

Article

Global Well-Posedness and Analyticity of Generalized Porous Medium Equation in Fourier-Besov-Morrey Spaces with Variable Exponent

Muhammad Zainul Abidin *  and Jiecheng Chen

College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321004, China; jcchen@zjnu.edu.cn

* Correspondence: mzainulabidin@zjnu.edu.cn

Abstract: In this paper, we consider the generalized porous medium equation. For small initial data u_0 belonging to the Fourier-Besov-Morrey spaces with variable exponent, we obtain the global well-posedness results of generalized porous medium equation by using the Fourier localization principle and the Littlewood-Paley decomposition technique. Furthermore, we also show Gevrey class regularity of the solution.

Keywords: global well-posedness; analyticity; porous medium equation; Fourier-Besov-Morrey space with variable exponent



Citation: Abidin, M.Z.; Chen, J. Global Well-Posedness and Analyticity of Generalized Porous Medium Equation in Fourier-Besov-Morrey Spaces with Variable Exponent. *Mathematics* **2021**, *9*, 498. <https://doi.org/10.3390/math9050498>

Academic Editor: Pedro J. Miana

Received: 29 January 2021

Accepted: 26 February 2021

Published: 28 February 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

We consider the three dimensional generalized porous medium (GPM) equation:

$$\begin{cases} u_t + \mu \Lambda^\beta u = \nabla \cdot (u \nabla P u) & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

where $u = u(x, t)$ denotes the concentration or density. The dissipative coefficient $\mu = 0$ is the inviscid case, while $\mu > 0$ represents the viscous case. P is an abstract operator and the fractional Laplacian term Λ^β is the Fourier Transform defined by $\widehat{\Lambda^\beta u} = |\zeta|^\beta \hat{u}$. For simplicity we chose $\mu = 1$.

The Equation (1) was first introduced by Zhou et al. [1]. Actually, by applying the fractional dissipative term $\mu \Lambda^\beta$ to the continuity equation $u_t + \nabla \cdot (uV) = 0$, Caffarelli et al. [2] obtained the Equation (1), where the velocity V derives from the potential, $V = -\nabla p$, and the velocity potential or pressure is associated with u by the abstract operator $p = Pu$ [3]. There are a number of physical applications of porous medium of equations, mainly to describe processes involving fluid flow, heat transfer or diffusion. May be the best known of them is the description of the flow of an isentropic gas through a porous medium. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory and other fields. For detailed study related to the physical significance of (1), we refer the reader to [4] and the references therein.

The abstract form pressure term Pu gives a good suitability in many cases. The simplest case comes from a model in ground water in filtration [5,6]. For more general case $\mu = 0$ and $Pu = (-\Delta)^{-s} u = \Lambda^{-2s} u$, $0 < s < 1$, Zhou et al. [7] obtained the strong solution for Equation (1) in Besov spaces $B_{p,\infty}^\beta$ and for any initial data in $B_{1,\infty}^\beta$ the local solution is obtained. Lin and zhang [8] considered (1) for $s = 1$ (critical case), which gives the mean field equation. For more details in this direction we refer the reader to [1,9] and the references therein.

Another similar model appears in the aggregation equation, which explains a aggregation phenomena and collective motion in mechanics of continuous media and bi-

ology [10,11]. This equation has many applications in various applied sciences such as chemistry, biology, physics and population dynamics. In the aggregation equation, the operator P can also be written as convolution operator with kernel $\mathcal{K}$ as $Pu = \mathcal{K} * u$. The typical kernels are exponential potential $-e^{-|x|}$ [12] and the Newton Potential $|x|^\gamma$ [13]. For more results related to the well-posedness and blowup criterion, we refer the reader to [14,15] and the references therein.

Furthermore, using the same initial data, we can write the Equation (1) as:

$$\begin{aligned} u_t + \mu \Lambda^\beta u + v \cdot \nabla u &= -u(\nabla \cdot v); \\ v &= -\nabla Pu. \end{aligned} \quad (2)$$

Then the Equation (2) can be compared to the geostrophic model and the convective velocity for the generalized porous medium equation is not absolutely divergence-free. In addition, if the divergence-free vector function v satisfies the equation ($\nabla \cdot v = 0$), the Equation (2) will hold the quasi-geostrophic (Q-G) equation [16,17].

One of the main problems to the Equation (1) is the singularity of the abstract form pressure term Pu that establish the well-posedness or gives the blow up solution. Zhou et al. [1] established the local well-posedness for large initial data in Besov spaces and the global solution for small initial data. They also given a blowup criterion for the solution. Li and Rodrigo [12] obtained well-posedness and the blowup criterion for the solution of Equation (1) associated with the pressure $Pu = \mathcal{K} * u$, where $\mathcal{K}(x) = e^{-|x|}$ belongs to Sobolev space. Moreover, their work is further extended by Wu and Zhang [18] to the case $\mathcal{K}(x) = e^{-|x|}$ and $\nabla \mathcal{K} \in W^{1,1}$. Xiao and Zhou [3] have shown the local well-posedness for large initial data in Fourier-Besov spaces and obtained the global well-posedness for small initial data for $\nabla \mathcal{K} \in L^1$. The controllability of convolution $\mathcal{K} * u$ and its gradient $\nabla \mathcal{K} * u$ in Besov spaces leads to these result.

Inspired by the above works, we obtain the global well-posedness and analyticity for Equation (1) and show the Gevrey class regularity of the solution in homogenous Fourier-Besov-Morrey spaces with variable exponent by considering $\nabla \mathcal{K} \in L^1$. The Fourier-Besov spaces goes back to the work of Konieczny and Yoneda [19] in the context of Navier-Stokes equation (NSE) with Coriolis force. Fourier-Besov spaces and Fourier-Besov-Morrey spaces have been extensively considered by many authors in order to deal with the well-posedness, self-similar solution, regularity etc. For more study in this direction, we refer the reader to [20–22] and the references therein.

The variable exponent Lebesgue spaces $L^{p(\cdot)}$, comes from Orlicz [23] and further developed by Musielak [24] and Nakano [25]. However the recent improvement begins with Kovacik and Rakosnik [26] and further advancement by Cruzuribe [27] and Diening [28]. The main reason of variable exponent function spaces is their applications in fluid dynamics [29], image processing [30] and partial differential equations [31]. For detailed study related to Besov spaces with variable exponent and Besov-Morrey spaces with variable exponent, we refer the reader to [32–37] and the references therein.

2. Preliminaries

In this section, we recall some definitions of various functions spaces with variable exponent, basic facts about dyadic decomposition, and some useful propositions. Throughout this paper, $f \lesssim g$ denotes that $f \leq Cg$ for some constant $C > 0$. We obtained the global well-posedness result and prove Gevrey class regularity in Sections 3 and 4, respectively. The conclusion is given in Section 5.

Definition 1. Let $\mathcal{P}_0$ denotes the set of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$ such that

$$0 < p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) = p_+ < \infty.$$

The Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is measurable, } \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty \right\},$$

with Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

In order to differentiate between constant and variable exponent, we denote constant exponent by p , and variable exponent by $p(\cdot)$. Also $(L^{p(\cdot)}(\mathbb{R}^n), \|f\|_{L^{p(\cdot)}})$ is a Banach space.

