals are to minimize the completion time and maximize the net present value simultaneously.

Under traditional games, each operator is either completely participated or entirely outside of participation with other operators. Back by a dose of reality in many real situations, however, such as changes in the current situation, differences in the target type to be addressed, or changes in the behavior of other operators, operators may therefore have to cope with them by adopting different work levels (strategies) at any given time. Under *multi-choice games*, each operator is permitted to operate with infinite working levels (or strategies). Thus, a multi-choice game could be treated as a generalization of a traditional game. By determining overall outcomes for a given operator on a multi-choice game, Hwang and Liao [6], Liao [7,8] and Nouweland et al. [9] proposed several generalized rules for several traditional allocation rules respectively.

On the other hand, the importance or influence brought by operators may vary depending on many subjective and objective factors, such as the size of the constituency represented by a congressman, the contribution arisen by a member of a hospital, and the bargaining power of a business personnel may vary. Besides, lack of symmetry might arise if alternative bargaining skills for alternative operators are formed. In line with the previous interpretations, one would now crave that arbitrary usability might be shared among the operators and its working levels in proportion to *weights*. Weights rise up involuntary in the context of allocating-usability. For instance, one might be dealing with an issue of usability allocation among investing plans. Thus, the weights could be appointed to the profitability of the different options of all plans. In the issue of distributing travel charges among various areas attended, the weights could be the amount of days used at each one (cf. Shapley [10]). In general, weights might be apportioned to the "operators" or the "levels" to distinguish the differences among the operators or its working levels respectively.

Under the axiomatic processes for allocation rules under cooperative games, *stability* (or consistency) is a critical notion of advantageous rules. The notion after this kind of stability could be described as follows: For a specific game, operators might progress anticipations of the game and may be agreeable to consent the computation of its remunerations to be built upon these anticipations. The allocation rule satisfies stability if it allots coincident remunerations to operators under the initial situation as it does to operators of the imaginary reduced situation. Therefore, stability is an essential factor of the inner "robustness" of compromises. Stability has been always analyzed under many subjects by considering *reduced games*, such as bargaining issues, resource distribution issues, and so on. Based on notion the *equal allocation of non-separable costs* (EANSC, Ransmeier [11]), Liao et al. [12] introduce two allocation rules by assigning weights to the operators and its activity levels (decisions) respectively under multiattribute multi-choice situations. Inspired by the axiomatic techniques due to Moulin [13], Liao et al. [12] also proposed an extended *complement-reduced game* to manifest that these two allocation rules present fair mechanisms for allocating usability.

The above mentioned existing outcomes generate one motivation:

- Whether different rule concepts might be considered by applying weights to the operators and its working levels (strategies) simultaneously under multiattribute multi-choice situations.

The article discusses the motivation. The main outcomes of this article are as follows.

1. Three rules, the *weighted lower-aggregate multiattribute index* (WLAMI), the *weighted regular multiattribute index* (WRMI), and the *weighted upper-aggregate multiattribute index* (WUAMI), are introduced by assigning weights to the operators and its working levels (strategies) simultaneously in Section 2. These three rules are weighted generalizations

of the maximal multiattribute index (MMI) due to Liao et al. [12] under multiattribute multi-choice games.

2. In order to realize the rationality for these weighted rules, we consider an extended reduction to manifest the following outcomes in Section 3:
   - The WLAMI is the only rule matching the axioms of *multiattribute weighted lower-aggregate principle* and *multiattribute bilateral stability*;
   - The WRMI is the only rule matching the axioms of *multiattribute weighted regular principle* and *multiattribute bilateral stability*;
   - The WUAMI is the only rule matching the axioms of *multiattribute weighted upper-aggregate principle* and *multiattribute bilateral stability*.
3. Under some situations, the legitimacy or fairness of the weights may be questioned. Therefore, it is a reasonable concept to use the relative maximal marginal variations as weights under different situations naturally. Maximal marginal variations instead of weights naturally in Section 4, different allocation rules and related axiomatic outcomes are also introduced under multiattribute multi-choice situations.

2. Preliminaries

2.1. Definitions and Notations

Assume that UO is the universe of operators, for example, the set formed by humans throughout the Earth. Any $s \in UO$ is said to be a member of UO, for example, a human of Earth. For $s \in UO$ and $f_s \in \mathbb{N}$, we define $F_s = \{0, 1, \ldots, f_s\}$ to be the working level (strategy) set of operator s and $F_s^+ = F_s \setminus \{0\}$, where 0 means no operation. Assume that $O \subseteq UO$ is the largest set of all members of an interactive system in UO, for example, all citizens of a country on Earth. Let $F^O = \prod_{s\in O} F_s$ be the product set of the working level sets for every operators of O. For every $K \subseteq O$, a operator-alliance K corresponds in a standard mode to the multi-choice alliance $e^K \in F^O$, which is the vector matching $e_s^K = 1$ if $s \in K$, and $e_s^K = 0$ if $s \in O \setminus K$. Denote 0_O the zero vector of $\mathbb{R}^O$. For $m \in \mathbb{N}$, we also define 0_m to be the zero vector of $\mathbb{R}^m$ and $\Gamma_m = \{1, 2, \ldots, m\}$.

A **multi-choice game** is denoted to be (O, f, h), where $O \neq \emptyset$ is a finite set of operators, $f = (f_s)_{s\in O} \in F^O$ is a vector that presents the number of working levels for every operator, and $h : F^O \to \mathbb{R}$ is a mapping with $h(0_O) = 0$ which apportions to each working level vector $\lambda = (\lambda_s)_{s\in O} \in F^O$ the benefit that the operators can receive when each operator s operates at level λ_s. A **multiattribute multi-choice game** is denoted by (O, f, H^m), where $m \in \mathbb{N}$, $H^m = (h^t)_{t\in\Gamma_m}$ and (O, f, h^t) is a multi-choice game for every $t \in \Gamma_m$. We also denote the family of all multiattribute multi-choice games to be Δ.

A **rule** is a map τ apportioning to each $(O, f, H^m) \in \Delta$ an element:

$$\tau(O, f, H^m) = \left(\tau^t(O, f, H^m)\right)_{t\in\Gamma_m},$$

where $\tau^t(O, f, H^m) = \left(\tau_s^t(O, f, H^m)\right)_{s\in O} \in \mathbb{R}^O$ and $\tau_s^t(O, f, H^m)$ is the remuneration of the operator s when s operates in (O, f, h^t). Let $(O, f, H^m) \in \Delta$, $K \subseteq O$ and $\lambda \in \mathbb{R}^O$, we set that $S(\lambda) = \{s \in O | \lambda_s \neq 0\}$ and $\lambda_K \in \mathbb{R}^K$ to be the restriction of λ to K. Given $s \in O$, we also define λ_{-s} to stand for $\lambda_{O\setminus\{s\}}$. Furthermore, $\gamma = (\lambda_{-s}, c) \in \mathbb{R}^O$ is defined by $\gamma_{-s} = \lambda_{-s}$ and $\gamma_s = c$.

