

## Article

# Global Persistence of the Unit Eigenvectors of Perturbed Eigenvalue Problems in Hilbert Spaces: The Odd Multiplicity Case

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**Abstract:** We study the persistence of eigenvalues and eigenvectors of perturbed eigenvalue problems in Hilbert spaces. We assume that the unperturbed problem has a nontrivial kernel of odd dimension and we prove a Rabinowitz-type global continuation result. The approach is topological, based on a notion of degree for oriented Fredholm maps of index zero between real differentiable Banach manifolds.

**Keywords:** eigenvalues; eigenvectors; nonlinear spectral theory; topological degree; bifurcation

**MSC:** 47J10; 47A75; 47H11; 55M25



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## 1. Introduction

Nonlinear spectral theory is a research field of increasing interest, which finds application to properties of the structure of the solution set of differential equations, see e.g., [1,2].

In this context, a nontrivial question consists of studying nonlinear perturbations of linear problems and in investigating the so-called “persistence” of eigenvalues and eigenvectors.

More precisely, let  $G$  and  $H$  denote two real Hilbert spaces. By a “perturbed eigenvalue problem” we mean a system of the following type:

$$\begin{cases} Lx + sN(x) = \lambda Cx \\ x \in S, \end{cases} \quad (1)$$

where  $s, \lambda$  are real parameters,  $L, C: G \rightarrow H$  are bounded linear operators,  $S$  denotes the unit sphere of  $G$ , and  $N: S \rightarrow H$  is a nonlinear map. We call *solution* of (1) a triple  $(s, \lambda, x) \in \mathbb{R} \times \mathbb{R} \times S$  satisfying the above system. The element  $x \in S$  is then said a *unit eigenvector* corresponding to the *eigenpair*  $(s, \lambda)$  of (4), and the set of solutions of (1) will be denoted by  $\Sigma \subseteq \mathbb{R} \times \mathbb{R} \times S$ .

To investigate the topological properties of  $\Sigma$ , we consider (1) as a (nonlinear) perturbation of the eigenvalue problem

$$\begin{cases} Lx = \lambda Cx \\ x \in S, \end{cases} \quad (2)$$

where we assume that the operator  $L - \lambda C \in \mathcal{L}(G, H)$  is invertible for some  $\lambda \in \mathbb{R}$ . When  $\lambda \in \mathbb{R}$  is such that  $\text{Ker}(L - \lambda C)$  is nontrivial, we call  $\lambda$  an *eigenvalue* of the equation  $L = \lambda C$  or, equivalently, of problem (2). A solution  $(\lambda, x)$  of (2) will be called an *eigenpoint*; in this case  $\lambda$  and  $x$  are, respectively, an eigenvalue and a *unit eigenvector* of the equation  $Lx = \lambda Cx$ .

Let  $(\lambda_*, x_*)$  be an eigenpoint of (2) and suppose that the following conditions hold:

- (H1)  $C$  is a compact operator,
- (H2)  $\text{Ker}(L - \lambda_* C)$  is odd dimensional,
- (H3)  $\text{Img}(L - \lambda_* C) \cap C(\text{Ker}(L - \lambda_* C)) = \{0\}$ .

Under assumptions (H1)–(H3) our main result, Theorem 5 below, asserts that

- in the set  $\Sigma$  of the solutions of (1), the connected component containing  $(0, \lambda_*, x_*)$  is either unbounded or includes a trivial solution  $(0, \lambda^*, x^*)$  with  $\lambda^* \neq \lambda_*$ .

The proof of Theorem 5, which can be thought of as a Rabinowitz-type global continuation result [3], is based on a preliminary study of the “unperturbed” problem (2). In particular, notice that the eigenpoints of (2) coincide with the solutions of the equation

$$\psi(\lambda, x) = 0,$$

where  $\psi$  is the  $H$ -valued function  $(\lambda, x) \mapsto Lx - \lambda Cx$  defined on the cylinder  $\mathbb{R} \times S$ , which is a smooth 1-codimensional submanifold of the Hilbert space  $\mathbb{R} \times G$ . A crucial point is then to evaluate the topological degree of the map  $\psi$ . Since the domain of  $\psi$  is a manifold, we cannot apply the classical Leray–Schauder degree. Instead, we use a notion of topological degree for oriented Fredholm maps of index zero between real differentiable Banach manifolds, developed by two authors of this paper, and whose construction and properties are summarized in Section 3 for the reader’s convenience. Such a notion of degree has been introduced in [4] (see also [5–7] for additional details).

Taking advantage of the odd multiplicity assumption (H2), of condition (H1) on the compactness of  $C$ , and of the transversality condition (H3), we are then able to apply a result of [8] concerning the case of *simple eigenvalues*. Precisely, call  $\lambda_* \in \mathbb{R}$  a *simple eigenvalue* of (2) if there exists  $x_* \in S$  such that  $\text{Ker}(L - \lambda_* C) = \mathbb{R}x_*$  and  $H = \text{Img}(L - \lambda_* C) \oplus \mathbb{R}Cx_*$ . In [8] we proved that

- if  $\lambda_*$  is a simple eigenvalue of (2) and  $x_*$  and  $-x_*$  are the two corresponding unit eigenvectors, then the “twin” eigenpoints  $p_* = (\lambda_*, x_*)$  and  $\bar{p}_* = (\lambda_*, -x_*)$  are isolated zeros of  $\psi$ . Moreover, under the assumption that the operator  $C$  is compact, they give the same contribution to the  $bf$ -degree, which is either 1 or  $-1$ , depending on the orientation of  $\psi$ .

Such an assertion generalizes, to the infinite dimensional case, an analogous result in [9] concerning a “classical eigenvalue problem” in  $\mathbb{R}^k$ . Let us point out that the result in [9] is based on the notion of Brouwer degree for maps between finite dimensional oriented manifolds, whereas, as already stressed, the extension to the infinite-dimensional setting of [8] requires a degree for Fredholm maps of index zero acting between Banach manifolds, as the one introduced in [4]. To apply this degree we need the unit sphere  $S$  to be a smooth manifold: for this reason, we restrict our study to Hilbert spaces instead of the more general Banach environment.

The study of the local [10–16] as well as global [5,8,9,17,18] persistence property when the eigenvalue  $\lambda_*$  is not necessarily simple has been performed in recent papers by the authors, also in collaboration with R. Chiappinelli. In particular, a first pioneering result in this sense is due to Chiappinelli [19], who proved the existence of the local persistence of eigenvalues and eigenvectors, in Hilbert spaces, in the case of a simple isolated eigenvalue.

Among others, let us quote our paper [18] in which we tackled a problem very similar to the one we consider here. The main result of [18] regards, roughly speaking, the global persistence property of the *eigenpairs*  $(s, \lambda)$  of (1), in the  $s\lambda$ -plane, under the *odd multiplicity*

assumption. Thus, the result we obtain here on the global persistence of the solutions  $(s, \lambda, x)$  of (1) was, in some sense, implicitly conjectured in [18].

The present paper generalizes the “global persistence” property of solution triples which either in finite-dimensional or infinite-dimensional case, has been studied in [5,8,9,17] in the case of a simple eigenvalue. Since it is known that the persistence property need not hold if  $\lambda_*$  is an eigenvalue of even multiplicity, it is natural to investigate the odd-multiplicity case. However such an extension is not trivial and is based on advanced degree-theoretical tools.

We close the paper with some illustrating examples showing, in particular, that the odd dimensionality of  $\text{Ker}(L - \lambda_* C)$  cannot be removed, the other assumptions remaining valid.

## 2. Preliminaries

In this section we recall some notions that will be used in the sequel. We mainly summarize some concepts which are needed for the construction of the topological degree for oriented Fredholm maps of index zero between real differentiable Banach manifolds introduced in [4], here called *bf*-degree to distinguish it from the Leray–Schauder degree, called *LS*-degree (see [5–7] for additional details).

It is necessary to begin by focusing on the preliminary concept of orientation for Fredholm maps of index zero between manifolds. The starting point is an algebraic notion of orientation for Fredholm linear operators of index zero.

Consider two real Banach spaces  $E$  and  $F$  and denote by  $\mathcal{L}(E, F)$  the space of the bounded linear operators from  $E$  into  $F$  with the usual operator norm. If  $E = F$ , we write  $\mathcal{L}(E)$  instead of  $\mathcal{L}(E, E)$ . By  $\text{Iso}(E, F)$  we mean the subset of  $\mathcal{L}(E, F)$  of the invertible operators, and we write  $GL(E)$  instead of  $\text{Iso}(E, E)$ . The subspace of  $\mathcal{L}(E, F)$  of the compact operators will be denoted by  $\mathcal{K}(E, F)$ , or simply by  $\mathcal{K}(E)$  when  $F = E$ . Finally,  $\mathcal{F}(E, F)$  will stand for the vector subspace of  $\mathcal{L}(E, F)$  of the operators having finite dimensional image (recall that in the infinite dimensional context,  $\mathcal{F}(E, F)$  is not closed in  $\mathcal{L}(E, F)$ ). We shall write  $\mathcal{F}(E)$  when  $F = E$ .