Since the $L^{p(\cdot)}$ does not have the same desired properties like L^p . So, to ensure that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, we postulate the following standard conditions:

(1) (locally log-Hölder continuous) There exists a constant $C_{\log}(p)$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + |x - y|^{-1})}, \text{ for any } x, y \in \mathbb{R}^n \text{ and } x \neq y.$$

(2) (locally log-Hölder continuous) There exists a constant $C_{\log}(p)$ and some constant independent of x such that

$$|p(x) - p_{\infty}| \leq \frac{C_{\log}(p)}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

$C^{\log}(\mathbb{R}^n)$ denote the set of all functions $p(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying (1) and (2).

Definition 2. Let $p(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $0 < p^- \leq p(x) \leq h(x) \leq \infty$, the Morrey space with variable exponent $M_{p(\cdot)}^{h(\cdot)} := M_{p(\cdot)}^{h(\cdot)}(\mathbb{R}^n)$ be the set of measurable functions on $\mathbb{R}^n$ with finite quasinorm

$$\|f\|_{M_{p(\cdot)}^{h(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r > 0} \|r^{\frac{n}{h(x)} - \frac{n}{p(x)}} f \chi_{B(x_0, r)}\|_{L^{p(\cdot)}}.$$

By the definition of $L^{p(\cdot)}$ quasinorm $\|f\|_{M_{p(\cdot)}^{h(\cdot)}}$ also has the following form

$$\|f\|_{M_{p(\cdot)}^{h(\cdot)}} := \sup_{x_0 \in \mathbb{R}^n, r > 0} \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)}(r^{\frac{n}{h(x)} - \frac{n}{p(x)}} \frac{f}{\lambda} \chi_{B(x_0, r)}) \leq 1 \right\}.$$

Now we present some essential lemmas from [36].

Lemma 1. Let $p(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $p(x) \leq h(x)$ and f be any measurable function. Then

$$\|f\|_{M_{p(\cdot)}^{h(\cdot)}} := \inf \left\{ \lambda > 0 : \sup_{x_0 \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)}(r^{\frac{n}{h(x)} - \frac{n}{p(x)}} \frac{f}{\lambda} \chi_{B(x_0, r)}) \leq 1 \right\}.$$

Lemma 2. $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. For any measurable function f

$$\sup_{x \in \mathbb{R}^n, r > 0} \varrho_{p(\cdot)}(f \chi_{B(x, r)}) = \varrho_{p(\cdot)}(f).$$

Lemma 3. If $p(\cdot) \in \mathcal{P}_0^n$, then $\|f\|_{M_{p(\cdot)}^{p(\cdot)}} = \|f\|_{L^{p(\cdot)}}$.

Now we recall dyadic decomposition of $\mathbb{R}^n$. Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ be the two non negative radial functions such that

$$\begin{aligned}\operatorname{supp} \chi &\subset \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3}\}, \\ \operatorname{supp} \varphi &\subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, \\ \chi(\xi) + \sum_{i \geq 0} \varphi(2^{-i}\xi) &= 1, \quad \xi \in \mathbb{R}^n, \\ \sum_{i \in \mathbb{Z}} \varphi(2^{-i}\xi) &= 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.\end{aligned}$$

We denote $\varphi_i(\xi) = \varphi(2^{-i}\xi)$ and $h_i = \mathcal{F}^{-1}\varphi_i$ to define the frequency localization as follows:

$$\begin{aligned}\Delta_i f &:= \mathcal{F}^{-1}\varphi_i \mathcal{F}f = \int_{\mathbb{R}^n} h_i(y) f(x-y) dy, \\ S_i f &= \sum_{j \leq i-1} \Delta_j f,\end{aligned}$$

where $\Delta_i = S_i - S_{i-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^i\}$ and S_i is a frequency to the ball $\{|\xi| \lesssim 2^i\}$. we can easily obtain that

$$\Delta_i \Delta_j \equiv 0, \text{ if } |i-j| \geq 2 \text{ and } \Delta_i(\Delta_{j-1} \Delta_j) \equiv 0, \text{ if } |i-j| \geq 5.$$

Definition 3. Let $p(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, the mixed Lebesgue-sequence space is defined as the set of all sequences $\{f_i\}_{i \in \mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that

$$\|\{f_i\}_{i \in \mathbb{Z}}\|_{lq(\cdot)(M_{p(\cdot)}^{h(\cdot)})} := \inf \left\{ \mu > 0, \varrho_{lq(\cdot)(M_{p(\cdot)}^{h(\cdot)})} \left(\left\{ \frac{f_i}{\mu} \right\}_{i \in \mathbb{Z}} \right) \leq 1 \right\} < \infty,$$

where

$$\varrho_{lq(\cdot)(M_{p(\cdot)}^{h(\cdot)})}(\{f_i\}_{i \in \mathbb{Z}}) := \sum_{i \in \mathbb{Z}} \inf \left\{ \lambda_i > 0, \int_{\mathbb{R}^n} \left(\frac{|r^{\frac{n}{h(x)} - \frac{n}{p(x)}} f_i \chi_{B(x_0, r)}|}{\lambda^{\frac{1}{q(x)}}} \right)^{p(x)} dx \leq 1 \right\}.$$

Notice that if $q_+ < \infty$ and $p(x) \leq q(x)$, then

$$\varrho_{\ell q(\cdot)(M_{p(\cdot)}^{h(\cdot)})}(\{f_j\}_{j \in \mathbb{N}_0}) = \sum_{j \in \mathbb{N}_0} \sup_{x_0 \in \mathbb{R}^n, r > 0} \|(|r^{\frac{n}{h(x)} - \frac{n}{p(x)}} f_j| \chi_{B(x_0, r)})^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

Definition 4. Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p^- \leq p(x) \leq h(x) \leq \infty$. We define the homogeneous Besov-Morrey space with variable exponent $\dot{\mathcal{N}}_{p(\cdot), h(\cdot), q(\cdot)}^{s(\cdot)}$ by

$$\dot{\mathcal{N}}_{p(\cdot), h(\cdot), q(\cdot)}^{s(\cdot)} = \{f \in \mathcal{D}'(\mathbb{R}^n) : \|f\|_{\dot{\mathcal{N}}_{p(\cdot), h(\cdot), q(\cdot)}^{s(\cdot)}} < \infty\},$$

with norm

$$\|f\|_{\dot{\mathcal{N}}_{p(\cdot), h(\cdot), q(\cdot)}^{s(\cdot)}} := \|\{2^{js(\cdot)} \Delta_j f\}_{j \in \mathbb{Z}}\|_{\ell q(\cdot)(M_{p(\cdot)}^{h(\cdot)})}.$$

The space $\mathcal{D}'(\mathbb{R}^n)$ is the dual space of

$$\mathcal{D}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha f)(0) = 0, \forall \alpha\},$$