Liao et al. [12] provided a multi-choice generalized EANSC under multiattribute situation as follows.

Definition 1. *The* **maximal EANSC (MEANSC)**, $\overline{\beta}$, *is defined by:*

$$\overline{\beta_s^t}(O, f, H^m) = \beta_s^t(O, f, H^m) + \frac{1}{|O|} \cdot \left[h^t(f) - \sum_{k\in O} \beta_k^t(O, f, H^m)\right]$$

for every $(O, f, H^m) \in \Delta$, *for every* $t \in \Gamma_m$, *and for every* $s \in O$. *The value* $\beta_s^t(O, f, H^m) = \max_{q\in F_s^+}\{h^t(f_{-s}, q) - h^t(f_{-s}, 0)\}$ *is the* **maximal lower-aggregate marginal variation** *among all working levels of operator s in (O, f, h^t). Here we apply bounded multi-choice games, considered as the games (O, f, h^t) such that, there exists $N_h \in \mathbb{R}$ such that $h^t(\lambda) \leq N_h$ for every $\lambda \in F^O$. We apply it to guarantee that $\beta_s(O, f, h^t)$ is well-defined. Under the notion of $\overline{\beta}$, all operators firstly receive its maximal marginal variations, and further introduce the rest of usability equally.*

A rule τ matches **multiattribute effectiveness (MECE)** if for every $(O, f, H^m) \in \Delta$ and for every $t \in \Gamma_m$, $\sum_{s\in O} \tau_s^t(O, f, H^m) = h^t(f)$. A rule τ matches **multiattribute principle for games (MPFG)** if $\tau(O, f, H^m) = \overline{\beta}(O, f, H^m)$ for every $(O, f, H^m) \in \Delta$ with $|O| \leq 2$. Axiom MECE presents that all operators allocate whole the usability completely. Axiom MPFG is a generalized form of the two-agent principle condition of Hart and Mas-Colell [14].

Moulin [13] considered the reduced game as that in which each alliance in the subgroup could attain remunerations to its operators only if they are agreeing with the original remunerations to "total" the operators out of the subgroup. A generalized Moulin reduction under multiattribute multi-choice games is considered as follows.

Let $(O, f, H^m) \in \Delta$, $K \subseteq O$ and τ be a rule. The **reduced game** $(K, f_K, H^m_{K,\tau})$ is defined by $H^m_{K,\tau} = (h^t_{K,\tau})_{t\in\Gamma_m}$ and for every $\lambda \in F^K$,

$$h^t_{K,\tau}(\lambda) = \begin{cases} 0 & \text{if } \lambda = 0_K, \\ h^t(\lambda, f_{O\setminus K}) - \sum\limits_{s\in O\setminus K} \tau_s^t(O, f, H^m) & \text{otherwise,} \end{cases}$$

Furthermore, a rule τ matches **multiattribute bilateral stability (MBSTA)** if $\tau_s^t(K, f_K, H^m_{K,\tau}) = \tau_s^t(O, f, H^m)$ for every $(O, f, H^m) \in \Delta$, for every $t \in \Gamma_m$, for every $K \subseteq O$ with $|K| = 2$ and for every $s \in K$.

As mentioned in the Introduction, weights rise up involuntarily in the context of usability allocation. For instance, we might be dealing with an issue of usability allocation among investing plans. Thus, the weights could be appointed to the profitability of the alternative options of all plans. Also, weights are contained in contracts approved by the proprietors of a townhouse and adopted to distribute the cost of maintaining or building common apparatus. Another application is patent pooling or data among trading companies where the scope of the trading companies, examined for instance by its market shares, could be natural weights. In general, weights might be apportioned to the "operators" or the "levels" to distinguish the dissimilarity among the operators or its working levels respectively. If $d : U \to \mathbb{R}^+$ be a positive map, then d is said to be a **weight map for operators**. If $w : \cup_{s\in U} F_s^+ \to \mathbb{R}^+$ be a positive map, then w is said to be a **weight map for levels**. By these two kinds of weight maps, two weighted extensions of the MEANSC is considered by Liao et al. [12] as follows.

Definition 2.

- *The* **1-maximal weighted allocation of non-separable costs (1-MWANSC)**, *η^d, is considered by for every $(O, f, H^m) \in \Delta$, for every weight map for operators d, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\eta_s^{d,t}(O, f, H^m) = \beta_s^t(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in O} d(k)} \cdot \Big[h^t(f) - \sum_{k\in O} \beta_k^t(O, f, H^m)\Big].$$

By definition of η^d, all operators get its maximal lower-aggregate marginal variations firstly, and further distribute the remaining usability proportionally by weights for operators.

- *The* **2-maximal weighted allocation of non-separable costs (2-MWANSC)**, η^w, *is considered by for every* $(O, f, H^m) \in \Delta$, *for every weight map for operators* w, *for every* $t \in \Gamma_m$ *and for every operator* $s \in O$,

$$\eta_s^{w,t}(O, f, H^m) = \beta_s^{w,t}(O, f, H^m) + \frac{1}{|O|} \cdot \left[h^t(f) - \sum_{k \in O} \beta_k^{w,t}(O, f, H^m)\right],$$

where $\beta_s^{w,t}(O, f, H^m) = \max_{q \in F_s^+} \{w(q) \cdot [h^t(f_{-s}, q) - h^t(f_{-s}, 0)]\}$ *is the* **maximal weighted lower-aggregate marginal variation** *among all working levels of operator* s. *By definition of* $\eta^{w,t}$, *all operators get its maximal weighted lower-aggregate marginal variations firstly, and further distribute the remaining usability equally.*

A rule τ matches **1-weighted principle for games (1WPFG)** if $\tau(O, f, H^m) = \eta^d(O, f, H^m)$ for every $(O, f, H^m) \in \Delta$ with $|O| \leq 2$ and for every weight map for operators d. A rule τ matches **2-weighted principle for games (2WPFG)** if $\tau(O, f, H^m) = \eta^w(O, f, H^m)$ for every $(O, f, H^m) \in \Delta$ with $|O| \leq 2$ and for every weight map for levels w.