Recall that an operator  $T \in \mathcal{L}(E, F)$  is said to be *Fredholm* (see e.g., [20]) if its kernel,  $\text{Ker } T$ , and its cokernel,  $\text{coKer } T = F/T(E)$ , are both finite dimensional. The *index* of a Fredholm operator  $T$  is the integer

$$\text{ind } T = \dim(\text{Ker } T) - \dim(\text{coKer } T).$$

In particular, any invertible linear operator is Fredholm of index zero. Observe also that if  $T \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^s)$ , then  $\text{ind } T = k - s$ .

The subset of  $\mathcal{L}(E, F)$  of the Fredholm operators will be denoted by  $\Phi(E, F)$ ; while  $\Phi_n(E, F)$  will stand for the set  $\{T \in \Phi(E, F) : \text{ind } T = n\}$ . By  $\Phi(E)$  and  $\Phi_n(E)$  we will designate, respectively,  $\Phi(E, E)$  and  $\Phi_n(E, E)$ .

We recall some important properties of Fredholm operators.

- (F1) If  $T \in \Phi(E, F)$ , then  $\text{Im } T$  is closed in  $F$ .
- (F2) The composition of Fredholm operators is Fredholm and its index is the sum of the indices of all the composite operators.
- (F3) If  $T \in \Phi_n(E, F)$  and  $K \in \mathcal{K}(E, F)$ , then  $T + K \in \Phi_n(E, F)$ .
- (F4) For any  $n \in \mathbb{Z}$ , the set  $\Phi_n(E, F)$  is open in  $\mathcal{L}(E, F)$ .

Let  $T \in \mathcal{L}(E)$  be given. If  $I - T \in \mathcal{F}(E)$ , where  $I \in \mathcal{L}(E)$  is the identity, we say that  $T$  is an *admissible operator* (for the determinant). The symbol  $\mathcal{A}(E)$  will stand for the affine subspace of  $\mathcal{L}(E)$  of the admissible operators.

It is known (see [21]) that the determinant of an operator  $T \in \mathcal{A}(E)$  is well defined as follows:  $\det T := \det T|_{\hat{E}}$ , where  $T|_{\hat{E}}$  is the restriction (as domain and as codomain) to any finite dimensional subspace  $\hat{E}$  of  $E$  containing  $\text{Im}(I - T)$ , with the understanding that  $\det T|_{\hat{E}} = 1$  if  $\hat{E} = \{0\}$ . As one can check, the function  $\det: \mathcal{A}(E) \rightarrow \mathbb{R}$  inherits most of the properties of the classical determinant. For more details, see e.g., [22].

Let  $T \in \Phi_0(E, F)$  be given. As in [5], we will say that an operator  $K \in \mathcal{F}(E, F)$  is a *companion* of  $T$  if  $T + K$  is invertible (in previous papers, e.g., in [4], it was used the word *corrector* instead of *companion*).

Observe in particular that any  $T \in \text{Iso}(E, F)$  has a *natural companion*: that is, the zero operator  $0 \in \mathcal{L}(E, F)$ . This fact was crucial in [4] for the construction of the *bf*-degree.

Given  $T \in \Phi_0(E, F)$ , we denote by  $\mathcal{C}(T)$  the (nonempty) subset of  $\mathcal{F}(E, F)$  of all the companions of  $T$ . The following definition establishes a partition of  $\mathcal{C}(T)$  in two equivalence classes and is a key step for the definition of orientation given in [4].

**Definition 1** (Equivalence relation). *Two companions  $K_1$  and  $K_2$  of an operator  $T \in \Phi_0(E, F)$  are equivalent (more precisely,  $T$ -equivalent) if the admissible operator  $(T + K_2)^{-1}(T + K_1)$  has positive determinant.*

**Definition 2** (Orientation). *An orientation of  $T \in \Phi_0(E, F)$  is one of the two equivalence classes of  $\mathcal{C}(T)$ , denoted by  $\mathcal{C}_+(T)$  and called the class of positive companions of the oriented operator  $T$ . The set  $\mathcal{C}_-(T) = \mathcal{C}(T) \setminus \mathcal{C}_+(T)$  of the negative companions is the opposite orientation of  $T$ .*

Some further definitions are in order.

**Definition 3** (Natural orientation). *Any  $T \in \text{Iso}(E, F)$  admits the natural orientation: the one given by considering the trivial operator of  $\mathcal{L}(E, F)$  as a positive companion.*

**Definition 4** (Oriented composition). *The oriented composition of two oriented operators,  $T_1 \in \Phi_0(E_1, E_2)$  and  $T_2 \in \Phi_0(E_2, E_3)$ , is the operator  $T_2 T_1$  with the orientation given by considering  $K = (T_2 + K_2)(T_1 + K_1) - T_2 T_1$  as a positive companion whenever  $K_1$  and  $K_2$  are positive companions of  $T_1$  and  $T_2$ , respectively.*

Observe that the oriented composition is associative and, consequently, this notion can be extended to the composition of three (or more) oriented operators.

**Definition 5** (Sign of an oriented operator). *Let  $T \in \Phi_0(E, F)$  be an oriented operator. Its sign is the integer*

$$\text{sign } T = \begin{cases} +1 & \text{if } T \text{ is invertible and naturally oriented,} \\ -1 & \text{if } T \text{ is invertible and not naturally oriented,} \\ 0 & \text{if } T \text{ is not invertible.} \end{cases}$$

A crucial fact in the definition of oriented map and the consequent construction of the *bf*-degree is that

- the orientation of any operator  $T_* \in \Phi_0(E, F)$  induces an orientation of the operators in a neighborhood of  $T_*$ .

In fact, since  $\text{Iso}(E, F)$  is open in  $\mathcal{L}(E, F)$ , for any companion  $K$  of  $T_*$  we have that  $T + K$  is invertible when  $T$  is sufficiently close to  $T_*$ . Thus, because of property (F3) of the Fredholm operators, any such  $T$  belongs to  $\Phi_0(E, F)$ . Consequently,  $K$  is as well a companion of  $T$ .

**Definition 6.** *Let  $\Gamma: X \rightarrow \Phi_0(E, F)$  be a continuous map defined on a metric space  $X$ . A pre-orientation of  $\Gamma$  is a function that to any  $x \in X$  assigns an orientation  $\omega(x)$  of  $\Gamma(x)$ . A pre-orientation (of  $\Gamma$ ) is an orientation if it is continuous, in the sense that given any  $x_* \in X$ , there exist  $K \in \omega(x_*)$  and a neighborhood  $W$  of  $x_*$  such that  $K \in \omega(x)$  for all  $x \in W$ . The map  $\Gamma$  is said to be orientable if it admits an orientation, and oriented if an orientation has been chosen. In particular, a subset  $Y$  of  $\Phi_0(E, F)$  is orientable or oriented if so is the inclusion map  $Y \hookrightarrow \Phi_0(E, F)$ .*

Observe that the set  $\hat{\Phi}_0(E, F)$  of the oriented operators of  $\Phi_0(E, F)$  has a natural topology, and the natural projection  $\pi: \hat{\Phi}_0(E, F) \rightarrow \Phi_0(E, F)$  is a 2-fold covering space (see [7] for details). Therefore, an orientation of a map  $\Gamma$  as in Definition 6 could be regarded as a lifting  $\hat{\Gamma}$  of  $\Gamma$ . This implies that if the domain  $X$  of  $\Gamma$  is simply connected and locally path connected, then  $\Gamma$  is orientable.

Let  $f: U \rightarrow F$  be a  $C^1$ -map defined on an open subset of  $E$ , and denote by  $df_x \in \mathcal{L}(E, F)$  the Fréchet differential of  $f$  at a point  $x \in U$ .

We recall that  $f$  is said to be *Fredholm of index  $n$* , called  $\Phi_n$ -map and hereafter also denoted by  $f \in \Phi_n$ , if  $df_x \in \Phi_n(E, F)$  for all  $x \in U$ . Therefore, if  $f \in \Phi_0$ , Definition 6 and the continuity of the differential map  $df: U \rightarrow \Phi_0(E, F)$  suggest the following

**Definition 7** (Orientation of a  $\Phi_0$ -map in Banach spaces). *Let  $U$  be an open subset of  $E$  and  $f: U \rightarrow F$  a Fredholm map of index zero. A pre-orientation or an orientation of  $f$  are, respectively, a pre-orientation or an orientation of  $df$ , according to Definition 6. The map  $f$  is said to be orientable if it admits an orientation, and oriented if an orientation has been chosen.*

**Remark 1.** *A very special  $\Phi_0$ -map is given by an operator  $T \in \Phi_0(E, F)$ . Thus, for  $T$  there are two different notions of orientations: the algebraic one and that in which  $T$  is seen as a  $C^1$ -map, according to Definitions 2 and 7, respectively. In each case  $T$  admits exactly two orientations (in the second one this is due to the connectedness of the domain  $E$ ). Hereafter, we shall tacitly assume that the two notions agree. Namely,  $T$  has an algebraic orientation  $\omega$  if and only if its differential  $dT_x: \dot{x} \mapsto T\dot{x}$  has the  $\omega$  orientation for all  $x \in E$ .*

Let us summarize how the notion of orientation can be given for maps acting between real Banach manifolds. In the sequel, by *manifold* we shall mean, for short, a smooth Banach manifold embedded in a real Banach space.