Definition 5. Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p^- \leq p(x) \leq h(x) \leq \infty$. We define the homogeneous Fourier-Besov-Morrey space with variable exponent $\mathcal{FN}_{p(\cdot), h(\cdot), q(\cdot)}^{s(\cdot)}$ by

$$\mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)} = \left\{ f \in \mathcal{D}'(\mathbb{R}^n); \quad \|f\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)}} < \infty \right\},$$

with norm

$$\|f\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)}} := \|\{2^{js(\cdot)} \varphi_j \widehat{f}\}_{j \in \mathbb{Z}}\|_{\ell^{q(\cdot)}(M_{p(\cdot)}^{h(\cdot)})}^\infty.$$

Definition 6. Let $T \in (0, T]$ and $1 \leq r, \gamma \leq \infty$. We define the Chemin-Lerner type homogeneous Fourier-Besov-Morrey space with variable exponent $\mathcal{L}^\gamma(0, T; \mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)})$ by

$$\mathcal{L}^\gamma(0, T; \mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)}) = \left\{ f \in \mathcal{D}'(\mathbb{R}^n); \quad \|f\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)}} < \infty \right\},$$

with norm

$$\|f\|_{\mathcal{L}^\gamma(0, T; \mathcal{FN}_{p(\cdot),h(\cdot),q(\cdot)}^{s(\cdot)})} = \left(\sum_{i \in \mathbb{Z}} \|2^{is(\cdot)} \Delta_i f\|_{L_T^\gamma M_{p(\cdot)}^{h(\cdot)}}^r \right)^{\frac{1}{r}}.$$

Proposition 1. The following inclusions are established for the Morrey spaces with variable exponent.

(1) ([37]) let $p(\cdot), p_1(\cdot), p_2(\cdot), h(\cdot), h_1(\cdot), h_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, such that $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$, $\frac{1}{h(x)} = \frac{1}{h_1(x)} + \frac{1}{h_2(x)}$. Then there exists a constant C depending only on p_- and p_+ such that

$$\|fg\|_{M_{p(\cdot)}^{h(\cdot)}} \leq C_{p,p_1} \|f\|_{M_{p_1(\cdot)}^{h_1(\cdot)}} \|g\|_{M_{p_2(\cdot)}^{h_2(\cdot)}}$$

holds for every $f \in M_{p_1(\cdot)}^{h_1(\cdot)}$ and $g \in M_{p_2(\cdot)}^{h_2(\cdot)}$.

(2) ([37]) Let $p, p_0, p_1, h, h_0, h_1 \in \mathcal{P}_0$, $0 < q < \infty$ and $s_0, s_1 \in L^\infty \cap C^{log}(\mathbb{R}^n)$ with $s_0 > s_1$. If $\frac{1}{q}$ and

$$s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$$

are locally log- Hölder continuous, then

$$\mathcal{N}_{p_0(\cdot),h_0(\cdot),q}^{s_0(\cdot)} \hookrightarrow \mathcal{N}_{p_1(\cdot),h_1(\cdot),q}^{s_1(\cdot)}.$$

(3) ([36]) For $p(\cdot) \in C^{log}(\mathbb{R}^n)$ and $\psi \in L^1(\mathbb{R}^n)$, assume $\Psi(x) = \sup_{y \notin B(0,|x|)} |\psi(y)|$ is integrable. Then

$$\|f * \psi_\varepsilon\|_{M_{p(\cdot)}^{h(\cdot)}(\mathbb{R}^n)} \leq C \|f\|_{M_{p(\cdot)}^{h(\cdot)}(\mathbb{R}^n)} \|\Psi\|_{L^1(\mathbb{R}^n)},$$

for all $f \in M_{p(\cdot)}^{h(\cdot)}(\mathbb{R}^n)$, where $\psi_\varepsilon = \frac{1}{\varepsilon^n} \psi(\frac{\cdot}{\varepsilon})$ and C depends only on n .

Next, we recall the paradifferential calculus which enables us to define a generalized product between distributions. The paraproduct between u and v is defined by

$$T_u v := \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v.$$

Then we have the formal decomposition:

$$uv = T_u v + T_v u + R(u, v),$$

with

$$R(u, v) = \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v \text{ and } \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

This decomposition is called Bony's paraproduct decomposition.

Proposition 2. Let $s > 0$, $1 \leq \rho \leq \infty$, $p(\cdot), h(\cdot), q(\cdot) \in C^{log} \cap \mathcal{P}_0(\mathbb{R}^n)$, $\frac{1}{h(\cdot)} = \frac{1}{h_1(\cdot)} + \frac{1}{h_2(\cdot)}$ and $\frac{1}{\rho} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$. Then we have

$$\|u\partial_k P v\|_{\dot{N}_{\rho, h(\cdot), 1}^s} \lesssim \|u\|_{\dot{N}_{p(\cdot), h_1(\cdot), 1}^0} \|v\|_{\dot{N}_{q(\cdot), h_2(\cdot), 1}^s} + \|v\|_{\dot{N}_{p(\cdot), h_1(\cdot), 1}^0} \|u\|_{\dot{N}_{q(\cdot), h_2(\cdot), 1}^s}.$$

Proof. Applying Bony's paraproduct decomposition for some fixed $i \in \mathbb{Z}$, we can write

$$\begin{aligned} \Delta_i(u\partial_k P v) &= \sum_{|j-i| \leq 4} \Delta_i(S_{j-1}u\Delta_j(\partial_k P v)) + \sum_{|j-i| \leq 4} \Delta_i(S_{j-1}(\partial_k P v)\Delta_j u) \\ &\quad + \sum_{j \geq i-2} \Delta_i(\Delta_j u \tilde{\Delta}_j(\partial_k P v)) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

To prove this proposition, we estimate the above three terms separately. According to the Proposition 1, we have

$$\|2^{is} \Delta_i(S_{j-1}u\Delta_j(\partial_k P v))\|_{M_{\rho}^{h(\cdot)}} \lesssim \|S_{j-1}u\|_{M_{p(\cdot)}^{h_1(\cdot)}} \|2^{is} \Delta_j(\partial_k P v)\|_{M_{q(\cdot)}^{h_2(\cdot)}}.$$

Then we have

$$\|2^{is} \Delta_i(S_{j-1}u\Delta_j(\partial_k P v))\|_{M_{\rho}^{h(\cdot)}} \lesssim \|S_{j-1}u\|_{M_{p(\cdot)}^{h_1(\cdot)}} \|2^{is} \Delta_j v\|_{M_{q(\cdot)}^{h_2(\cdot)}}.$$

Considering I_1 , we have

$$\|2^{is} I_1\|_{M_{\rho}^{h(\cdot)}} \lesssim \sum_{|j-i| \leq 4} \|S_{j-1}u\|_{M_{p(\cdot)}^{h_1(\cdot)}} \|2^{is} \Delta_j v\|_{M_{q(\cdot)}^{h_2(\cdot)}}.$$