Several axiomatic results of the MEANSC, the 1-MWANSC and the 2-MWANSC are proposed by Liao et al. [12] as follows.

Theorem 1.

- *On* Δ, *the MEANSC is the only rule matching MPFG and MBSTA;*
- *On* Δ, *the 1-MWANSC is the only rule matching 1WPFG and MBSTA;*
- *On* Δ, *the 2-MWANSC is the only rule matching 2WPFG and MBSTA.*

2.2. Motivating and Practical Examples

As mentioned in the Introduction, each performer might be admitted to operate with alternative levels under real-world situations respectively. On the other hand, multiattribute analysis is a notion of multiple criterion investigations that is concerned with situations involving simultaneously more than one objective to be optimized. Multiattribute analysis also has been adopted in various issues, including biomedical sciences, environmental analysis, information engineering, strategical management sciences, and logistics where efficacious strategies need to be adopted in the presence of trade-offs among several objectives. For instance, minimizing cost while maximizing comfort while marketing a central air conditioning mode, and maximizing efficacy whilst minimizing emission of pollutants and sources consumption are exemplifications of multiattribute efficacious issues involving respectively various objectives. Under various cases, there might be more than three objectives. Hence, we focus on the framework of multiattribute multi-choice schemes throughout this paper. Nevertheless, it might not be appropriate under various situations if arbitrary additional fixed usability should be shared equally among the operators who are concerned. Thus, it is reasonable that weights could be appointed to operators or its working levels, and arbitrary fixed usability should be shared according to these weights.

Next, a concise motivating example of multiattribute multi-choice schemes will be presented under the situation of "management". Let O be a set of all operators of a multiattribute management organization (O, f, H^m). The function h^t could be pondered as an usability function which appoints to each level vector $\alpha = (\alpha_s)_{s \in O} \in F^O$ the worth that the performers can gain if each performer s operates at operational strategy $\alpha_s \in F_s$ under sub-management organization (O, f, h^t). Modeled by this notion, the multiattribute management organization (O, f, H^m) could be generalized to be a multiattribute multi-choice game, with h^t being every characteristic mapping and F_s being the collection of total operational strategies of the operator s.

Subsequently, we further present a practical application of power evaluation under a legislature. Let O be a collection of total agents of a legislature in a sovereign state. Under the legislature, all agents of the legislature are chosen via voting or recommendation from parties. All agents possess the power to discuss, establish, propose, and veto all acts. They dedicate alternative grades of attention and participation to alternative acts depending upon related academic expertise and the public judgment they exhibit. The level of involvement is also intimately associated with the agreement tactic generated for the interests of alternative political parties. For the aforementioned considerations, tactics applied by each agent of the legislature show distinct tactics of participation and certain amounts of multiplicity. The mapping h^t could be pondered as a power evaluation which appoints to every tactic vector $\alpha = (\alpha_s)_{s\in O} \in F^O$ the power that the agents could dedicate if every agent s takes operational strategy $\alpha_s \in F_s$ in an acts affairs committee. Modeled by this notion, an acts affairs committee could be pondered as a multi-choice game (O, f, h^t), with h^t being every characteristic mapping and F_s being the collection of total operational tactics of the agent s. The legislature operational scheme could be generalized as a multiattribute multi-choice game (O, f, H^m). To evaluate the influence of each agent in the legislature, applying the power indexes in Definitions 1 and 2, one could assess the maximal influence each legislature agent has accumulated over previous act meetings based on multi-tactics.The remaining shared power distribution should be shared equally by all agents, which is the MEANSC presented in Definition 1. As every legislature agent possesses alternative academic expertise and represents alternative public judgments, they naturally carry alternative levels of significance under various situations thus, they generate alternative weights of agents from a weight mapping d. The rest of shared power distribution also should be shared in proportion to the weight derived for each agent, which is the 1-MWANSC presented in Definition 2. As each tactic has alternative operational significance, these tactics naturally carry alternative levels of significance under various situations thus, these tactics generate alternative weights from a weight mapping w. The representative influence of each agent should be computed by its maximal weighted variation. The remaining shared power distribution might also be shared equally by all agents, which is the 2-MWANSC presented in Definition 2.

3. Different Weighted Extension

As we mention in the Introduction, we introduce different extensions of the MEANSC in this section by simultaneously applying weights to the operators and its working levels (strategies). Based on MBSTA, we further characterize these weighted rules.

Definition 3.

- *The* **weighted lower-aggregate multiattribute index (WLAMI)**, $\beta^{d,w}$, *is defined by for every* $(O, f, H^m) \in \Delta$, *for every weight map for operators* d, *for every weight map for levels* w, *for every* $t \in \Gamma_m$ *and for every operator* $s \in O$,

$$\beta_s^{d,w,t}(O, f, H^m) = \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in O} d(k)} \cdot \big[h^t(f) - \sum_{k\in O} \beta_k^{w,t}(O, f, H^m)\big].$$

  By definition of $\beta^{d,w}$, *all operators get its maximal weighted lower-aggregate marginal variations firstly, and further distribute the remaining usability proportionally by weights for operators.*

- *The* **weighted regular multiattribute index (WRMI)**, $\alpha^{d,w}$, *is defined by for every* $(O, f, H^m) \in \Delta$, *for every weight map for operators* d, *for every weight map for levels* w, *for every* $t \in \Gamma_m$ *and for every operator* $s \in O$,

$$\alpha_s^{d,w,t}(O, f, H^m) = \alpha_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in O} d(k)} \cdot \big[h^t(f) - \sum_{k\in O} \alpha_k^{w,t}(O, f, H^m)\big],$$

  where $\alpha_s^{w,t}(O, f, H^m) = \max\limits_{q\in F_s^+}\{w(q) \cdot [h^t(f_{-s}, q) - h^t(f_{-s}, q-1)]\}$ *is the* **maximal weighted regular marginal variation** *among all working levels of operator s. By defi-*

nition of $\alpha^{d,w}$, all operators get its maximal weighted regular marginal variations firstly, and further distribute the remaining usability proportionally by weights for the operators.