Given a manifold  $\mathcal{M}$  and a point  $x \in \mathcal{M}$ , the tangent space of  $\mathcal{M}$  at  $x$  will be denoted by  $T_x\mathcal{M}$ . If  $\mathcal{M}$  is embedded in a Banach space  $\tilde{E}$ ,  $T_x\mathcal{M}$  will be identified with a closed subspace of  $\tilde{E}$ , for example by regarding any tangent vector of  $T_x\mathcal{M}$  as the derivative  $\Gamma'(0)$  of a smooth curve  $\Gamma: (-1, 1) \rightarrow \mathcal{M}$  such that  $\Gamma(0) = x$ .

Assume that  $f: \mathcal{M} \rightarrow \mathcal{N}$  is a  $C^1$ -map between two manifolds, respectively embedded in  $\tilde{E}$  and  $\tilde{F}$  and modelled on  $E$  and  $F$ . As in the flat case,  $f$  is said to be *Fredholm of index  $n$*  (written  $f \in \Phi_n$ ) if so is the differential  $df_x: T_x\mathcal{M} \rightarrow T_{f(x)}\mathcal{N}$ , for any  $x \in \mathcal{M}$  (see [23]).

Given  $f \in \Phi_0$ , suppose that to any  $x \in \mathcal{M}$  it is assigned an orientation  $\omega(x)$  of  $df_x$  (also called *orientation of  $f$  at  $x$* ). As above, the function  $\omega$  is called a *pre-orientation of  $f$* , and an *orientation* if it is continuous, in a sense to be specified (see Definition 9).

**Definition 8.** *The pre-oriented composition of two (or more) pre-oriented maps between manifolds is given by assigning, at any point  $x$  of the domain of the composite map, the composition of the orientations (according to Definition 4) of the differentials in the chain representing the differential at  $x$  of the composite map.*

Assume that  $f: \mathcal{M} \rightarrow \mathcal{N}$  is a  $C^1$ -diffeomorphism. Thus, for any  $x \in \mathcal{M}$ , we may take as  $\omega(x)$  the natural orientation of  $df_x$  (recall Definition 3). This pre-orientation of  $f$  turns out to be continuous according to Definition 9 below (it is, in some sense, constant).

From now on, unless otherwise stated, *any diffeomorphism will be considered oriented with the natural orientation*. In particular, in a composition of pre-oriented maps, all charts and parametrizations of a manifold will be tacitly assumed to be naturally oriented.

**Definition 9** (Orientation of a  $\Phi_0$ -map between manifolds). *Let  $f: \mathcal{M} \rightarrow \mathcal{N}$  be a  $\Phi_0$ -map between two manifolds modelled on  $E$  and  $F$ , respectively. A pre-orientation of  $f$  is an orientation if it is continuous in the sense that given any two charts,  $\varphi: U \rightarrow E$  of  $\mathcal{M}$  and  $\zeta: V \rightarrow F$  of  $\mathcal{N}$ , such that  $f(U) \subseteq V$ , the pre-oriented composition*

$$\zeta \circ f \circ \varphi^{-1}: U \rightarrow V$$



is an oriented map according to Definition 7.

The map  $f$  is said to be orientable if it admits an orientation, and oriented if an orientation has been chosen.

For example any local diffeomorphism  $f: \mathcal{M} \rightarrow \mathcal{N}$  admits the *natural orientation*, given by assigning the natural orientation to the operator  $df_x$ , for any  $x \in \mathcal{M}$  (see Definition 3).

In contrast, a very simple example of non-orientable  $\Phi_0$ -map is given by a constant map from the 2-dimensional projective space into  $\mathbb{R}^2$  (see [7]).

**Notation 1.** Let  $D$  be a subset of the product  $X \times Y$  of two metric spaces. Given  $x \in X$ , we call  $x$ -slice of  $D$  the set  $D_x = \{y \in Y : (x, y) \in D\}$ . Moreover, if  $f: D \rightarrow Z$  is a map into a metric space  $Z$ , we denote by  $f_x: D_x \rightarrow Z$  the partial map of  $f$  defined by  $f_x = f(x, \cdot)$ .

As with the case of a single map, one can define a notion of orientation of a continuous family of  $\Phi_0$ -maps depending on a parameter  $s \in [0, 1]$ . To be precise, one has the following

**Definition 10** (Oriented  $\Phi_0$ -homotopy). A  $\Phi_0$ -homotopy between two Banach manifolds  $\mathcal{M}$  and  $\mathcal{N}$  is a  $C^1$ -map  $h: [0, 1] \times \mathcal{M} \rightarrow \mathcal{N}$  such that for any  $s \in [0, 1]$ , the partial map  $h_s = h(s, \cdot)$  is Fredholm of index zero. An orientation of  $h$  is a continuous function  $\omega$  that to any  $(s, x) \in [0, 1] \times \mathcal{M}$  assigns an orientation  $\omega(s, x)$  to the differential  $d(h_s)_x \in \Phi_0(T_x \mathcal{M}, T_{h(s,x)} \mathcal{N})$ , where “continuous” means that, given any chart  $\varphi: U \rightarrow E$  of  $\mathcal{M}$ , a subinterval  $J$  of  $[0, 1]$ , and a chart  $\zeta: V \rightarrow F$  of  $\mathcal{N}$  such that  $h(J \times U) \subseteq V$ , the pre-orientation of the map  $\Gamma: J \times U \rightarrow \Phi_0(E, F)$  that to any  $(s, x) \in J \times U$  assigns the pre-oriented composition

$$d(\zeta \circ h_s \circ \varphi^{-1})_x = d\zeta_{h(s,x)} d(h_s)_x (d\varphi_x)^{-1}$$

is an orientation, according to Definition 6.

The homotopy  $h$  is said to be orientable if it admits an orientation, and oriented if an orientation has been chosen.

If a  $\Phi_0$ -homotopy  $h$  has an orientation  $\omega$ , then any partial map  $h_s = h(s, \cdot)$  has a compatible orientation  $\omega(s, \cdot)$ . Conversely, one has the following

**Proposition 1** ([4,7]). Let  $h: [0, 1] \times \mathcal{M} \rightarrow \mathcal{N}$  be a  $\Phi_0$ -homotopy, and assume that one of its partial maps, say  $h_s$ , has an orientation. Then, there exists and is unique an orientation of  $h$  which is compatible with that of  $h_s$ . In particular, if two maps from  $\mathcal{M}$  to  $\mathcal{N}$  are  $\Phi_0$ -homotopic, then they are both orientable or both non-orientable.

As a consequence of Proposition 1, one gets that any  $C^1$ -map  $f: \mathcal{M} \rightarrow \mathcal{M}$  which is  $\Phi_0$ -homotopic to the identity is orientable, since so is the identity (even when  $\mathcal{M}$  is finite dimensional and not orientable).

The  $bf$ -degree, introduced in [4], satisfies the three fundamental properties listed below: *Normalization, Additivity and Homotopy Invariance*. In [6], by means of an axiomatic approach, it is proved that the  $bf$ -degree is the only possible integer-valued function that satisfies these three properties.

More in detail, the  $bf$ -degree is defined in a class of *admissible triples*. Given an oriented  $\Phi_0$ -map  $f: \mathcal{M} \rightarrow \mathcal{N}$ , an open (possibly empty) subset  $U$  of  $\mathcal{M}$ , and a target value  $y \in \mathcal{N}$ , the triple  $(f, U, y)$  is said to be *admissible* for the  $bf$ -degree provided that  $U \cap f^{-1}(y)$  is compact. From the axiomatic point of view, the  $bf$ -degree is an integer-valued function,  $\deg_{bf}$ , defined on the class of all the admissible triples, that satisfies the following three *fundamental properties*.

- (Normalization) If  $f: \mathcal{M} \rightarrow \mathcal{N}$  is a naturally oriented diffeomorphism onto an open subset of  $\mathcal{N}$ , then

$$\deg_{bf}(f, \mathcal{M}, y) = 1, \quad \forall y \in f(\mathcal{M}).$$

- (Additivity) Let  $(f, U, y)$  be an admissible triple. If  $U_1$  and  $U_2$  are two disjoint open subsets of  $U$  such that  $U \cap f^{-1}(y) \subseteq U_1 \cup U_2$ , then

$$\deg_{bf}(f, U, y) = \deg_{bf}(f|_{U_1}, U_1, y) + \deg_{bf}(f|_{U_2}, U_2, y).$$

- (Homotopy Invariance) Let  $h: [0, 1] \times \mathcal{M} \rightarrow \mathcal{N}$  be an oriented  $\Phi_0$ -homotopy, and  $\Gamma: [0, 1] \rightarrow \mathcal{N}$  a continuous path. If the set

$$\{(s, x) \in [0, 1] \times \mathcal{M} : h(s, x) = \gamma(s)\}$$

is compact, then  $\deg_{bf}(h(s, \cdot), \mathcal{M}, \gamma(s))$  does not depend on  $s \in [0, 1]$ .

Other useful properties are deduced from the fundamental ones (see [6] for details). Here we mention some of them.