Similarly, for I_2 we have

$$\|2^{is} I_2\|_{M_{\rho}^{h(\cdot)}} \lesssim \sum_{|j-i| \leq 4} \|S_{j-1}u\|_{M_{p(\cdot)}^{h_1(\cdot)}} \|2^{is} \Delta_j v\|_{M_{q(\cdot)}^{h_2(\cdot)}}.$$

To estimate I_3 , using Proposition 1 and obtain

$$\|\Delta_i(\Delta_j u \tilde{\Delta}_j)(\partial_k P v)\|_{M_{\rho}^{h(\cdot)}} \lesssim \|\Delta_j u\|_{M_{q(\cdot)}^{h_2(\cdot)}} \|\tilde{\Delta}_j(\partial_k P v)\|_{M_{p(\cdot)}^{h_1(\cdot)}}.$$

Then we get

$$\|\Delta_i(\Delta_j u \tilde{\Delta}_j)(\partial_k P v)\|_{M_{\rho}^{h(\cdot)}} \lesssim \|\Delta_j u\|_{M_{q(\cdot)}^{h_2(\cdot)}} \|\tilde{\Delta}_j v\|_{M_{p(\cdot)}^{h_1(\cdot)}}.$$

Considering I_3 , we have

$$\begin{aligned} \|2^{is} I_3\|_{M_{\rho}^{h(\cdot)}} &\lesssim \sum_{j \geq i-2} \|\Delta_j u\|_{M_{q(\cdot)}^{h_2(\cdot)}} \|\tilde{\Delta}_j v\|_{M_{p(\cdot)}^{h_1(\cdot)}} \\ &= \sum_{j \geq i-2} 2^{(i-j)s} \|\Delta_j u\|_{M_{q(\cdot)}^{h_2(\cdot)}} \|\tilde{\Delta}_j v\|_{M_{p(\cdot)}^{h_1(\cdot)}}. \end{aligned}$$

By combining the above estimates yields the result. $\square$

Proposition 3. Let $s > 0$, $1 \leq \gamma, \gamma_1, \gamma_2, h, h_1, h_2, p, q, r, \rho \leq \infty$, $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{q}$, $\frac{1}{h} = \frac{1}{h_1} + \frac{1}{h_2}$ and $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$. Then we have

$$\|uv\|_{\mathcal{L}^{\gamma} \dot{N}_{\rho, h, q}^s} \lesssim \|u\|_{\mathcal{L}^{\gamma_1} \dot{N}_{p, h_1, q}^s} \|v\|_{\mathcal{L}^{\gamma_2} M_r^{h_2}} + \|v\|_{\mathcal{L}^{\gamma_1} \dot{N}_{p, h_2, q}^s} \|u\|_{\mathcal{L}^{\gamma_2} M_r^{h_1}}.$$

Proof. Replacing $M_\rho^{h(\cdot)}$ in the proof of Proposition 2 by $\mathcal{L}^\gamma M_\rho^{h(\cdot)}$, we can obtain the desired inequality. $\square$

3. The Well-Posedness

In this section, we obtain the global well-posedness of generalized porous equations by using contraction mapping in critical Banach spaces. To insure the global well-posedness for small initial data u_0 , the following lemma is very important.

Lemma 4. (Lemma 5.5, [38]) Let X be a Banach space with norm $\|\cdot\|$ and $B : X \times X \rightarrow X$ be a bounded linear operator satisfying $\|B(u, v)\| \leq \eta \|u\| \|v\|$ for any $u, v \in X$ and a constant $\eta > 0$. Then for any $y \in X$ such that $4\eta \|y\| < 1$, the equation $x = y + B(u, v)$ has a solution $x \in X$. In particular, the solution is such that $\|x\| \leq 2\|y\|$ and it is the only one such that $\|x\| < \frac{1}{2\eta}$.

The main tool to prove the result is to obtain the priori estimates of the Equation (1). First, we prove the linear estimates of (1). To obtain this, we consider the linear homogeneous dissipative equation

$$\begin{cases} u_t + \mu \Lambda^\beta = f(x, t) & \text{in } \mathbb{R}^3 \times \mathbb{R}^+, \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^3, \end{cases} \quad (3)$$

for which we show the following lemma.

Lemma 5. Let $I = [0, T)$, $T \in (0, \infty]$, $1 \leq \gamma, q \leq \infty$, $p(\cdot), p_1(\cdot), h(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ for $p_1(\cdot) \leq p(\cdot)$, $p(\cdot) \leq h(\cdot) < \infty$ and $s(\cdot) \in C^{log}(\mathbb{R}^n)$. Assume that $u_0 \in \mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)}}$ and $f \in \mathcal{L}^\gamma(I; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma} - \beta})$. Then the Cauchy problem (3) has a unique solution $u \in \mathcal{L}^\infty(I; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)}}) \cap \mathcal{L}^\gamma(I; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma}})$ such that for all $\gamma_1 \in [\gamma, \infty]$

$$\|u\|_{\mathcal{L}^{\gamma_1}(I; \mathcal{FN}_{p_1(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1}})} \lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)}}} + \|f\|_{\mathcal{L}^\gamma(I; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma} - \beta})}. \quad (4)$$

Moreover, if $q < \infty$, then $u \in \mathcal{C}(I; \mathcal{FN}_{p_1(\cdot), h(\cdot), q}^{s(\cdot) + \frac{3}{p'(\cdot)}})$.

Proof. Clearly, solution to the Cauchy problem (3) can be written by the following integral equation:

$$u(x, t) = e^{-t\Lambda^\beta} u_0 + \int_0^t e^{-(t-\tau)\Lambda^\beta} f(x, \tau) d\tau. \quad (5)$$

The Fourier Transform of Equation (5) gives

$$\widehat{u}(x, t) = e^{-t|\cdot|^\beta} \widehat{u}_0 + \int_0^t e^{-(t-\tau)|\cdot|^\beta} \widehat{f}(\cdot, \tau) d\tau. \quad (6)$$

Multiplying $2^{i(s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i$ both sides of Equation (6), we have

$$\begin{aligned} 2^{i(s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i \widehat{u}(x, t) &= 2^{i(s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i e^{-t|\cdot|^\beta} \widehat{u}_0 \\ &\quad + \int_0^t 2^{i(s(\cdot) + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i e^{-(t-\tau)|\cdot|^\beta} \widehat{f}(\cdot, \tau) d\tau. \end{aligned} \quad (7)$$

Applying $\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})$ on both sides of Equation (7), we have

$$\begin{aligned}
& \left\| 2^{i(s(\cdot) + \frac{3}{p_1(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i \widehat{u} \right\|_{\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& \leq \left\| 2^{i(s(\cdot) + \frac{3}{p_1(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i e^{-t|\cdot|^\beta} \widehat{u}_0 \right\|_{\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& \quad + \left\| \int_0^t 2^{i(s(\cdot) + \frac{3}{p_1(\cdot)} + \frac{\beta}{\gamma_1})} e^{-(t-\tau)|\cdot|^\beta} \varphi_i \widehat{f}(\cdot, t) d\tau \right\|_{\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& = I_1 + I_2.
\end{aligned} \tag{8}$$