- *The* **weighted upper-aggregate multiattribute index (WUAMI)***, $\gamma^{d,w}$, is defined by for every $(O,f,H^m)\in\Delta$, for every weight map for operators d, for every weight map for levels w, for every $t\in\Gamma_m$, and for every operator $s\in O$,*

$$\gamma_s^{d,w,t}(O,f,H^m)=\gamma_s^{w,t}(O,f,H^m)+\frac{d(s)}{\sum\limits_{k\in O}d(k)}\cdot\Big[h^t(f)-\sum_{k\in O}\gamma_k^{w,t}(O,f,H^m)\Big],$$

where $\gamma_s^{w,t}(O,f,H^m)=\max\limits_{q\in F_s^+}\{w(q)\cdot[h^t(f)-h^t(f_{-s},q-1)]\}$ is the **maximal weighted upper-aggregate marginal variation** *among all working levels of operator s. By definition of $\alpha^{d,w}$, all operators get maximal weighted upper-aggregate marginal variations firstly, and further distribute the remaining usability proportionally by weights for operators.*

Lemma 1. *The rules $\beta^{d,w}$, $\alpha^{d,w}$, $\gamma^{d,w}$ match MECE.*

Proof of Lemma 1. Let $(O,f,H^m)\in\Delta$, $t\in\Gamma_m$, d be weight map for operators and w be weight map for levels. By Definition 3,

$$\begin{aligned}
&\sum_{s\in O}\beta_s^{d,w,t}(O,f,H^m)\\
=\;&\sum_{s\in O}\beta_s^{w,t}(O,f,H^m)+\sum_{s\in O}\Big[\frac{d(s)}{\sum_{k\in O}d(k)}\cdot\big[h^t(f)-\sum_{k\in O}\beta_k^{w,t}(O,f,H^m)\big]\Big]\\
=\;&\sum_{s\in O}\beta_s^{w,t}(O,f,H^m)+\frac{\sum_{s\in O}d(s)}{\sum_{k\in O}d(k)}\cdot\big[h^t(f)-\sum_{k\in O}\beta_k^{w,t}(O,f,H^m)\big]\\
=\;&\sum_{s\in O}\beta_s^{w,t}(O,f,H^m)+h^t(f)-\sum_{k\in O}\beta_k^{w,t}(O,f,H^m)\\
=\;&h^t(f).
\end{aligned}$$

The proof is finished. Similarly, it is easy to manifest that the rules $\alpha^{d,w}$, $\gamma^{d,w}$ also match MECE. □

Lemma 2. *The rules $\beta^{d,w}$, $\alpha^{d,w}$, $\gamma^{d,w}$ match MBSTA.*

Proof of Lemma 2. Let $(O,f,H^m)\in\Delta$, $K\subseteq O$, $t\in\Gamma_m$, d be weight map for operators and w be weight map for levels. Let $|O|\geq 2$ and $|K|=2$. By Definition 3,

$$\begin{aligned}
&\beta_s^{d,w,t}(K,f_K,H^m_{K,\beta^{d,w}})\\
=\;&\beta_s^{w,t}(K,f_K,H^m_{K,\beta^{d,w}})+\frac{d(s)}{\sum_{k\in K}d(k)}\cdot\big[h^t_{K,\beta^{d,w}}(f_K)-\sum_{k\in K}\beta_k^{w,t}(K,f_K,H^m_{K,\beta^{d,w}})\big]
\end{aligned}\tag{1}$$

for every $s\in K$ and for every $t\in\Gamma_m$. By definitions of $\beta^{w,t}$ and $h^t_{K,\beta^{d,w}}$,

$$\begin{aligned}
\beta_s^{w,t}(K,f_K,H^m_{K,\beta^{d,w}})&=\max_{q\in F_s^+}\{w(q)\cdot[h^t_{K,\beta^{d,w}}(f_{K\setminus\{s\}},q)-h^t_{K,\beta^{d,w}}(f_{K\setminus\{s\}},0)]\}\\
&=\max_{q\in F_s^+}\{w(q)\cdot[h^t(f_{-s},q)-h^t(f_{-s},0)]\}\\
&=\beta_s^{w,t}(O,f,H^m).
\end{aligned}\tag{2}$$

By Equations (1) and (2) and definitions of $h^t_{K,\beta^{d,w}}$ and $\beta^{d,w}$,

$$\begin{aligned}
& \beta_s^{d,w,t}(K, f_K, H^m_{K,\beta^{d,w}}) \\
= \; & \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \cdot \left[h^t_{K,\beta^{d,w}}(f_K) - \sum_{k\in K} \beta_k^{w,t}(O, f, H^m)\right] \\
= \; & \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \cdot \left[h^t(f) - \sum_{k\in O\setminus K} \beta_k^{d,w,t}(O, f, H^m) - \sum_{k\in K} \beta_k^{w,t}(O, f, H^m)\right] \\
= \; & \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \cdot \left[\sum_{k\in K} \beta_k^{d,w,t}(O, f, H^m) - \sum_{k\in K} \beta_k^{w,t}(O, f, H^m)\right] \\
& \textbf{(MECE of } \beta^{d,w}\textbf{)} \\
= \; & \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{k\in K} d(k)} \cdot \left[\frac{\sum\limits_{k\in K} d(k)}{\sum\limits_{p\in O} d(p)} \cdot \left[h^t(f) - \sum_{p\in O} \beta_p^{w,t}(O, f, H^m)\right]\right] \\
= \; & \beta_s^{w,t}(O, f, H^m) + \frac{d(s)}{\sum\limits_{p\in O} d(p)} \cdot \left[h^t(f) - \sum_{p\in O} \beta_p^{t}(O, f, H^m)\right] \\
= \; & \beta_s^{d,w,t}(O, f, H^m)
\end{aligned}$$

for every $s \in K$ and for every $t \in \Gamma_m$. The proof is finished. Similarly, it is easy to manifest that the rules $\alpha^{d,w}$, $\gamma^{d,w}$ also match MBSTA. □

Inspired by the work of Hart and Mas-Colell [14], we adopt MBSTA to characterize the WMMI. A rule τ matches **weighted lower-aggregate principle (WLAP)** if $\tau(O, b, H^m) = \beta^{d,w}(O, b, H^m)$ for every $(O, b, H^m) \in \Delta$ with $|O| \leq 2$, for every weight map for operators d and for every weight map for levels w. A rule τ matches **weighted regular principle (WRP)** if $\tau(O, b, H^m) = \alpha^{d,w}(O, b, H^m)$ for every $(O, b, H^m) \in \Delta$ with $|O| \leq 2$, for every weight map for operators d and for every weight map for levels w. A rule τ matches **weighted upper-aggregate principle (WUAP)** if $\tau(O, b, H^m) = \gamma^{d,w}(O, b, H^m)$ for every $(O, b, H^m) \in \Delta$ with $|O| \leq 2$, for every weight map for operators d and for every weight map for levels w.

Theorem 2.