- (Localization) If  $(f, U, y)$  is an admissible triple, then

$$\deg(f, U, y) = \deg(f|_U, U, y).$$

- (Existence) If  $(f, U, y)$  is admissible and  $\deg_{bf}(f, U, y) \neq 0$ , then the equation  $f(x) = y$  admits at least one solution in  $U$ .
- (Excision) If  $(f, U, y)$  is admissible and  $V$  is an open subset of  $U$  such that  $f^{-1}(y) \cap U \subseteq V$ , then

$$\deg(f, U, y) = \deg(f, V, y).$$

In some sense, given an admissible triple  $(f, U, y)$ , the integer  $\deg_{bf}(f, U, y)$  is an algebraic count of the solutions in  $U$  of the equation  $f(x) = y$ . In fact, from the fundamental properties one gets the following

- (Computation Formula) If  $(f, U, y)$  is admissible and  $y$  is a regular value for  $f$  in  $U$ , then the set  $U \cap f^{-1}(y)$  is finite and

$$\deg_{bf}(f, U, y) = \sum_{x \in U \cap f^{-1}(y)} \text{sign}(df_x).$$

Another useful property that can be deduced from the fundamental ones is the

- (Topological Invariance) If  $(f, U, y)$  is admissible and  $g: \mathcal{N} \rightarrow \mathcal{O}$  is a naturally oriented diffeomorphism onto a manifold  $\mathcal{O}$ , then

$$\deg_{bf}(f, U, y) = \deg_{bf}(g \circ f, U, g(y)).$$

Some further notation and definitions are in order.

**Notation 2.** Hereafter we will use the shorthand notation  $\deg_{bf}(f, U)$  instead of  $\deg_{bf}(f, U, 0)$ , where  $f: \mathcal{M} \rightarrow F$  is an oriented  $\Phi_0$ -map from a manifold into a Banach space,  $U$  is an open subset of  $\mathcal{M}$ , and  $0$  is the null vector of  $F$ . Analogously,  $\deg_{LS}(f, U)$  means the Leray–Schauder degree  $\deg_{LS}(f, U, 0)$ , where  $U$  is an open bounded subset of a Banach space  $E$ ,  $f: \bar{U} \rightarrow E$  is a compact vector field defined on the closure of  $U$ , and  $0$  is the null vector of  $E$ .

**Definition 11.** Let  $X$  be a metric space and  $\mathcal{K} \subseteq \mathcal{A} \subseteq X$ . We shall say that  $\mathcal{K}$  is an isolated subset of  $\mathcal{A}$  if it is compact and relatively open in  $\mathcal{A}$ . Thus, there exists an open subset  $U$  of  $X$  such that  $U \cap \mathcal{A} = \mathcal{K}$ . The set  $U$  is called an isolating neighborhood of  $\mathcal{K}$  among (the elements of)  $\mathcal{A}$ .

**Definition 12.** Let  $f: \mathcal{M} \rightarrow F$  be an oriented  $\Phi_0$ -map from a manifold into a Banach space. If  $\mathcal{K}$  is an isolated subset of  $f^{-1}(0)$ , we shall call contribution of  $\mathcal{K}$  to the  $bf$ -degree of  $f$  the integer  $\deg_{bf}(f, U)$ , where  $U \subseteq \mathcal{M}$  is any isolating neighborhood of  $\mathcal{K}$  among  $f^{-1}(0)$ . The excision property of the degree implies that  $\deg_{bf}(f, U)$  does not depend on the isolating neighborhood  $U$ .

Regarding Definition 12, we observe that the finite union of isolated subsets of  $f^{-1}(0)$  is still an isolated subset. Moreover, from the excision and the additivity properties of the  $bf$ -degree one gets that the contribution to the  $bf$ -degree of this union is the sum of the single contributions of these subsets.

### 3. The Eigenvalue Problem and the Associated Topological Degree

Let, hereafter,  $G$  and  $H$  denote two real Hilbert spaces and consider the eigenvalue problem

$$\begin{cases} Lx = \lambda Cx \\ x \in S, \end{cases} \quad (3)$$

where  $\lambda$  is a real parameter,  $L, C: G \rightarrow H$  are bounded linear operators, and  $S$  denotes the unit sphere of  $G$ .

To prevent the problem from being meaningless, we will always assume that the operator  $L - \lambda C \in \mathcal{L}(G, H)$  is invertible for some  $\lambda \in \mathbb{R}$ .

When  $\lambda \in \mathbb{R}$  is such that  $\text{Ker}(L - \lambda C)$  is nontrivial, then  $\lambda$  is called an *eigenvalue* of the equation  $L = \lambda C$  or, equivalently, of problem (3).

A solution  $(\lambda, x)$  of (3) will also be called an *eigenpoint*. In this case,  $\lambda$  and  $x$  are, respectively, an eigenvalue and a *unit eigenvector* of the equation  $Lx = \lambda Cx$ .

Notice that the eigenpoints are the solutions of the equation

$$\psi(\lambda, x) = 0,$$

where  $\psi$  is the  $H$ -valued function  $(\lambda, x) \mapsto Lx - \lambda Cx$  defined on the cylinder  $\mathbb{R} \times S$ , which is a smooth 1-codimensional submanifold of the Hilbert space  $\mathbb{R} \times G$ .

By  $\mathcal{S}$  we will denote the set of the eigenpoints of (3). Therefore, given any  $\lambda \in \mathbb{R}$ , the  $\lambda$ -slice  $\mathcal{S}_\lambda = \{x \in S : (\lambda, x) \in \mathcal{S}\}$  of  $\mathcal{S}$  coincides with  $S \cap \text{Ker}(L - \lambda C)$ .

Thus,  $\mathcal{S}_\lambda$  is nonempty if and only if  $\lambda$  is an eigenvalue of problem (3). In this case,  $\mathcal{S}_\lambda$  will be called the *eigensphere* of (3) corresponding to  $\lambda$  or, simply, the  $\lambda$ -*eigensphere*. Observe that  $\mathcal{S}_\lambda$  is a sphere whose dimension equals that of  $\text{Ker}(L - \lambda C)$  minus one. The nonempty subset  $\{\lambda\} \times \mathcal{S}_\lambda$  of the cylinder  $\mathbb{R} \times S$  will be called an *eigenet* of (3).

**Remark 2.** The assumption that  $L - \lambda C$  is invertible for some  $\lambda \in \mathbb{R}$  implies that, for any  $\lambda \in \mathbb{R}$ , the restriction of  $C$  to the (possibly trivial) kernel of  $L - \lambda C$  is injective.

Remark 2 can be proved arguing by contradiction. In fact, assume that the assertion is false. Then, there are  $\lambda_* \in \mathbb{R}$  and a nonzero vector

$$x_* \in \text{Ker}(L - \lambda_* C) \cap \text{Ker } C.$$

This implies that, for any  $\lambda$ , the operator  $L - \lambda C$  is non-injective and, consequently, non-invertible, in contrast to the assumption. In fact, for any  $\lambda$ , one has

$$(L - \lambda C)x_* = (L - \lambda_* C)x_* - (\lambda - \lambda_*)Cx_* = 0.$$

**Remark 3.** If the operator  $C$  is compact, then, from the assumption that  $L - \lambda C$  is invertible for some  $\hat{\lambda} \in \mathbb{R}$ , it follows that  $L - \lambda C$  is Fredholm of index zero for any  $\lambda \in \mathbb{R}$  and, consequently, the set of the eigenvalues of problem (3) is discrete. Moreover,  $\text{Ker}(L - \lambda C)$  is always finite dimensional, and so is the intersection

$$\text{Im}(L - \lambda C) \cap C(\text{Ker}(L - \lambda C)).$$



Consequently, if this intersection is the singleton  $\{0\}$ , taking into account Remark 2 and the fact that  $L - \lambda C \in \Phi_0(G, H)$ , one has

$$H = \text{Img}(L - \lambda C) \oplus C(\text{Ker}(L - \lambda C)).$$

To prove Remark 3 notice that, if  $L - \hat{\lambda}C$  is invertible, then it is trivially Fredholm of index zero. Now, given any  $\lambda \in \mathbb{R}$ , one has

$$(L - \lambda C) = (L - \hat{\lambda}C) - (\lambda - \hat{\lambda})C.$$

Thus, because of the compactness of  $C$ , from property (F3) of Fredholm operators, one gets that  $L - \lambda C$  is also Fredholm of index zero. Finally, the set of the eigenvalues of problem (3) is discrete since so is, according to the spectral theory of linear operators, the set of the characteristic values of  $(L - \hat{\lambda}C)^{-1}C$ .

Because of Remark 3, from now until the end of this section we assume that the operator  $C$  is compact.

Observe that the function  $\psi$  defined above is the restriction to  $\mathbb{R} \times S$  of the nonlinear smooth map

$$\bar{\psi}: \mathbb{R} \times G \rightarrow H, \quad (\lambda, x) \mapsto Lx - \lambda Cx.$$

According to Remark 3, any partial map  $\bar{\psi}_\lambda: G \rightarrow H$  of  $\bar{\psi}$  is Fredholm of index zero. Since the map  $\bar{\sigma}: \mathbb{R} \times G \rightarrow G$  given by  $\bar{\sigma}(\lambda, x) = x$  is clearly  $\Phi_1$ , the same holds true, because of the property (F2) of Fredholm operators, for the composition  $\bar{\psi} = \bar{\psi}_\lambda \circ \bar{\sigma}$ . Consequently, again because of property (F2), one has that the restriction  $\psi$  of  $\bar{\psi}$  to the 1-codimensional submanifold  $\mathbb{R} \times S$  of  $\mathbb{R} \times G$  is  $\Phi_0$ .