We estimate the above two terms separately. Let $p^* = \frac{p(x)p_1(x)}{p(x)-p_1(x)}$, using Proposition 1 and considering $p_1(\cdot) \leq p(\cdot)$, we get

$$\begin{aligned}
& \left\| 2^{i(s(\cdot) + \frac{3}{p_1(\cdot)} + \frac{\beta}{\gamma_1})} \varphi_i e^{-t|\cdot|^\beta} \widehat{u}_0 \right\|_{\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& \lesssim \sum_{j=0, \pm 1} \left\| 2^{i(s(\cdot) + \frac{3}{p'(\cdot)})} \varphi_i \widehat{u}_0 \right\|_{M_{p(\cdot)}^{h(\cdot)}} \left\| r^{-\frac{3}{p^*}} 2^{\frac{3}{p_1(\cdot)} - \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma_1}} \varphi_{i+j} e^{-t|\cdot|^\beta} \widehat{u}_0 \right\|_{\mathcal{L}^{\gamma_1}(I; L^{p^*})} \\
& \lesssim \sum_{j=0, \pm 1} \left\| 2^{i(s(\cdot) + \frac{3}{p'(\cdot)})} \varphi_i \widehat{u}_0 \right\|_{M_{p(\cdot)}^{h(\cdot)}} \left\| r^{-\frac{3}{p^*}} 2^{\frac{\beta}{\gamma_1}} e^{-t2^{\beta(i+j)}} \right\|_{\mathcal{L}^{\gamma_1}(I)} \left\| 2^{3i(\frac{1}{p_1(\cdot)} - \frac{1}{p'(\cdot)})} \varphi_{i+j} \right\|_{L^{p^*}} \\
& \lesssim \sum_{j=0, \pm 1} \left\| 2^{3i(\frac{1}{p_1(\cdot)} - \frac{1}{p'(\cdot)})} \varphi_{i+j} \right\|_{L^{p^*}} \left\| 2^{i(s(\cdot) + \frac{3}{p'(\cdot)})} \varphi_i \widehat{u}_0 \right\|_{M_{p(\cdot)}^{h(\cdot)}} \\
& \lesssim \left\| 2^{i(s(\cdot) + \frac{3}{p'(\cdot)})} \varphi_i \widehat{u}_0 \right\|_{M_{p(\cdot)}^{h(\cdot)}},
\end{aligned} \tag{9}$$

where we have used the following fact in the above estimate

$$\begin{aligned}
& \sum_{j=0, \pm 1} \left\| 2^{3i(\frac{1}{p_1(\cdot)} - \frac{1}{p'(\cdot)})} \varphi_{i+j} \right\|_{L^{p^*}} \\
& = \sum_{j=0, \pm 1} \left\| 2^{3i(\frac{1}{p_1(\cdot)} - \frac{1}{p'(\cdot)})} \varphi_{i+j} \right\|_{L^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}}} \\
& = \sum_{j=0, \pm 1} \inf \left\{ \lambda > 0 : \int \left| \frac{2^{3i(\frac{1}{p_1(\cdot)} - \frac{1}{p'(\cdot)})} \varphi_{i+j}}{\lambda} \right|^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}} dx \leq 1 \right\} \\
& = \sum_{j=0, \pm 1} \inf \left\{ \lambda > 0 : \int \left| \frac{\varphi_{i+j}}{\lambda} \right|^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}} 2^{-3i} dx \leq 1 \right\} \\
& = \sum_{j=0, \pm 1} \inf \left\{ \lambda > 0 : \int \left| \frac{\varphi_{i+j}}{\lambda} \right|^{\frac{p(2^i \cdot) p_1(2^i \cdot)}{p(2^i \cdot) - p_1(2^i \cdot)}} dx \leq 1 \right\} \lesssim C.
\end{aligned}$$

To estimate I_2 , using Proposition 1, we have

$$\begin{aligned}
& \left\| \int_0^t 2^{i(s(\cdot) + \frac{3}{p_1'(\cdot)} + \frac{\beta}{\gamma_1})} e^{-(t-\tau)|\cdot|^\beta} \varphi_i \widehat{f}(\cdot, t) d\tau \right\|_{\mathcal{L}^{\gamma_1}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& \lesssim \sum_{j=0, \pm 1} \left\| \int_0^t 2^{i(s(\cdot) + \frac{3}{p_1'(\cdot)} + \frac{\beta}{\gamma} - \beta)} \varphi_i \widehat{f}(\cdot, t) \right\|_{M_{p(\cdot)}^{h(\cdot)}} \\
& \quad \cdot \left\| r^{\frac{-3}{p^*}} \varphi_{i+j} e^{-(t-\tau)2^{\beta(i+j)}} 2^{\beta i(1 + \frac{1}{\gamma_1} - \frac{1}{\gamma}) + 3i(\frac{1}{p_1'(\cdot)} - \frac{1}{p'(\cdot)})} \right\|_{L^{p^*} d\tau} \Big\|_{\mathcal{L}^{\gamma_1}(I)} \\
& \lesssim \sum_{j=0, \pm 1} \left\| 2^{i(s(\cdot) + \frac{3}{p_1'(\cdot)} + \frac{\beta}{\gamma} - \beta)} \varphi_i \widehat{f}(\cdot, t) \right\|_{\mathcal{L}^{\gamma}(I; M_{p_1(\cdot)}^{h(\cdot)})} \left\| \varphi_{i+j} 2^{3i(\frac{1}{p_1'(\cdot)} - \frac{1}{p'(\cdot)})} \right\|_{L^{p^*}} \\
& \quad \cdot \left\| r^{\frac{-3}{p^*}} e^{-t2^{\beta(i+j)}} 2^{\beta i(1 + \frac{1}{\gamma_1} - \frac{1}{\gamma})} \right\|_{\mathcal{L}^{(1 + \frac{1}{\gamma_1} - \frac{1}{\gamma})^{-1}}(\mathbb{R}_+)} \\
& \lesssim \sum_{j=0, \pm 1} \left\| \varphi_{i+j} 2^{3i(\frac{1}{p_1'(\cdot)} - \frac{1}{p'(\cdot)})} \right\|_{L^{p^*}} \left\| 2^{i(s(\cdot) + \frac{3}{p_1'(\cdot)} + \frac{\beta}{\gamma} - \beta)} \varphi_i \widehat{f}(\cdot, t) \right\|_{\mathcal{L}^{\gamma}(I; M_{p_1(\cdot)}^{h(\cdot)})} \\
& \lesssim \left\| 2^{i(s(\cdot) + \frac{3}{p_1'(\cdot)} + \frac{\beta}{\gamma} - \beta)} \varphi_i \widehat{f}(\cdot, t) \right\|_{\mathcal{L}^{\gamma}(I; M_{p_1(\cdot)}^{h(\cdot)})}. \tag{10}
\end{aligned}$$

Using the inequalities (9), (10) in (8) and taking the l^q -norm, we can obtain the desired estimate (4). $\square$