1. *On Δ, the WLAMI is the only rule matching WLAP and MBSTA;*
2. *On Δ, the WRMI is the only rule matching WRP and MBSTA;*
3. *On Δ, the WUAMI is the only rule matching WUAP and MBSTA.*

Proof of Theorem 2. By Lemma 2, the rules $\beta^{d,w}$, $\alpha^{d,w}$, $\gamma^{d,w}$ match MBSTA. Clearly, the rules $\beta^{d,w}$, $\alpha^{d,w}$, $\gamma^{d,w}$ match WLAP, WRP, and WUAP respectively.

To demonstrate the uniqueness of outcome 1, suppose that τ matches WLAP and MBSTA. By WLAP and MBSTA of τ, it is easy to clarify that τ also matches MECE, hence we omit it. Let $(O, f, H^m) \in \Delta$, d be weight map for operators and w be weight map for levels. By WLAP of τ, $\tau(O, f, H^m) = \beta^{d,w}(O, f, H^m)$ if $|O| \leq 2$. The condition $|O| > 2$: Let $s \in O$, $t \in \Gamma_m$ and $K = \{s, p\}$ with $p \in O \setminus \{s\}$.

$$\begin{aligned}
& \tau_s^t(O, f, H^m) - \beta_s^{d,w,t}(O, f, H^m) \\
= \; & \tau_s^t(K, f_K, H^m_{K,\tau}) - \beta_s^{d,w,t}(K, f_K, H^m_{K,\beta^{d,w}}) \\
& \textbf{(MBSTA of } \beta^{d,w,t} \textbf{ and } \tau\textbf{)} \\
= \; & \beta_s^{d,w,t}(K, f_K, H^m_{K,\tau}) - \beta_s^{d,w,t}(K, f_K, H^m_{K,\beta^{d,w},}). \\
& \textbf{(WLAP of } \tau\textbf{)}
\end{aligned} \tag{3}$$

Similar to Equation (2),

$$\beta_s^{w,t}(K, f_K, H^m_{K,\tau}) = \beta_s^{w,t}(O, f, H^m) = \beta_s^{w,t}(K, f_K, H^m_{K,\beta^{d,w},}). \tag{4}$$

By Equations (3) and (4),

$$\begin{aligned}
& \tau_s^t(O,f,H^m) - \beta_s^{d,w,t}(O,f,H^m) \\
= \; & \beta_s^{d,w,t}(K,f_K,H^m_{K,\tau}) - \beta_s^{d,w,t}(K,f_K,H^m_{K,\beta^{d,w,}}) \\
= \; & \tfrac{d(s)}{d(s)+d(p)} \cdot \left[h^t_{K,\tau}(f_K) - h^t_{K,\beta^{d,w}}(f_K)\right] \\
= \; & \tfrac{d(s)}{d(s)+d(p)} \cdot \left[\tau_s^t(O,f,H^m) + \tau_p^t(O,f,H^m) \right. \\
& \left. \quad - \beta_s^{d,w,t}(O,f,H^m) - \beta_p^{d,w,t}(O,f,H^m)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
& d(p) \cdot \left[\tau_s^t(O,f,H^m) - \beta_s^{d,w,t}(O,f,H^m)\right] \\
= \; & d(s) \cdot \left[\tau_p^t(O,f,H^m) - \beta_p^{d,w,t}(O,f,H^m)\right].
\end{aligned}$$

By MECE of $\beta^{d,w,t}$ and τ,

$$\begin{aligned}
& \left[\tau_s^t(O,f,H^m) - \beta_s^{d,w,t}(O,f,H^m)\right] \cdot \sum_{p\in O} d(p) \\
= \; & d(s) \cdot \sum_{p\in O} \left[\tau_p^t(O,f,H^m) - \beta_p^{d,w,t}(O,f,H^m)\right] \\
= \; & d(s) \cdot \left[h^t(f) - h^t(f)\right] \\
= \; & 0.
\end{aligned}$$

Hence, $\tau_s^t(O,f,H^m) = \beta_s^{d,w,t}(O,f,H^m)$ for every $s \in O$ and for every $t \in \Gamma_m$. Similarly, the proofs of outcomes 2 and 3 could be finished. □

In the following we exhibit some examples to display that every of the properties applied in Theorem 2 is independent of the rest of properties.

Example 1. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta$, for every weight map for operators d, for every weight map for levels w, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\tau_s^t(O,f,H^m) = \begin{cases} \beta_s^{d,w,t}(O,f,H^m) & \text{if } |O| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches WLAP, but it does not match MBSTA.

Example 2. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta$, for every weight map for operators d, for every weight map for levels w, for every $t \in \Gamma_m$, and for every operator $s \in O$,*

$$\tau_s^t(O,f,H^m) = \begin{cases} \alpha_s^{d,w,t}(O,f,H^m) & \text{if } |O| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches WRP, but it does not match MBSTA.

Example 3. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta$, for every weight map for operators d, for every weight map for levels w, for every $t \in \Gamma_m$, and for every operator $s \in O$,*

$$\tau_s^t(O,f,H^m) = \begin{cases} \gamma_s^{d,w,t}(O,f,H^m) & \text{if } |O| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches WUAP, but it does not match MBSTA.

Example 4. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta$, for every weight map for operators d, for every weight map for levels w, for every $t \in \Gamma_m$, and for every operator $s \in O$, $\tau_s^t(O,f,H^m) = 0$. Clearly, τ matches MBSTA, but it does not match WLAP, WRP, and WUAP.*

Example 5. *We consider the rule τ as follows. For every $(O, f, H^m) \in \Delta$, for every weight map for operators d, for every weight map for levels w, for every $t \in \Gamma_m$,and for every operator $s \in O$, $\tau_s^t(O, f, H^m) = 0$. Clearly, τ matches MBSTA, but it does not match MWP.*

4. Other Generalizations and Revised Stability

In Sections 2 and 3, several weighted generalizations are generalized by simultaneously using weights to the operators and its working levels (strategies). But sometimes the fairness or legitimacy of the weight functions might be questioned. The weights to the operators and its working levels (strategies) might be apportioned artificially. Therefore, it is a reasonable concept to use the relative maximal marginal variations as weights under different circumstances naturally.

"Maximal marginal variations" instead of "weights", several generalizations could be considered as follows.

Definition 4.