Notice that, if  $\dim G = 1$ , the cylinder  $\mathbb{R} \times S$  is disconnected: it is the union of two horizontal lines,  $\mathbb{R} \times \{-1\}$  and  $\mathbb{R} \times \{1\}$ .

Because of this, to make some statements simpler, from now on, unless otherwise stated, we assume that the dimension of the space  $G$  is greater than 1.

In this case, the cylinder  $\mathbb{R} \times S$  is connected, and simply connected if  $\dim G > 2$ . It is actually contractible if  $G$  is infinite dimensional. Therefore, the  $\Phi_0$ -map  $\psi$ , defined above, is orientable and admits exactly two orientations.

We choose one of them and hereafter we assume that  $\psi$  is oriented.

**Remark 4.** Let  $\hat{\lambda} \in \mathbb{R}$  be such that  $L - \hat{\lambda}C$  is invertible and let  $Z: H \rightarrow G$  denote its inverse. Then, given any  $\lambda \in \mathbb{R}$ , the two equations

- $\bar{\psi}_\lambda(x) = (L - \lambda C)x = 0 \in H$ ,
- $\bar{\eta}_\lambda(x) = Z\bar{\psi}_\lambda(x) = (I - (\lambda - \hat{\lambda})ZC)x = 0 \in G$

are equivalent ( $I$  being the identity on  $G$ ). Therefore, if  $B$  denotes the unit ball of  $G$ , the Leray–Schauder degree with target  $0 \in G$ ,  $\deg_{LS}(\bar{\eta}_\lambda, B)$ , of the compact vector field  $\bar{\eta}_\lambda$  is well defined whenever  $\lambda$  is not an eigenvalue of the equation  $Lx = \lambda Cx$ .

Observe that, as a consequence of the homotopy invariance property of the Leray–Schauder degree, the function  $\lambda \mapsto \deg_{LS}(\bar{\eta}_\lambda, B)$  is constant on any interval in which it is defined. Moreover, in these intervals,  $\deg_{LS}(\bar{\eta}_\lambda, B)$  is either 1 or  $-1$ , since the equation  $\bar{\eta}_\lambda(x) = 0$  has only one solution: the regular point  $0 \in G$ .

**Remark 5.** Let  $U$  be an isolating neighborhood of a compact subset of the set  $S$  of the eigenpoints of (3), and let  $Z: H \rightarrow G$  be as in Remark 4. Then  $\deg_{bf}(\psi, U) = \deg_{bf}(\eta, U)$ , provided that the map  $\eta = Z\psi$  is the oriented composition obtained by considering  $Z$  as a naturally oriented diffeomorphism.

Concerning possible relations between the LS-degree of  $\bar{\eta}_\lambda$  and the bf-degree of  $\psi$  (or, equivalently, of  $\eta = Z\psi$ ), we believe that the following conjecture is true.

- (C) Let  $[\alpha, \beta]$  be a compact (nontrivial) real interval such that the extremes are not eigenvalues of  $Lx = \lambda Cx$ . Then the  $bf$ -degree of  $\psi$  (or, equivalently, of  $\eta = Z\psi$ ) on the open subset  $U = (\alpha, \beta) \times S$  of  $\mathbb{R} \times S$  is different from zero if and only if  $\deg_{LS}(\bar{\eta}_\alpha, B) \neq \deg_{LS}(\bar{\eta}_\beta, B)$ .

In support of the above conjecture we observe that both the conditions

$$\deg_{bf}(\psi, U) \neq 0 \quad \text{and} \quad \deg_{LS}(\bar{\eta}_\alpha, B) \neq \deg_{LS}(\bar{\eta}_\beta, B)$$

imply the existence of at least one eigenpoint  $p_* = (\lambda_*, x_*) \in U$ . The first one because of the existence property of the  $bf$ -degree and the last one due to the homotopy invariance property of the  $LS$ -degree.

**Definition 13.** An eigenpoint  $(\lambda_*, x_*)$  of (3) is said to be simple provided that the operator  $T = L - \lambda_* C$  is Fredholm of index zero and satisfies the conditions:

- (1)  $\text{Ker } T = \mathbb{R}x_*$ ,
- (2)  $Cx_* \notin \text{Img } T$ .

We point out that, if an eigenpoint  $p_* = (\lambda_*, x_*)$  is simple, then the corresponding eigenset  $\{\lambda_*\} \times S_{\lambda_*}$  is disconnected. In fact, it has only two elements:  $p_*$  and its twin eigenpoint  $\bar{p}_* = (\lambda_*, -x_*)$ , which is as well simple.

The following theorem obtained in [5] was essential in the proofs of some results in [5] concerning perturbations of (3), as problem (4) in the next section.

**Theorem 1.** In addition to the compactness of  $C$ , assume that  $p_* = (\lambda_*, x_*)$  and  $\bar{p}_* = (\lambda_*, -x_*)$  are two simple twin eigenpoints of (3). Then, the contributions of  $p$  and  $\bar{p}$  to the  $bf$ -degree of  $\psi$  are equal: they are both either 1 or  $-1$  depending on the orientation of  $\psi$ . Consequently, if  $U$  is an isolating neighborhood of the eigenset  $\{\lambda_*\} \times S_{\lambda_*}$ , one has  $\deg_{bf}(\psi, U) = \pm 2$ .

We close this section strictly devoted to the unperturbed eigenvalue problem (3) with a consequence of Theorem 1, which will be crucial in the proof of our main result (Theorem 5 in Section 4).

**Theorem 2.** Let  $\lambda_* \in \mathbb{R}$ , put  $T = L - \lambda_* C$ , and suppose that

- (H1)  $C$  is a compact operator,
- (H2)  $\text{Ker } T$  is odd dimensional,
- (H3)  $\text{Img } T \cap C(\text{Ker } T) = \{0\}$ .

Then, given (in  $\mathbb{R} \times S$ ) an isolating neighborhood  $U$  of the eigenset  $\{\lambda_*\} \times S_{\lambda_*}$ , one has  $\deg_{bf}(\psi, U) \neq 0$ .

**Proof.** Because of the assumption  $\text{Img } T \cap C(\text{Ker } T) = \{0\}$ , as well as the fact that  $T$  is Fredholm of index zero, we can split the spaces  $G$  and  $H$  as follows:

$$G = G_1 \oplus G_2 \quad \text{with} \quad G_1 = (\text{Ker } T)^\perp \quad \text{and} \quad G_2 = \text{Ker } T;$$

$$H = H_1 \oplus H_2 \quad \text{with} \quad H_1 = \text{Img } T \quad \text{and} \quad H_2 = C(\text{Ker } T).$$

With these splittings,  $T$  and  $C$  can be represented in block matrix form as follows:

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{pmatrix}.$$

The operators  $T_{11}: G_1 \rightarrow H_1$  and  $C_{22}: G_2 \rightarrow H_2$  are isomorphisms (the second one because of Remark 2), while  $C_{11}: G_1 \rightarrow H_1$  and  $C_{21}: G_1 \rightarrow H_2$  are, respectively, compact and finite dimensional.

We can equivalently regard the equation  $\bar{\psi}(\lambda, x) = 0$  as  $Z\bar{\psi}(\lambda, x) = 0$ , where  $Z: H \rightarrow G$  is an isomorphism. We choose  $Z$  as follows:

$$Z = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & C_{22}^{-1} \end{pmatrix}.$$

Given any  $\lambda \in \mathbb{R}$ , the operator  $\bar{\psi}_\lambda = L - \lambda C \in \mathcal{L}(G, H)$  can be written as  $T - (\lambda - \lambda_*)C$ . Therefore, putting  $\bar{\eta} = Z\bar{\psi}: \mathbb{R} \times G \rightarrow G$ , the partial map  $\bar{\eta}_\lambda: G \rightarrow G$  (see Notation 1) can be represented as

$$\bar{\eta}_\lambda = \begin{pmatrix} I_{11} - (\lambda - \lambda_*)\hat{C}_{11} & 0 \\ -(\lambda - \lambda_*)\hat{C}_{21} & \lambda_* I_{22} - \lambda I_{22} \end{pmatrix},$$

where  $I$  is the identity on  $G = G_1 \oplus G_2$  and  $\hat{C} = ZC$  (observe that  $\hat{C}_{22}$  coincides with the identity  $I_{22} \in \mathcal{L}(G_2)$ ).

This shows that, given any  $\lambda \in \mathbb{R}$ , the endomorphism  $\bar{\eta}_\lambda: G \rightarrow G$  is a compact vector field. Therefore, its Leray–Schauder degree on the unit ball  $B$  of  $G$  is well defined whenever  $\lambda$  is not an eigenvalue of the equation  $Lx = \lambda Cx$ , and this happens when  $\lambda$  is close to, but different from,  $\lambda_*$ . Since  $G_2$  is odd dimensional and, because of assumption (H3), the geometric and algebraic multiplicities of  $\lambda_*$  coincide, the function  $\lambda \mapsto \deg_{LS}(\bar{\eta}_\lambda, B)$  has a sign-jump crossing  $\lambda_*$ . Therefore, if conjecture (C) above were true, we would have done. So we need to proceed differently.