Theorem 1. Let $p(\cdot), h(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $h(\cdot) \leq p(\cdot)$, $2 \leq p(\cdot) \leq \frac{6}{5-2\beta}$, $1 \leq q < \frac{3}{\beta-1}$, $1 < \beta \leq \frac{5}{2}$ and $1 \leq \gamma \leq \infty$ then there exists a constant $\delta > 0$ such that for any $u_0 \in \mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)}}(\mathbb{R}^n)$ satisfies $\|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)}}(\mathbb{R}^n)} < \delta$, the Equation (1) has a unique global solution

$$u \in \mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2} - \beta + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\infty}(\mathbb{R}_+; \mathcal{N}_{2, 2, q}^{\frac{5}{2} - \beta}),$$

such that

$$\|u\|_{\mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2} - \beta + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\infty}(\mathbb{R}_+; \mathcal{N}_{2, 2, q}^{\frac{5}{2} - \beta})} \lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)}}}. \tag{11}$$

Proof. For $H_{\beta}(t) = e^{-t\Lambda^{\beta}}$, we can write the solution $u(x, t)$ of the Equation (1) in the following integral form

$$u = H_{\beta}(t)u_0 + \int_0^t H_{\beta}(t - \tau) \nabla \cdot (u \nabla Pu) d\tau. \tag{12}$$

Next we define

$$Y := \mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), q}^{1-\beta + \frac{3}{p'(\cdot)} + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\gamma}(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2} - \beta + \frac{\beta}{\gamma}}) \cap \mathcal{L}^{\infty}(\mathbb{R}_+; \mathcal{N}_{2, 2, q}^{\frac{5}{2} - \beta})$$

and consider the mapping below

$$\psi : u \rightarrow H_{\beta}(t)u_0 + \int_0^t H_{\beta}(t - \tau) \nabla \cdot (u \nabla Pu) d\tau. \tag{13}$$

We have to show that the above mapping is a contraction mapping. First, we can write

$$\begin{aligned}
\|\psi(u)\|_Y & \leq \|H_{\beta}(t)u_0\|_Y + \left\| \int_0^t H_{\beta}(t - \tau) \nabla \cdot (u \nabla Pu) d\tau \right\|_Y \\
& = I_1 + I_2.
\end{aligned} \tag{14}$$

To estimate I_1 , using Lemma 5 with $f \equiv 0$ and considering the hypothesis $p(\cdot) \geq 2$, we obtain

$$\begin{aligned} \|H_\beta(t)u_0\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma}})} &\lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}}}, \\ \|H_\beta(t)u_0\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{2,2,q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} &\lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}}}, \\ \|H_\beta(t)u_0\|_{\mathcal{L}^\infty(\mathbb{R}_+; \mathcal{FN}_{2,2,q}^{\frac{5}{2}-\beta})} &\lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}}}. \end{aligned}$$

Hence, we get

$$\|H_\beta(t)u_0\|_Y \lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}}} < \delta. \quad (15)$$

To estimate I_2 , let $\bar{p}_\alpha(\cdot) = \frac{6p(\cdot)}{6-(5-2\beta)p(\cdot)}$, using Propositions 1 and 3, then

$$\begin{aligned} &\left\| \int_0^t H_\beta(t-\tau) \nabla \cdot (u \nabla Pu) d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma}})} \\ &\lesssim \left\| \int_0^t 2^{i(1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i e^{-(t-\tau)|\cdot|^\beta} \nabla \cdot (\widehat{u \nabla Pu}) d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})} \Big\|_{l^q} \\ &\lesssim \left\| \int_0^t \|r^{\frac{-3}{\bar{p}_\alpha(\cdot)}} 2^{i(2-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i e^{-(t-\tau)|\cdot|^\beta} \|_{L^{\bar{p}_\alpha(\cdot)}} \|\Delta_i(u \nabla Pu)\|_{M_{p(\cdot)}^{\frac{6}{2\beta+1}}} d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+)} \Big\|_{l^q} \\ &\lesssim \left\| \int_0^t 2^{i(\frac{\beta}{\gamma}+\frac{5}{2})} e^{-(t-\tau)2^{\beta i}} \|r^{\frac{-3}{\bar{p}_\alpha(\cdot)}} 2^{-3i\frac{1}{\bar{p}_\alpha(\cdot)}} \varphi_i\|_{L^{\bar{p}_\alpha(\cdot)}} \|\Delta_i(u \nabla Pu)\|_{M_{p(\cdot)}^{\frac{6}{2\beta+1}}} d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+)} \Big\|_{l^q} \\ &\lesssim \left\| 2^{i(\frac{\beta}{\gamma}+\frac{5}{2}-\beta)} \|\Delta_i(u \nabla Pu)\|_{M_{p(\cdot)}^{\frac{6}{2\beta+1}}} \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+)} \|e^{-t2^{\beta i}} 2^{\beta i}\|_{\mathcal{L}^1(\mathbb{R}_+)} \Big\|_{l^q} \\ &\lesssim \|u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \|u\|_{\mathcal{L}^\infty(\mathbb{R}_+; L^{\frac{3}{\beta-1}})} \\ &\lesssim \|u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \|u\|_{\mathcal{L}^\infty(\mathbb{R}_+; B_{2,q}^{\frac{5}{2}-\beta})} \\ &\lesssim \|u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \|u\|_{\mathcal{L}^\infty(\mathbb{R}_+; \mathcal{N}_{2,2,q}^{\frac{5}{2}-\beta})}. \end{aligned} \quad (16)$$

Similarly, we have

$$\begin{aligned} &\left\| \int_0^t H_\beta(t-\tau) \nabla \cdot (u \nabla Pu) d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \\ &\lesssim \|u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \|u\|_{\mathcal{L}^\infty(\mathbb{R}_+; \mathcal{N}_{2,2,q}^{\frac{5}{2}-\beta})} \end{aligned} \quad (17)$$

and

$$\begin{aligned} &\left\| \int_0^t H_\beta(t-\tau) \nabla \cdot (u \nabla Pu) d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{p(\cdot), h(\cdot), q}^{\frac{5}{2}-\beta})} \\ &\lesssim \|u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{N}_{2, h(\cdot), q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}})} \|u\|_{\mathcal{L}^\infty(\mathbb{R}_+; \mathcal{N}_{2,2,q}^{\frac{5}{2}-\beta})}. \end{aligned} \quad (18)$$

Hence, we obtain

$$\left\| \int_0^t H_\beta(t-\tau) \nabla \cdot (u \nabla Pu) d\tau \right\|_Y \lesssim \|u\|_Y \|u\|_Y. \quad (19)$$

Using the estimates (15) and (19) in (14), we have

$$\begin{aligned}\|\psi(u)\|_Y &\leq C_1 \|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}} + C_2 \|u\|_Y \|u\|_Y \\ &\leq C_1 \|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}} + C_2 \delta_0^2.\end{aligned}$$

Choosing $\delta_0 < \frac{1}{2\max\{C_1, C_2\}}$ and for any $u_0 \in \mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}$ with

$$\|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}} < \frac{\delta_0}{2\max\{C_1, C_2\}},$$

we get

$$\|\psi(u)\|_Y \leq \frac{\delta_0}{2} + \frac{\delta_0}{2} = \delta_0.$$

Using Lemma 4, we can directly obtain that Equation (1) has a unique global solution with $\|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}} < \delta$ for sufficiently small δ . $\square$

4. Gevrey Class Regularity

In this section, we show the gevrety class regularity for Equation (1). Many researchers have studied analyticity with respect to Navier-Stokes equations. For more study related to Gevrey class regularity we refer the reader to [39,40]. In order to prove the spatial analyticity, the following lemma is very helpful.