1. *The* **1-interior multi-choice multiattribute index (1IMMI)**, *η^1, is considered by for every $(O, f, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\eta_s^{1,t}(O, f, H^m) = \beta_s^t(O, f, H^m) + \frac{\beta_s^t(O, f, H^m)}{\sum\limits_{k \in O} \beta_k^t(O, f, H^m)} \cdot \left[h^t(f) - \sum_{k \in O} \beta_k^t(O, f, H^m)\right],$$

where $\Delta^{1*} = \{(O, f, H^m) \in \Delta | \sum\limits_{k \in O} \beta_k^t(O, f, H^m) \neq 0$ for every $t \in \Gamma_m\}$. By definition of η^1, all operators get its maximal lower-aggregate marginal variations firstly, and further distribute the remaining usability proportionally by its maximal lower-aggregate marginal variations.

2. *The* **2-interior multi-choice multiattribute index (2IMMI)**, *η^2, is considered by for every $(O, f, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\eta_s^{2,t}(O, f, H^m) = \alpha_s^t(O, f, H^m) + \frac{\alpha_s^t(O, f, H^m)}{\sum\limits_{k \in O} \alpha_k^t(O, f, H^m)} \cdot \left[h^t(f) - \sum_{k \in O} \alpha_k^t(O, f, H^m)\right],$$

where $\alpha_s^t(O, f, H^m) = \max\limits_{q \in F_s^+}\{\cdot[h^t(f_{-s}, q) - h^t(f_{-s}, q-1)]\}$ is the maximal regular marginal variation and $\Delta^{2*} = \{(O, f, H^m) \in \Delta | \sum\limits_{k \in O} \alpha_k^t(O, f, H^m) \neq 0$ for every $t \in \Gamma_m\}$. By definition of η^2, all operators get its maximal regular marginal variations firstly, and further distribute the remaining usability proportionally by its maximal regular marginal variations.

3. *The* **3-interior multi-choice multiattribute index (3IMMI)**, *η^3, is considered by for every $(O, f, H^m) \in \Delta^*$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\eta_s^{3,t}(O, f, H^m) = \gamma_s^t(O, f, H^m) + \frac{\gamma_s^t(O, f, H^m)}{\sum\limits_{k \in O} \gamma_k^t(O, f, H^m)} \cdot \left[h^t(f) - \sum_{k \in O} \gamma_k^t(O, f, H^m)\right],$$

where $\gamma_s^t(O, f, H^m) = \max\limits_{q \in F_s^+}\{\cdot[h^t(f) - h^t(f_{-s}, q-1)]\}$ is the maximal upper-aggregate marginal variation and $\Delta^{3*} = \{(O, f, H^m) \in \Delta | \sum\limits_{k \in O} \gamma_k^t(O, f, H^m) \neq 0$ for every $t \in \Gamma_m\}$. By definition of η^3, all operators get the maximal upper-aggregate marginal variations firstly, and further distribute the remaining usability proportionally by its maximal upper-aggregate marginal variations.

In the following, we would like to characterize the 1IFMI, 2IFMI, and 3IFMI by applying stability. A rule τ matches **1-multiattribute interior principle (1MIP)** if $\tau(O, b, H^m) = \eta^1(O, b, H^m)$ for every $(O, b, H^m) \in \Delta$ with $|O| \leq 2$. A rule τ matches **2-multiattribute inte-**

rior principle (2MIP) if $\tau(O,b,H^m) = \eta^2(O,b,H^m)$ for every $(O,b,H^m) \in \Delta$ with $|O| \leq 2$. A rule τ matches **3-multiattribute interior principle (3MIP)** if $\tau(O,b,H^m) = \eta^3(O,b,H^m)$ for every $(O,b,H^m) \in \Delta$ with $|O| \leq 2$.

It is trivial to verify that $\sum_{k \in K} \beta^t_k(O,f,H^m) = 0$ (or $\sum_{k \in K} \alpha^t_k(O,f,H^m) = 0$, $\sum_{k \in K} \gamma^t_k(O,f,H^m) = 0$) for some $(O,f,H^m) \in \Delta$, for some $K \subseteq O$, and for some $t \in \Gamma_m$, i.e., $\eta^{1,t}(K, f_K, H^m_{K,\eta})$ (or $\eta^{2,t}(K, f_K, H^m_{K,\eta})$, $\eta^{3,t}(K, f_K, H^m_{K,\eta})$) does not exist for some $(O,f,H^m) \in \Delta$, for some $K \subseteq O$ and for some $t \in \Gamma_m$. So, we focus on the multiattribute revised stability as follows. A rule τ matches **multiattribute revised-stability (MRSTA)** if $(K, f_K, H^m_{K,\tau})$ and $\tau(K, f_K, H^m_{K,\tau})$ exist for some $(O,f,H^m) \in \Delta$, for some $K \subseteq O$ and for some $t \in \Gamma_m$, it holds that $\tau_s(K, f_K, H^m_{K,\tau}) = \tau_s(O,f,H^m)$ for every $s \in K$.

Similar to Theorems 1 and 2, related axiomatic outcomes of η^1, η^2, η^3 also could be provided as follows.

Theorem 3.

1. *The rules η^1, η^2, η^3 match MECE on Δ^{1*}, Δ^{2*}, Δ^{3*} respectively;*
2. The rules η^1, η^2, η^3 match MRSTA on Δ^{1*}, Δ^{2*}, Δ^{3*} respectively;
3. *On Δ^{1*}, the 1IFMI is the only rule matching 1MIP and MRSTA;*
4. *On Δ^{2*}, the 2IFMI is the only rule matching 2MIP and MRSTA;*
5. *On Δ^{3*}, the 3IFMI is the only rule matching 3MIP and MRSTA.*

Proof of Theorem 3. The proof is similar to Lemmas 1, 2, and Theorem 2. □

In the following we give some examples to display that every of the properties applied in Theorem 3 is independent of the rest of properties.

Example 6. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta^{1*}$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\tau^t_s(O,f,H^m) = \begin{cases} \eta^{1,t}_s(O,f,H^m) & \text{if } |O| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches 1MIP, but it does not match MRSTA.

Example 7. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta^{2*}$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\tau^t_s(O,f,H^m) = \begin{cases} \eta^{2,t}_s(O,f,H^m) & \text{if } |O| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches 2MIP, but it does not match MRSTA.

Example 8. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta^{3*}$, for every $t \in \Gamma_m$ and for every operator $s \in O$,*

$$\tau^t_s(O,f,H^m) = \begin{cases} \eta^{3,t}_s(O,f,H^m) & \text{if } |O| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, τ matches 3MIP, but it does not match MRSTA.