We consider an isolating neighborhood of the eigenset  $\{\lambda_*\} \times S_{\lambda_*}$  of the type  $U = (\alpha, \beta) \times S$  and we approximate the family of operators  $\bar{\eta}_\lambda, \lambda \in [\alpha, \beta]$ , with a family  $\bar{\eta}_\lambda^\varepsilon \in \mathcal{L}(G), \lambda \in [\alpha, \beta]$ , having in  $(\alpha, \beta)$  only simple eigenvalues; the number of them equal to the dimension of  $G_2 = \text{Ker } T$ .

First of all we point out that

- the operator  $I_{11} - (\lambda - \lambda_*)\hat{C}_{11} \in \mathcal{L}(G_1)$  is invertible for all  $\lambda \in [\alpha, \beta]$ , since otherwise the equation  $Lx = \lambda Cx$  would have eigenvalues different from  $\lambda_*$  in the interval  $[\alpha, \beta]$ .

Now, given  $\varepsilon > 0$  such that  $(\lambda_* - \varepsilon, \lambda_* + \varepsilon) \subset (\alpha, \beta)$ , we choose a linear operator  $A^\varepsilon \in \mathcal{L}(G_2)$  with the following properties:

- in the operator norm, the distance between  $A^\varepsilon$  and  $\lambda_* I_{22}$  is less than  $\varepsilon$ ,
- the eigenvalues of  $A^\varepsilon$  are real and simple,
- any eigenvalue  $\lambda$  of  $A^\varepsilon$  is such that  $|\lambda - \lambda_*| < \varepsilon$ .

For any  $\lambda \in \mathbb{R}$  we define  $\bar{\eta}_\lambda^\varepsilon \in \mathcal{L}(G_1 \oplus G_2)$  by

$$\bar{\eta}_\lambda^\varepsilon = \begin{pmatrix} I_{11} - (\lambda - \lambda_*)\hat{C}_{11} & 0 \\ -(\lambda - \lambda_*)\hat{C}_{21} & A^\varepsilon - \lambda I_{22} \end{pmatrix}.$$

Then, any eigenvalue  $\lambda$  of  $A^\varepsilon$  is as well an eigenvalue of the equation  $\bar{\eta}_\lambda^\varepsilon(x) = 0$ , and viceversa provided that  $\lambda \in [\alpha, \beta]$ . Therefore,  $\bar{\eta}_\lambda^\varepsilon(x) = 0$  has exactly  $n = \dim(G_2)$  simple eigenvalues in the interval  $(\alpha, \beta)$ . Consequently, the function

$$\eta^\varepsilon: \mathbb{R} \times S \rightarrow G, \quad (\lambda, x) \mapsto \bar{\eta}_\lambda^\varepsilon(x)$$

has exactly  $n$  eigensets in the open subset  $U = (\alpha, \beta) \times S$  of the cylinder  $\mathbb{R} \times S$ , all of them corresponding to a simple eigenvalue. Therefore, according to Theorem 1, the contribution of each of them to  $\deg_{bf}(\eta^\varepsilon, U)$  is either 2 or  $-2$ . Consequently, taking into account that  $n$  is odd, one gets  $\deg_{bf}(\eta^\varepsilon, U) \neq 0$ .

Let the isomorphism  $Z$  be naturally oriented and let the restriction  $\eta$  of  $\bar{\eta}$  to the manifold  $\mathbb{R} \times S$  be oriented according to the composition  $Z\psi$ . Thus, because of the topological invariance property of the  $bf$ -degree, we get

$$\deg_{bf}(\eta, U) = \deg_{bf}(\psi, U).$$

Hence, it remains to show that, if  $\varepsilon > 0$  is sufficiently small, then

$$\deg_{bf}(\eta^\varepsilon, U) = \deg_{bf}(\eta, U).$$

In fact, this is a consequence of the homotopy invariance property of the  $bf$ -degree. To see this it is sufficient to show that (if  $\varepsilon$  is small) the homotopy  $h: [0, 1] \times \bar{U} \rightarrow G$ , defined by  $h(t, \lambda, x) = t\eta^\varepsilon(\lambda, x) + (1 - t)\eta(\lambda, x)$ , is admissible, i.e.,

$$h(t, \lambda, x) \neq 0 \text{ for } t \in [0, 1] \text{ and } (\lambda, x) \in \partial U = \{(\lambda, x) \in [\alpha, \beta] \times S : \lambda = \alpha \text{ or } \lambda = \beta\}.$$

Let us prove that this is true for the left boundary of  $U$ ; that is, for  $\lambda = \alpha$ . The argument for  $\lambda = \beta$  will be the same.

We need to show that (if  $\varepsilon$  is small) the linear operator  $A_t = t\bar{\eta}_\alpha^\varepsilon + (1 - t)\bar{\eta}_\alpha$  of  $\mathcal{L}(G)$  is invertible for any  $t \in [0, 1]$ . In fact, since  $A_0 = \bar{\eta}_\alpha$  is invertible, and the set of the invertible operators of  $\mathcal{L}(G)$  is open, this holds true for all  $A_t$  provided that  $\varepsilon$  is sufficiently small.  $\square$

#### 4. The Perturbed Eigenvalue Problem and Global Continuation

Here, as in Section 3,  $G$  and  $H$  denote two real Hilbert spaces,  $L, C: G \rightarrow H$  are bounded linear operators,  $S$  is the unit sphere of  $G$  and, as in problem (3), the operator  $L - \lambda C$  is invertible for some  $\lambda \in \mathbb{R}$ .

Consider the perturbed eigenvalue problem

$$\begin{cases} Lx + sN(x) = \lambda Cx \\ x \in S, \end{cases} \quad (4)$$

where  $N: S \rightarrow H$  is a  $C^1$  compact map and  $s$  is a real parameter.

A solution of (4) is a triple  $(s, \lambda, x) \in \mathbb{R} \times \mathbb{R} \times S$  satisfying (4). The element  $x \in S$  is a unit eigenvector corresponding to the eigenpair  $(s, \lambda)$ .

The set of solutions of (4) will be denoted by  $\Sigma$  and  $\mathcal{E}$  is the subset of  $\mathbb{R}^2$  of the eigenpairs. Notice that  $\mathcal{E}$  is the projection of  $\Sigma$  into the  $s\lambda$ -plane and the  $s = 0$  slice  $\Sigma_0$  of  $\Sigma$  is the same as the set  $\mathcal{S} = \psi^{-1}(0)$  of the eigenpoints of (3), where  $\psi$  has been defined in the previous section.

A solution  $(s, \lambda, x)$  of (4) is regarded as *trivial* if  $s = 0$ . In this case,  $p = (\lambda, x)$  is the corresponding eigenpoint of problem (3). When  $p$  is simple, the triple  $(0, \lambda, x) \in \Sigma$  will be as well said to be *simple*. A nonempty subset of  $\Sigma$  of the type  $\{0\} \times \{\lambda\} \times \mathcal{S}_\lambda$  will be called a *solution-sphere*.

We consider the subset  $\{(s, \lambda, x) \in \Sigma : s = 0\} = \{0\} \times \Sigma_0 = \{0\} \times \mathcal{S}$  of the trivial solutions of  $\Sigma$  as a *distinguished subset*. Thus, it makes sense to call a solution  $q_* = (0, \lambda_*, x_*)$  of (4) a *bifurcation point* if any neighborhood of  $q_*$  in  $\Sigma$  contains nontrivial solutions.

We say that a bifurcation point  $q_* = (0, \lambda_*, x_*)$  is *global* (in the sense of Rabinowitz [3]) if in the set of nontrivial solutions there exists a connected component, called *global (bifurcating) branch*, whose closure in  $\Sigma$  contains  $q_*$  and it is either unbounded or includes a trivial solution  $q^* = (0, \lambda^*, x^*)$  with  $\lambda^* \neq \lambda_*$ . In the second case  $q^*$  is as well a global bifurcation point.

A meaningful case is when a bifurcation point  $q_* = (0, \lambda_*, x_*)$  belongs to a connected solution-sphere  $\{0\} \times \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ . In this case, the dimension of  $\mathcal{S}_{\lambda_*}$  is positive and we will simply say that  $x_*$  is a bifurcation point. In fact, 0 and  $\lambda_*$  being known,  $x_*$  can be regarded as an *alias* of  $q_*$ .

For a necessary condition as well as some sufficient conditions for a point  $x_*$  of a connected eigensphere to be a bifurcation point see [11]. Other results regarding the

existence of bifurcation points belonging to even-dimensional eigenspheres can be found in [5,8,9,12,13,15–18].

As already pointed out, if the operator  $C$  is compact, then  $\psi: \mathbb{R} \times S \rightarrow H$  is Fredholm of index zero, and this is crucial for the global results regarding the perturbed eigenvalue problem (4).

Because of this, from now on, unless otherwise stated, we will tacitly assume that the linear operator  $C$  is compact.

We define the  $C^1$ -map

$$\psi^+: \mathbb{R} \times \mathbb{R} \times S \rightarrow H, \quad (s, \lambda, x) \mapsto \psi(\lambda, x) + sN(x),$$

in which  $\psi: \mathbb{R} \times S \rightarrow H$ , as in Section 3, is given by  $\psi(\lambda, x) = Lx - \lambda Cx$ . Therefore the set  $(\psi^+)^{-1}(0)$  of the zeros of  $\psi^+$  coincides with  $\Sigma$ .