Lemma 6. (Lemma 4.1 [41]) Let $0 < s \leq t < \infty$ and $0 \leq \beta \leq 2$. Then the following inequality holds

$$t|x|^{\frac{\beta}{2}} - \frac{1}{2}(t^2 - s^2)|x|^{\beta} - s|x - y|^{\frac{\beta}{2}} - s|y|^{\frac{\beta}{2}} \leq \frac{1}{2}$$

for all $x, y \in \mathbb{R}^n$.

Theorem 2. Let $p(\cdot), h(\cdot) \in C^{log}(\mathbb{R}^n) \cap \mathcal{P}_0(\mathbb{R}^n)$, $h(\cdot) \leq p(\cdot)$, $2 \leq p(\cdot) \leq \frac{6}{5-2\beta}$, $1 \leq q < \frac{3}{\beta-1}$, then there exist a constant $\varepsilon > 0$ such that for all initial data $u_0 \in \mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}$ satisfying $\|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}} < \varepsilon$. The Equation (1) has a unique analytic solution u in the sense that

$$\|e^{\sqrt{t}|D|^{\frac{\beta}{2}}} u\|_{\mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma}}) \cap \mathcal{L}^\gamma(\mathbb{R}_+; \mathcal{FN}_{2,h(\cdot),q}^{\frac{5}{2}-\beta+\frac{\beta}{\gamma}}) \cap \mathcal{L}^\infty(\mathbb{R}_+; \mathcal{FN}_{2,2,q}^{\frac{5}{2}-\beta})} \lesssim \|u_0\|_{\mathcal{FN}_{p(\cdot),h(\cdot),q}^{1-\beta+\frac{3}{p'(\cdot)}}}$$

Proof. Assume $\omega(x, t) = e^{\sqrt{t}|D|^{\frac{\beta}{2}}} u(x, t)$ and using Equation (12), we obtain

$$\omega(x, t) = e^{\sqrt{t}|D|^{\frac{\beta}{2}}} H_\beta(t) u_0 + \int_0^t e^{\sqrt{t}|D|^{\frac{\beta}{2}}} H_\beta(t - \tau) \nabla \cdot (u \nabla P u) d\tau.$$

It is easy to obtain

$$\begin{aligned}
& \left\| 2^{i(1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i \widehat{w} \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})} \\
& \lesssim \left\| e^{\sqrt{t}|\xi|^{\frac{\beta}{2}}-t|\xi|^\beta} 2^{i(1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i \widehat{u}_0 \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})} \\
& + \left\| 2^{i(2-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i \int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^\beta} \int_{\mathbb{R}^n} e^{\sqrt{t}|\xi|^{\frac{\beta}{2}}-\frac{1}{2}(t-\tau)|\xi|^\beta-\sqrt{\tau}(|\xi-\eta|^{\frac{\beta}{2}}+|\eta|^{\frac{\beta}{2}})} \right. \\
& \quad \left. \left(\nabla \cdot (u(\xi-\eta, \tau) \nabla Pu(\eta, \tau)) \right) d\eta d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})}.
\end{aligned}$$

Using $e^{\sqrt{t}|\xi|^{\frac{\beta}{2}}-\frac{1}{2}t|\xi|^\beta} = e^{\frac{1}{2}(\sqrt{t}|\xi|^{\frac{\beta}{2}}-1)^2+\frac{1}{2}} \leq e^{\frac{1}{2}}$ and Lemma 6, we obtain

$$\begin{aligned}
& \lesssim \left\| e^{-\frac{1}{2}t|\xi|^\beta} 2^{i(1-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i \widehat{u}_0 \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})} \\
& + \left\| 2^{i(2-\beta+\frac{3}{p'(\cdot)}+\frac{\beta}{\gamma})} \varphi_i \int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^\beta} \left(\nabla \cdot (u(\xi-\eta, \tau) \nabla Pu(\eta, \tau)) \right) d\tau \right\|_{\mathcal{L}^\gamma(\mathbb{R}_+; M_{p(\cdot)}^{h(\cdot)})}.
\end{aligned}$$

The remaining part of the proof is similar to the proof of Theorem 3.3, therefore the details can be omitted. $\square$

5. Conclusions

In this paper, we considered the generalized porous medium equations. Young's inequality is one of the important tools to obtain the global well-posedness result of such equations. In the previous work of Xiao and Zhou [3], they obtained the local well-posedness for large initial data and the global well-posedness for small initial data in Fourier-Besov spaces. We can't use Young's inequality in variable exponent function spaces in order to obtain the global well-posedness result. We overcame with this problem and obtained the global well-posedness results of this equation for small initial data u_0 belonging to the homogeneous Fourier-Besov-Morrey spaces with variable exponent. In addition, we also shown the Gevrey class regularity of the solution.

Author Contributions: Methodology, M.Z.A.; Project administration, J.C.; Software, M.Z.A.; Supervision, J.C.; Writing—original draft, M.Z.A.; Writing—review & editing, J.C. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported by NSF of China (No. 12071437).

Acknowledgments: The authors is highly thankful to the Editor-in-Chief and the anonymous referees for their valuable comments and suggestions for the improvement of our manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zhou, X.; Xiao, W.; Zheng, T. Well-posedness and blowup criterion of generalized porous medium equation in Besov spaces. *Electron. J. Differ. Equ.* **2015**, *2015*, 1–14.
2. Caffarelli, L.; Vazquez, J.L. Nonlinear Porous Medium Flow with Fractional Potential Pressure. *Arch. Ration. Mech. Anal.* **2011**, *202*, 537–565. [[CrossRef](#)]
3. Xiao, W.; Zhou, X. On the Generalized Porous Medium Equation in Fourier-Besov Spaces. *J. Math. Study* **2020**, *53*, 316–328.
4. Vázquez, J.L. *The Porous Medium Equation: Mathematical Theory*; Oxford University Press: New York, NY, USA, 2007.
5. Bear, J. *Dynamics of Fluids in Porous Media*; Courier Corporation: Chelmsford, MA, USA, 2013.
6. Aronson, D.G. The porous medium equation. In *Nonlinear Diffusion Problems*; Springer: Berlin/Heidelberg, Germany, 1986; pp. 1–46.
7. Zhou, X.; Xiao, W.; Chen, J. Fractional porous medium and mean field equations in Besov spaces. *Electron. J. Differ. Equ.* **2014**, *2014*, 1–14.
8. Lin, F.; Zhang, P. On the hydrodynamic limit of Ginzburg-Landau wave vortices. *Commun. Pure Appl. Math.* **2002**, *55*, 831–856. [[CrossRef](#)]