Example 9. *We consider the rule τ as follows. For every $(O,f,H^m) \in \Delta^*$, for every $t \in \Gamma_m$, and for every operator $s \in O$, $\tau^t_s(O,f,H^m) = 0$. Clearly, τ matches MRSTA, but it does not match 1MIP, 2MIP, and 3MIP.*

In the following, we provide a numerical example which shows **(a)** how the new rules would allocate value differently than the old rules and **(b)** differently from each other. Let $(O, f, H^m) \in \Delta$ with $O = \{i, j, k\}$, $m = 2$, $f = (2, 1, 1)$, $F_i = \{0, 1_i, 2_i\}$, $F_j = \{0, 1_j\}$, $F_k = \{0, 1_k\}$, $d(i) = 5$, $d(j) = 1$, $d(k) = 2$, $w(1_i) = 3$, $w(2_i) = 4$, $w(1_j) = 7$, $w(1_k) = 4$.

Furthermore, let $h^1(2,1,1) = 5$, $h^1(1,1,1) = 7$, $h^1(2,1,0) = 3$, $h^1(2,0,1) = 2$, $h^1(2,0,0) = 9$, $h^1(1,1,0) = 3$, $h^1(1,0,1) = -4$, $h^1(0,1,1) = 4$, $h^1(1,0,0) = -1$, $h^1(0,1,0) = 2$, $h^1(0,0,1) = -3$, $h^2(2,1,1) = 9$, $h^2(1,1,1) = 3$, $h^2(2,1,0) = 5$, $h^2(2,0,1) = 6$, $h^2(2,0,0) = 4$, $h^2(1,1,0) = -3$, $h^2(1,0,1) = 4$, $h^2(0,1,1) = 3$, $h^2(1,0,0) = 7$, $h^2(0,1,0) = -2$, $h^2(0,0,1) = 3$ and $h^1(0,0,0) = 0 = h^2(0,0,0)$. By Definitions 1–4,

$$
\begin{array}{rclrclrcl}
\overline{\beta_i^1}(O,f,H^m) &=& 2, & \overline{\beta_j^1}(O,f,H^m) &=& 2, & \overline{\beta_k^1}(O,f,H^m) &=& 1,\\
\overline{\beta_i^2}(O,f,H^m) &=& \frac{14}{3}, & \overline{\beta_j^2}(O,f,H^m) &=& \frac{5}{3}, & \overline{\beta_k^2}(O,f,H^m) &=& \frac{8}{3},\\
\eta_i^{d,1}(O,f,H^m) &=& \frac{9}{8}, & \eta_j^{d,1}(O,f,H^m) &=& \frac{21}{8}, & \eta_k^{d,1}(O,f,H^m) &=& \frac{5}{4},\\
\eta_i^{d,2}(O,f,H^m) &=& \frac{7}{2}, & \eta_j^{d,2}(O,f,H^m) &=& \frac{5}{2}, & \eta_k^{d,2}(O,f,H^m) &=& 3,\\
\eta_i^{w,1}(O,f,H^m) &=& -2, & \eta_j^{w,1}(O,f,H^m) &=& 10, & \eta_k^{w,1}(O,f,H^m) &=& -3,\\
\eta_i^{w,2}(O,f,H^m) &=& \frac{20}{3}, & \eta_j^{w,2}(O,f,H^m) &=& \frac{11}{3}, & \eta_k^{w,2}(O,f,H^m) &=& \frac{-4}{3},\\
\beta_i^{d,w,1}(O,f,H^m) &=& \frac{-93}{8}, & \beta_j^{d,w,1}(O,f,H^m) &=& \frac{135}{8}, & \beta_k^{d,w,1}(O,f,H^m) &=& \frac{-1}{4},\\
\beta_i^{d,w,2}(O,f,H^m) &=& \frac{-17}{2}, & \beta_j^{d,w,2}(O,f,H^m) &=& \frac{29}{2}, & \beta_k^{d,w,2}(O,f,H^m) &=& 3,\\
\alpha_i^{d,w,1}(O,f,H^m) &=& \frac{-93}{8}, & \alpha_j^{d,w,1}(O,f,H^m) &=& \frac{135}{8}, & \alpha_k^{d,w,1}(O,f,H^m) &=& \frac{-1}{4},\\
\alpha_i^{d,w,2}(O,f,H^m) &=& \frac{-17}{2}, & \alpha_j^{d,w,2}(O,f,H^m) &=& \frac{29}{2}, & \alpha_k^{d,w,2}(O,f,H^m) &=& 3,\\
\gamma_i^{d,w,1}(O,f,H^m) &=& \frac{-111}{8}, & \gamma_j^{d,w,1}(O,f,H^m) &=& \frac{141}{8}, & \gamma_k^{d,w,1}(O,f,H^m) &=& \frac{5}{4},\\
\gamma_i^{d,w,2}(O,f,H^m) &=& \frac{-17}{2}, & \gamma_j^{d,w,2}(O,f,H^m) &=& \frac{29}{2}, & \gamma_k^{d,w,2}(O,f,H^m) &=& 3,\\
\eta_i^{1,1}(O,f,H^m) &=& \frac{15}{8}, & \eta_j^{1,1}(O,f,H^m) &=& \frac{15}{8}, & \eta_k^{1,1}(O,f,H^m) &=& \frac{5}{4},\\
\eta_i^{1,2}(O,f,H^m) &=& \frac{54}{13}, & \eta_j^{1,2}(O,f,H^m) &=& \frac{27}{13}, & \eta_k^{1,2}(O,f,H^m) &=& \frac{36}{13},\\
\eta_i^{2,1}(O,f,H^m) &=& \frac{15}{8}, & \eta_j^{2,1}(O,f,H^m) &=& \frac{15}{8}, & \eta_k^{2,1}(O,f,H^m) &=& \frac{5}{4},\\
\eta_i^{2,2}(O,f,H^m) &=& \frac{54}{13}, & \eta_j^{2,2}(O,f,H^m) &=& \frac{27}{13}, & \eta_k^{2,2}(O,f,H^m) &=& \frac{36}{13},\\
\eta_i^{3,1}(O,f,H^m) &=& \frac{5}{6}, & \eta_j^{3,1}(O,f,H^m) &=& \frac{5}{2}, & \eta_k^{3,1}(O,f,H^m) &=& \frac{5}{3},\\
\eta_i^{3,2}(O,f,H^m) &=& \frac{54}{13}, & \eta_j^{3,2}(O,f,H^m) &=& \frac{27}{13}, & \eta_k^{3,2}(O,f,H^m) &=& \frac{36}{13}.
\end{array}
$$