As shown in [5], because of the compactness of  $C$  and  $N$ , one gets that  $\psi^+$  is proper on any bounded and closed subset of its domain. Consequently, any bounded connected component of  $\Sigma$  is compact. This fact will be useful later.

Notice that  $\psi^+$  is the restriction to the manifold  $\mathbb{R} \times \mathbb{R} \times S$  of the nonlinear map

$$\bar{\psi}^+: \mathbb{R} \times \mathbb{R} \times G \rightarrow H, \quad (s, \lambda, x) \mapsto \bar{\psi}(\lambda, x) + s\bar{N}(x),$$

where  $\bar{\psi}$  is as in Section 3 and  $\bar{N}$  is the positively homogeneous extension of  $N$ .

The following result of [5] is crucial for proving the existence of global bifurcation points.

**Theorem 3.** *Given an open subset  $\Omega$  of  $\mathbb{R} \times \mathbb{R} \times S$ , let*

$$\Omega_0 = \{(\lambda, x) \in \mathbb{R} \times S : (0, \lambda, x) \in \Omega\}$$

*be its 0-slice. If  $\deg_{bf}(\psi, \Omega_0)$  is well defined and nonzero, then  $\Omega$  contains a connected set of nontrivial solutions whose closure in  $\Omega$  is non-compact and meets at least one trivial solution of (4).*

Corollary 1 below, which was deduced in [5] from Theorem 3, asserts that the contribution to the  $bf$ -degree of the 0-slice of any compact (connected) component of  $\Sigma$  is null. We will need this basic property later.

**Corollary 1.** *Let  $\mathcal{D}$  be a compact component of  $\Sigma$ , and let  $\mathcal{D}_0 \subset \mathbb{R} \times S$  be its (possibly empty) 0-slice. Then, if  $U \subset \mathbb{R} \times S$  is an isolating neighborhood of  $\mathcal{D}_0$ , one has  $\deg_{bf}(\psi, U) = 0$ .*

The following result, obtained in ([5] [Theorem 4.5]), regards the existence of a global branch of solutions emanating from a trivial solution of problem (4) which corresponds to a simple eigenpoint of (3).

**Theorem 4.** *If  $(\lambda_*, x_*)$  is a simple eigenpoint of problem (3), then, in the set  $\Sigma$  of the solutions of (4), the connected component containing  $(0, \lambda_*, x_*)$  is either unbounded or includes a trivial solution  $(0, \lambda^*, x^*)$  with  $\lambda^* \neq \lambda_*$ .*

We are now ready to prove our main result, which extends Theorem 4 and provides a global version of Theorem 3.9 in [15], the latter concerning the existence of local bifurcation points belonging to even dimensional eigenspheres.

**Theorem 5.** *In addition to the compactness of  $C$ , let  $(\lambda_*, x_*)$  be an eigenpoint of (3) and denote by  $T$  the non-invertible operator  $L - \lambda_* C$ . Assume that*

- $\text{Ker } T$  is odd dimensional,
- $\text{Im } T \cap C(\text{Ker } T) = \{0\}$ .

Then, in the set  $\Sigma$  of the solutions of (4), the connected component containing  $(0, \lambda_*, x_*)$  is either unbounded or includes a trivial solution  $(0, \lambda^*, x^*)$  with  $\lambda^* \neq \lambda_*$ .

**Proof.** Because of the compactness of  $C$ , according to Remark 3, the operator  $L - \lambda C$  is Fredholm of index zero for all  $\lambda \in \mathbb{R}$ . Moreover, the set of the eigenvalues of problem (3) is discrete. Consequently, the eigenset  $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ , which is compact and nonempty, is relatively open in the set  $S$  of the eigenpoints. Thus, it admits an isolating neighborhood  $U \subset \mathbb{R} \times S$  and, therefore,  $\deg_{bf}(\psi, U)$  is well defined.

Denote by  $\mathcal{D}$  the connected component of  $\Sigma$  containing  $(0, \lambda_*, x_*)$ . We may assume that  $\mathcal{D}$  is bounded. Thus, it is actually compact, since  $\psi^+$  is proper on any bounded and closed subset of  $\mathbb{R} \times \mathbb{R} \times S$ . We need to prove that  $\mathcal{D}$  contains a trivial solution  $(0, \lambda^*, x^*)$  with  $\lambda^* \neq \lambda_*$ .

Assume, by contradiction, that this is not the case. Then the 0-slice  $\mathcal{D}_0$  of  $\mathcal{D}$  is contained in the eigenset  $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ . We will show that this contradicts Corollary 1. We distinguish two cases:  $n = 1$  and  $n > 1$ , where  $n$  is the dimension of  $\text{Ker } T$ .

*Case  $n = 1$ .* Because of the assumption  $\text{Img } T \cap C(\text{Ker } T) = \{0\}$ , the eigenpoint  $p_* = (\lambda_*, x_*)$  is simple and  $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  has only two points:  $p_* = (\lambda_*, x_*)$  and  $\bar{p}_* = (\lambda_*, -x_*)$ . In this case, according to Theorem 1, the contribution to the  $bf$ -degree of any subset of  $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  is different from zero, and this, having assumed  $\mathcal{D}_0 \subseteq \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ , is incompatible with Corollary 1.

*Case  $n > 1$ .* The solution-sphere  $\{0\} \times \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  is connected and, consequently, it is contained in the component  $\mathcal{D}$  of  $\Sigma$ . Thus, the eigenset  $\{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  is contained in the slice  $\mathcal{D}_0$  of  $\mathcal{D}$ . Having assumed  $\mathcal{D}_0 \subseteq \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ , we get  $\mathcal{D}_0 = \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ . Hence, because of Theorem 2, given an isolating neighborhood  $U$  of  $\mathcal{D}_0$ , one gets  $\deg_{bf}(\psi, U) \neq 0$ , and we obtain a contradiction with Corollary 1.  $\square$

**Remark 6.** Under the notation and assumptions of Theorem 5 suppose, in addition, that  $\dim(\text{Ker } T) > 1$ . Then, the connected component  $\mathcal{D}$  containing  $(0, \lambda_*, x_*)$  contains as well the connected solution-sphere  $\mathcal{D}_* = \{0\} \times \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$ .

This implies that there exists at least one point  $\hat{q} = (0, \lambda_*, \hat{x}) \in \mathcal{D}_*$  which is in the closure  $\overline{\mathcal{D} \setminus \mathcal{D}_*}$  of the difference  $\mathcal{D} \setminus \mathcal{D}_*$ . Thus,  $\hat{q}$  (or, equivalently, its alias  $\hat{x} \in \mathcal{S}_{\lambda_*}$ ) is a global bifurcation point.

## 5. Some Illustrating Examples

In this section we provide three examples in  $\ell^2$  concerning Theorem 5. The dimensions of  $\text{Ker } T$  (where  $T = L - \lambda_* C$ ) are, respectively, 3, 2, and 1. The second example, in which  $\text{Ker } T$  is two dimensional, shows that in Theorem 5, as well as in Remark 6, the hypothesis of the odd dimensionality of  $\text{Ker } T$  cannot be removed, the other assumptions remaining valid.

Given a positive integer  $k$ , let  $T_k \in \mathcal{L}(\ell^2)$  be the bounded linear operator that to any  $x = (\xi_1, \xi_2, \xi_3, \dots) \in \ell^2$  associates the element

$$T_k x = (0, 0, \dots, 0, \xi_{k+1}, \xi_{k+2}, \dots),$$

in which the first  $k$  components are 0. Notice that  $T_k$  is Fredholm of index zero and its kernel is the  $k$ -dimensional space

$$\text{Ker } T_k = \{x \in \ell^2 : x = (\xi_1, \xi_2, \dots, \xi_k, 0, 0, \dots)\},$$

which is orthogonal to  $\text{Img } T_k$ .

Hereafter,  $C$  will be the well-known compact linear operator defined by

$$(\xi_1, \xi_2, \xi_3, \dots) \mapsto (\xi_1/1, \xi_2/2, \xi_3/3, \dots, \xi_n/n, \dots).$$



Given any compact (possibly nonlinear) map  $N: \ell^2 \rightarrow \ell^2$  of class  $C^1$ , consider the perturbed eigenvalue problem

$$\begin{cases} T_k x + sN(x) = \lambda Cx, \\ x \in S, \end{cases} \quad (5)$$

where  $S$  is the unit sphere of  $\ell^2$ . As before, we denote by  $\Sigma$  the set of solutions  $(s, \lambda, x)$  of (5).

Observe that, for any  $k \in \mathbb{N}$ ,  $k \geq 1$ ,  $\lambda_* = 0$  is an eigenvalue of the unperturbed equation  $T_k x = \lambda Cx$  and the condition  $\text{Im}g T_k \cap C(\text{Ker } T_k) = \{0\}$  is satisfied. Therefore, according to Theorem 5, given any positive odd integer  $k$ , any compact perturbing map  $N: \ell^2 \rightarrow \ell^2$  of class  $C^1$ , and any  $x_* \in S \cap \text{Ker } T_k$ , the connected component of  $\Sigma$  containing  $(0, 0, x_*)$  is either unbounded or encounters a trivial solution  $(0, \lambda^*, x^*)$  with  $\lambda^* \neq 0$ .