9. Biler, P.; Imbert, C.; Karch, G. Barenblatt profiles for a nonlocal porous medium equation. *Comptes Rendus Math.* **2011**, *349*, 641–645. [\[CrossRef\]](#)
10. Blanchet, A.; Carrillo, J.A.; Masmoudi, N. Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. *Commun. Pure Appl. Math.* **2008**, *61*, 1449–1481. [\[CrossRef\]](#)
11. Topaz, C.M.; Bertozzi, A.L.; Lewis, M.A. A nonlocal continuum model for biological aggregation. *Bull. Math. Biol.* **2006**, *68*, 1601. [\[CrossRef\]](#)
12. Li, D.; Rodrigo, J.L. Well-posedness and regularity of solutions of an aggregation equation. *Rev. Mat. Iberoam.* **2010**, *26*, 261–294. [\[CrossRef\]](#)
13. Li, D.; Zhang, X. Global wellposedness and blowup of solutions to a nonlocal evolution problem with singular kernels. *Commun. Pure Appl. Anal.* **2010**, *9*, 1591. [\[CrossRef\]](#)
14. Karch, G.; Suzuki, K. Blow-up versus global existence of solutions to aggregation equations. *Appl. Math.* **2011**, *38*, 243–258. [\[CrossRef\]](#)
15. Laurent, T. Local and global existence for an aggregation equation. *Commun. Partial. Differ. Equ.* **2007**, *32*, 1941–1964. [\[CrossRef\]](#)
16. Chen, Q.; Zhang, Z. Global well-posedness of the 2D critical dissipative quasi-geostrophic equation in the Triebel–Lizorkin spaces. *Nonlinear Anal. Theory Methods Appl.* **2007**, *67*, 1715–1725. [\[CrossRef\]](#)
17. Wang, H.; Zhang, Z. A frequency localized maximum principle applied to the 2D quasi-geostrophic equation. *Commun. Math. Phys.* **2011**, *301*, 105–129. [\[CrossRef\]](#)
18. Wu, G.; Zhang, Q. Global well-posedness of the aggregation equation with supercritical dissipation in Besov spaces. *ZAMM J. Appl. Math. Mech./Z. Angew. Math. Mech.* **2013**, *93*, 882–894. [\[CrossRef\]](#)
19. Konieczny, P.; Yoneda, T. On dispersive effect of the Coriolis force for the stationary Navier–Stokes equations. *J. Differ. Equ.* **2011**, *250*, 3859–3873. [\[CrossRef\]](#)
20. Iwabuchi, T.; Takada, R. Global well-posedness and ill-posedness for the Navier–Stokes equations with the Coriolis force in function spaces of Besov type. *J. Funct. Anal.* **2014**, *267*, 1321–1337. [\[CrossRef\]](#)
21. Liu, Q.; Zhao, J. Global well-posedness for the generalized magneto-hydrodynamic equations in the critical Fourier–Herz spaces. *J. Math. Anal. Appl.* **2014**, *420*, 1301–1315. [\[CrossRef\]](#)
22. Toumtilin, M. Global well-posedness and analyticity for generalized porous medium equation in critical Fourier–Besov–Morrey spaces. *Open J. Math. Anal.* **2019**, *3*, 71–80. [\[CrossRef\]](#)
23. Orlicz, W. Über eine gewisse Klasse von Räumen vom Typus B. *Bull. Int. Acad. Pol. Ser. A* **1932**, *8*, 207–220.
24. Musielak, J. Orlicz spaces and modular spaces. *Lect. Notes Math.* **1983**, *1034*, 1–216.
25. Nakano, H. *Topology and Linear Topological Spaces*; Maruzen Company: Tokyo, Japan, 1951; Volume 3.
26. Kováčik, O.; Rákosník, J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslov. Math. J.* **1991**, *41*, 592–618. [\[CrossRef\]](#)
27. Cruz-Uribe, D. Operator on variable-LP spaces. In *Seminar of Mathematical Analysis: Proceedings*; Universidad de Sevilla: Sevilla, Spain, 2003; p. 147.
28. Diening, L. Maximal Function on Generalized Lebesgue Spaces $L^{p(x)}$. *Math. Inequal. Appl.* **2004**, *7*, 245–253.
29. Ruzicka, M. *Electrorheological Fluids: Modeling and Mathematical Theory*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2000.
30. Chen, Y.; Levine, S.; Rao, M. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **2006**, *66*, 1383–1406. [\[CrossRef\]](#)
31. Fan, X. Global C_1, α regularity for variable exponent elliptic equations in divergence form. *J. Differ. Equ.* **2007**, *235*, 397–417. [\[CrossRef\]](#)
32. Almeida, A.; Hästö, P. Besov spaces with variable smoothness and integrability. *J. Funct. Anal.* **2010**, *258*, 1628–1655. [\[CrossRef\]](#)
33. Ru, S.; Abidin, M.Z. Global well-posedness of the incompressible fractional Navier–Stokes equations in Fourier–Besov spaces with variable exponents. *Comput. Math. Appl.* **2019**, *77*, 1082–1090. [\[CrossRef\]](#)
34. Abidin, M.Z.; Chen, J. Global well-posedness of the generalized rotating magnetohydrodynamics equations in variable exponent Fourier–Besov spaces. *J. Appl. Anal. Comput.* **2020**. [\[CrossRef\]](#)
35. Khan, A.A.; Bukhari, S.R.; Marin, M.; Ellahi, R. Effects of chemical reaction on third-grade MHD fluid flow under the influence of heat and mass transfer with variable reactive index. *Heat Transf. Res.* **2019**, *50*, 1061–1080. [\[CrossRef\]](#)
36. Almeida, A.; Caetano, A. Variable exponent Besov–Morrey spaces. *J. Fourier Anal. Appl.* **2020**, *26*, 1–42. [\[CrossRef\]](#)
37. Abidin, M.Z.; Chen, J. Global well-posedness for fractional Navier–Stokes equations in variable exponent Fourier–Besov–Morrey spaces. *Acta Math. Sci.* **2021**, *41*, 164–176. [\[CrossRef\]](#)
38. Bahouri, H.; Chemin, J.Y.; Danchin, R. *Fourier Analysis and Nonlinear Partial Differential Equations*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2011; Volume 343.
39. Bae, H.; Biswas, A.; Tadmor, E. Analyticity and decay estimates of the Navier–Stokes equations in critical Besov spaces. *Arch. Ration. Mech. Anal.* **2012**, *205*, 963–991. [\[CrossRef\]](#)
40. Foias, C.; Temam, R. Gevrey class regularity for the solutions of the Navier–Stokes equations. *J. Funct. Anal.* **1989**, *87*, 359–369. [\[CrossRef\]](#)
41. Ferreira, L.C.; Villamizar-Roa, E.J. Exponentially-stable steady flow and asymptotic behavior for the magnetohydrodynamic equations. *Commun. Math. Sci.* **2011**, *9*, 499–516. [\[CrossRef\]](#)