5. Conclusions

1. Differing from existing investigations, we introduced the WLAMI, WRMI, WUAMI, and related axiomatic outcomes by applying weights to the operators and its working levels (strategies) simultaneously under multiattribute multi-choice situations. Maximal marginal variations instead of weights naturally were discussed, the 1IFMI, 2IFMI, and 3IFMI and related axiomatic outcomes were also introduced under multiattribute multi-choice situations. One should compare related existing outcomes with the outcomes provided in this article.
    - The WLAMI, WRMI, WUAMI, 1IFMI, 2IFMI, 3IFMI, and related outcomes were proposed initially under multiattribute multi-choice games;
    - Rule concepts of traditional games have only considered non-participation or participation among all operators. In this article, we proposed two weighted multi-choice rules to analyze distribution mechanism under multiattribute situations;
    - Different from the SEANSC, the 1-SWANSC and 2-SWANSC due to Liao et al. [12] on multiattribute multi-choice games, we proposed the WLAMI, WRMI, WUAMI, 1IFMI, 2IFMI, and 3IFMI by applying weights to the operators and its working levels simultaneously.
        - Under the SEANSC and the 2-SWANSC, any additional fixed usability should be distributed equally among all operators.

- Under the 1-SWANSC, all operators receive its maximal marginal variation firstly.
- Clearly, the operators and its working levels are essential factors in the framework of multiattribute multi-choice games. Therefore, the weights should be considered simultaneously under the operators and its working levels. Under the WLAMI, WRMI, and WUAMI, all operators get different types of weighted maximal marginal variations firstly, and further distribute the remaining usability proportionally by weights for operators.
- However, the weights may be apportioned artificially. Under the 1IFMI, 2IFMI, and 3IFMI, all operators get different types of maximal marginal variations firstly, and further distribute the remaining usability proportionally by related maximal marginal variations.

2. The advantages of the rules proposed throughout this article are as follows.
   - Allocation rules of traditional games were only considered non-participation or participation among all operators. In this article, we considered that all operators possess different working levels of participation;
   - In a multitude of multi-choice game literature on allocation rules, although it might be also assumed that all operators possess different working levels of participation, most literature determined the value of a specific operator presented with a specific working level of participation, such as Hwang and Liao [15], Liao [16], and so on. In this article, we evaluated the overall value each operator exerts with different working levels of participation;
   - By considering many real-world situations, we propose the WLAMI, WRMI, and WUAMI to allocate additional fixed usability among the operators and its working levels in proportion to two types of weights simultaneously. Furthermore, several types of the maximal marginal variations were considered under the WLAMI, WRMI, and WUAMI respectively. Since the legitimacy or fairness of the weight functions may be questioned, the relative maximal marginal variations were applied as weights under different circumstances under the 1IFMI, 2IFMI, and 3IFMI naturally.
3. The disadvantages of the rules proposed throughout this article are as follows. As mentioned in the advantages above, every operator possesses different working levels. Although one could determine the overall value each operator exerts, it is impossible to evaluate the value of a specific operator with a specific working level of participation. Hence, in future research, we will consider alternative allocation rules built on the simultaneous consideration of the overall value and the specific working level of participation;
4. The outcomes of this article raise an additional motivation.
   - Other traditional rules might be generated by adopting the maximal marginal variations under multi-choice behavior and multiattribute situations.

   This is left to the readers.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Bednarczuk, E.M.; Miroforidis, J.; Pyzel, P. A multi-criteria approach to approximate solution of multiple-choice knapsack problem. *Comput. Optim. Appl.* **2018**, *70*, 889–910. [CrossRef]
2. Goli, A.; Zare, H.K.; Tavakkoli-Moghaddam, R.; Sadegheih, A. Hybrid artificial intelligence and robust optimization for a multi-objective product portfolio problem Case study: The dairy products industry. *Comput. Ind. Eng.* **2019**, *137*, 106090. [CrossRef]
3. Guarini, M.R.; Battisti, F.; Chiovitti, A. A methodology for the selection of multi-criteria decision analysis methods in real estate and land management processes. *Sustainability* **2018**, *10*, 507. [CrossRef]
4. Mustakerov, I.; Borissova, D.; Bantutov, E. Multiple-choice decision making by multicriteria combinatorial optimization. *Adv. Model. Optim.* **2018**, *14*, 729–737.
5. Tirkolaee, E.B.; Goli, A.; Hematian, M.; Sangaiah, A.K.; Han, T. Multi-objective multi-mode resource constrained project scheduling problem using Pareto-based algorithms. *Computing* **2019**, *101*, 547–570. [CrossRef]
6. Hwang, Y.A; Liao, Y.H. The unit-level-core for multi-choice games: The replicated core for TU games. *J. Glob. Optim.* **2010**, *47*, 161–171. [CrossRef]
7. Liao, Y.H. The maximal equal allocation of nonseparable costs on multi-choice games. *Econ. Bull.* **2008**, *3*, 1–8.
8. Liao, Y.H. The duplicate extension for the equal allocation of nonseparable costs. *Oper. Res. Int. J.* **2012**, *13*, 385–397. [CrossRef]
9. van den Nouweland, A.; Potters, J.; Tijs, S.; Zarzuelo, J.M. Core and related solution concepts for multi-choice games. *ZOR-Math. Methods Oper. Res.* **1995**, *41*, 289–311. [CrossRef]
10. Shapley, L.S. Discussant's Comment. In *Joint Cost Allocation*; Moriarity, S., Ed.; University of Oklahoma Press: Tulsa, OK, USA, 1982.
11. Ransmeier, J.S. *The Tennessee Valley Authority*; Vanderbilt University Press: Nashville, TN, USA, 1942.
12. Liao, Y.H.; Chung, L.Y.; Du, W.S.; Ho, S.C. Consistent solutions and related axiomatic results under multicriteria management systems. *Am. J. Math. Manag. Sci.* **2018**, *37*, 107–116. [CrossRef]
13. Moulin, H. On additive methods to share joint costs. *Jpn. Econ. Rev.* **1985**, *46*, 303–332. [CrossRef]
14. Hart, S.; Mas-Colell, A. Potential, value and consistency. *Econometrica* **1989**, *57*, 589–614. [CrossRef]
15. Hwang, Y.A.; Liao, Y.H. Potential approach and characterizations of a Shapley value in multi-choice games. *Math. Soc. Sci.* **2008**, *56*, 321–335. [CrossRef]
16. Liao, Y.H. Consonance, symmetry and extended outputs. *Symmetry* **2021**, *13*, 72. [CrossRef]