In the three examples below we will check whether or not the assertions of Theorem 5 and Remark 6 hold, by taking, for all of them, the same perturbing map. Namely,

$$N: \ell^2 \rightarrow \ell^2, \quad (\xi_1, \xi_2, \xi_3, \xi_4, \dots) \mapsto (-\xi_2, \xi_1, -\xi_4, \xi_3, 0, 0, 0, \dots).$$

**Example 1** ( $k = 3$ ). The eigenvalues of the unperturbed equation  $T_3 x = \lambda Cx$  are  $0, 4, 5, 6, \dots$ . The first one,  $\lambda_* = 0$ , has geometric and algebraic multiplicity 3 and all the other eigenvalues are simple.

A standard computation shows that, in the  $s\lambda$ -plane, the set  $\mathcal{E}$  of the eigenpairs has a connected subset  $\mathcal{E}_1$  satisfying the equation  $3s^2 + (\lambda - 2)^2/4 = 1$ , corresponding to eigenvectors of the type  $(0, 0, \xi_3, \xi_4, 0, 0, \dots)$ . The set  $\mathcal{E}_1$  is an ellipse with center  $(0, 2)$  and half-axes  $1/\sqrt{3}$  and 2. Observe that it includes the eigenpair  $(s, \lambda) = (0, 0)$ . All the other eigenpairs are the points of the horizontal lines  $\lambda = 5, \lambda = 6, \lambda = 7$ , etc. Thus, the connected component in  $\Sigma$  containing any trivial solution  $(0, \lambda, x)$  with eigenvalue  $\lambda \geq 5$  is unbounded, and this agrees with Theorem 5.

The above ellipse can be parametrized by  $s = (1/\sqrt{3}) \sin \theta$ ,  $\lambda = 2(1 - \cos \theta)$ ,  $\theta \in [0, 2\pi]$ , and for any  $\theta$  in the open interval  $(0, 2\pi)$ , the kernel of the equation

$$T_3 x + (1/\sqrt{3}) \sin \theta N x - 2(1 - \cos \theta) Cx = 0$$

is 1-dimensional and spanned by the vector

$$x(\theta) = (0, 0, (1/\sqrt{3}) \sin \theta, -(2/3)(1 - \cos \theta), 0, 0, \dots).$$

Since  $\mathcal{E}_1$  is bounded, so is the connected component  $\mathcal{D}$  of  $\Sigma$  containing the 2-dimensional solution-sphere  $\mathcal{D}_* = \{0\} \times \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  (recall that  $\lambda_* = 0$ ). As we shall see,  $\mathcal{D}$  includes the twin trivial solutions  $(0, \lambda^*, \pm x^*)$ , where

$$\lambda^* = 4 \quad \text{and} \quad x^* = x(\pi)/\|x(\pi)\| = (0, 0, 0, 1, 0, 0, \dots).$$

According to Remark 6, there exists at least one bifurcation point  $\hat{x} \in \mathcal{S}_{\lambda_*}$ . Actually, in this case one gets exactly two (global) bifurcation points. This is due to the fact that  $\mathcal{D} \setminus \mathcal{D}_*$  has two disjoint “twin” branches whose closures meet the solution-sphere  $\mathcal{D}_*$ . The branches can be parametrized with  $\theta \in (0, 2\pi)$  as follows:

$$\begin{aligned} q(\theta) &= ((1/\sqrt{3}) \sin \theta, 2(1 - \cos \theta), x(\theta)/\|x(\theta)\|), \\ \bar{q}(\theta) &= ((1/\sqrt{3}) \sin \theta, 2(1 - \cos \theta), -x(\theta)/\|x(\theta)\|). \end{aligned}$$

Then, if the following limits exist:

$$\lim_{\theta \rightarrow 0} q(\theta) \quad \text{and} \quad \lim_{\theta \rightarrow 0} \bar{q}(\theta),$$

we get the bifurcation points (as elements of  $\mathcal{D}_*$ ). Equivalently, to find the aliases of these points (that is, the corresponding elements in the eigensphere  $\mathcal{S}_{\lambda_*}$ ) we compute

$$\pm \lim_{\theta \rightarrow 0} (x(\theta) / \|x(\theta)\|)$$

obtaining  $\pm(0, 0, 1, 0, 0, \dots) \in \mathcal{S}_{\lambda_*}$ . In fact, to compute the limits, observe that  $\sin \theta = u_s(\theta)\theta$  and  $2(1 - \cos \theta) = u_c(\theta)\theta^2$ , where  $u_s$  and  $u_c$  are continuous functions such that  $u_s(0) = u_c(0) = 1$ . Hence, one quickly obtains

$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{(1/\sqrt{3}) \sin \theta}{\sqrt{(1/3) \sin^2 \theta + (4/9)(1 - \cos \theta)^2}} &= 1, \\ \lim_{\theta \rightarrow 0} \frac{-(2/3)(1 - \cos \theta)}{\sqrt{(1/3) \sin^2 \theta + (4/9)(1 - \cos \theta)^2}} &= 0. \end{aligned}$$

**Example 2** ( $k = 2$ ). The eigenvalues of the unperturbed equation  $T_2x = \lambda Cx$  are  $0, 3, 4, 5, 6, \dots$ . The first one,  $\lambda_* = 0$ , has geometric and algebraic multiplicity 2 and all the others are simple.

As in Example 1, for any eigenvalue  $\lambda \geq 5$ , one gets an horizontal line of eigenpairs containing  $(0, \lambda)$ . Moreover, as one can check, the trivial eigenpairs  $(0, 3)$  and  $(0, 4)$  are vertices of an ellipse of eigenpairs with center  $(0, 7/2)$  and half-axes  $1/\sqrt{48}$  and  $1/2$ , corresponding, as in Example 1, to eigenvectors of the type  $(0, 0, \xi_3, \xi_4, 0, \dots)$ . However, in a neighborhood of the origin of the  $s\lambda$ -plane there are no eigenpairs, except the isolated one  $(0, 0)$ . This means that the solution-circle  $\mathcal{D}_* = \{0\} \times \{\lambda_*\} \times \mathcal{S}_{\lambda_*}$  is an isolated subset of  $\Sigma$ . Therefore, the assertions of Theorem 5 and Remark 6 do not hold in this case. Moreover, according to Corollary 1, the contribution of  $\mathcal{D}_*$  to the bf-degree of the map  $\psi$  is zero.

In conclusion, in Theorem 5 and Remark 6, the assumption that  $\text{Ker } T$  is odd dimensional cannot be removed.

**Example 3** ( $k = 1$ ). In this case, the eigenvalues of the unperturbed problem are  $0, 2, 3, 4, 5, \dots$ . All of them are simple. As in the previous two examples, the  $s\lambda$ -plane contains infinitely many horizontal lines of eigenpairs. Their equations are  $\lambda = 5$ ,  $\lambda = 6$ ,  $\lambda = 7, \dots$

In addition to the horizontal lines, the set of the eigenpairs has two bounded components: an ellipse with center  $(0, 1)$  and half-axes  $1/\sqrt{2}$  and  $1$ , therefore containing  $(0, 0)$  and  $(0, 2)$ ; and, as in Example 2, an ellipse joining  $(0, 3)$  with  $(0, 4)$ , with center  $(0, 7/2)$  and half-axes  $1/\sqrt{48}$  and  $1/2$ .

Finally, one can check that, in accordance with Theorem 5, given any one of the two points of the 0-dimensional solution-sphere  $\{0\} \times \{0\} \times \mathcal{S}_0$ , its connected component in  $\Sigma$  is bounded and contains a point of  $\{0\} \times \{2\} \times \mathcal{S}_2$ . This agrees with Theorem 5.

## 6. Conclusions and Perspectives

In this paper, we obtain a Rabinowitz-type global continuation result, namely the above Theorem 5, for nontrivial solution triples of perturbed eigenvalue problems in Hilbert spaces under the assumption that the unperturbed problem has a nontrivial kernel of odd dimension.

Our result is based on a link between two apparently different topics, i.e., linear eigenvalue problems and topological degree theory. In particular, we apply the so called bf-degree introduced in [4] for oriented  $C^1$  Fredholm maps of index zero between real differentiable Banach manifolds, which extends the Brouwer degree for maps between oriented finite dimensional smooth manifolds, as well as the Leray-Schauder degree for  $C^1$  compact vector fields in Banach spaces.

Our work is inspired by some papers by R. Chiappinelli, in which the local persistence of eigenvalues and eigenvectors is studied in the case of a simple isolated eigenvalue, and this property is applied to investigate the structure of the solution set of differential equations (see also [5] for possible applications of our results to boundary value problems).

Recently, in [5] we obtained a global continuation result for nontrivial solution triples, under the assumption of the existence of a simple isolated eigenvalue. In spite of the apparent similarity, we stress that the present extension to the odd-dimensional case is not trivial. In fact, to prove Theorem 5 we have first to evaluate the  $bf$ -degree of a suitable nonlinear map in an open subset of the cylinder  $\mathbb{R} \times S$  ( $S$  being the unit sphere of the Hilbert space  $G$ ), see Theorem 2. If we were able to relate this degree with the Leray–Schauder degree of a suitable compact vector field in the unit ball of  $G$ , we could provide a simpler proof of Theorem 2. This is the content of the above conjecture (C), which will be the object of a future investigation.

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