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Prime i -Ideals in Ordered n -ary SemigroupsPatchara Pornsurat , Pakorn Palakawong na Ayutthaya and Bundit Pibaljommee *

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Abstract: We study the concept of i -ideal of an ordered n -ary semigroup and give a construction of the i -ideal of an ordered n -ary semigroup generated by its nonempty subset. Moreover, we study the notions of prime, weakly prime, semiprime and weakly semiprime ideals of an ordered n -ary semigroup.

Keywords: n -ary semigroup; ordered n -ary semigroup; prime ideal; weakly prime ideal; semiprime ideal

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1. Introduction

A semigroup, which is a well-known generalization of groups, is an algebraic structure appearing in a natural manner in some applications concerning the theory of automata formal languages and other branches of applied mathematics (for example, see [1–5]). A notable particular kind of semigroup is a semigroup together with a partially ordered relation, namely an ordered semigroup. Obviously, we are always able to see a semigroup S as an ordered semigroup together with the equality relation $\{(x, x) \in S \times S\}$. Some classical and notable researches on ordered semigroups are [6–9]. Additionally, Kehayopulu and Tsingelis defined several types of regularities on ordered semigroups in [10,11].

A ternary algebraic system was first introduced as a certain ternary algebraic system called a triplex, which turns out to be a ternary group by Lehmer [12] in 1932. As a generalization of a ternary group, the notion of a ternary semigroup was known to S. Banach. He showed that it is always possible to construct a ternary semigroup from a (binary) semigroup and gave an example in which a ternary semigroup does not necessarily reduce to a semigroup. Later, Santiago [13] mainly investigated the notions of an ideal and a bi-ideal of a ternary semigroup and used them to characterize a regular ternary semigroup. In 2012, Daddi and Pawar [14] defined the notions of ordered quasi-ideals and ordered bi-ideals of ordered ternary semigroups, which are special kinds of ternary semigroups and also presented regular ordered ternary semigroups in terms of several ideal-theoretical characterizations. Another type of regularity on ordered ternary semigroups, namely an intra-regular ordered ternary semigroup, was introduced and characterized by S. Lekkoksung and N. Lekkoksung in [15]. Several kinds of regularities of ordered ternary semigroups were also investigated and characterized in terms of many kinds of ordered ideals by Daddi in [16] and Pornsurat and Pibaljommee in [17].

A generalization of classical algebraic structures to n -ary structures was first introduced by Kasner [18] in 1904. In 1928, Dörnte [19] introduced and studied the notion of n -ary groups, which is a generalization of that one of groups. Later, Sioson [20] introduced a regular n -ary semigroup, which is an important generalization of a (binary) semigroup and verified its properties. In [21], Dudek and Groździńska investigated the nature of regular n -ary semigroups. Moreover, Dudek proved several results and gave many examples of n -ary groups in [22,23]. Furthermore, in [24], Dudek also studied potent elements of n -ary semigroups ($n \geq 3$) and investigated properties of ideals in which all elements are potent.

Prime, semiprime, weakly prime and weakly semiprime properties are interesting properties of ideals of semigroups and ordered semigroups. In 1970, Szász [25] showed

that a semigroup is intra-regular and the set of all ideals forms a chain if and only if every ideal is prime. Later, Kehayopulu [26] gave the same result in ordered semigroups. Furthermore, Kehayopulu also generalized her work on ordered semigroups [26] to ordered hypersemigroups in [27]. The concepts of prime and semiprime ideals were also studied on ternary semigroups by Shabir and Bashir [28] in 2009. In cases of quasi-ideals and bi-ideals of ternary semigroups, Shabir and Bano [29] introduced the notions of prime, semiprime and strongly prime properties of quasi-ideals and bi-ideals and also characterized ternary semigroups for which each bi-ideal is strongly prime.

In this work, we investigate some algebraic properties of an ordered n -ary semigroup, which is an n -ary semigroup together with a partially ordered relation and also a generalization of ordered semigroups. We also study the concept of an i -ideal of an ordered n -ary semigroup and give the construction of an i -ideal of an ordered n -ary semigroup generated by its nonempty subset. Then we generalize the concept of an i -ideal of an ordered n -ary semigroup to the concept of a Λ -ideal in order that an i -ideal is a special case of a Λ -ideal where $\Lambda = \{i\}$. Finally, we study the notions of prime, weakly prime, semiprime and weakly semiprime properties of a Λ -ideal where $\Lambda = \{1, n\}$ and use them to generalize some results in [26] to an ordered n -ary semigroup.

2. Preliminaries

Let $\mathbb{N}$ be the set of all natural numbers and $i, j, n \in \mathbb{N}$. A nonempty set S together with an n -ary operation given by $f : S^n \rightarrow S$, where $n \geq 2$, is called an n -ary groupoid [24]. For $1 \leq i < j \leq n$, the sequence $x_i, x_{i+1}, x_{i+2}, \dots, x_j$ of elements of S is denoted by x_i^j and if $x_i = x_{i+1} = \dots = x_j = x$, we write x^{j-i+1} instead of x_i^j . For $j < i$, we denote x_i^j as empty symbol and so is x^0 . So, the term

$$f(x_1, \dots, x_i, \underbrace{x, x, \dots, x}_j, x_{i+j+1}, \dots, x_n)$$

is able to be simply represented by

$$f(x_1^i, x^j, x_{i+j+1}^n).$$

The associative law [23] for the n -ary operation f on S is defined by

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

for all $1 \leq i \leq j \leq n$ and $x_1, \dots, x_{2n-1} \in S$. An n -ary groupoid (S, f) is called n -ary semigroup if the n -ary operation f satisfies the associative law.

By the associativity of an n -ary semigroup (S, f) , we define the map f_k where $k \geq 2$ by

$$f_k(x_1^{k(n-1)+1}) = \underbrace{f(f(\dots, f(f(x_1^n), x_{n+1}^{2n-1}), \dots), x_{(k-1)(n-1)+2}^{k(n-1)+1})}_{k \text{ terms}}$$

for any $x_1, \dots, x_{k(n-1)+1} \in S$. Hence, we note that $f_2(x_1^{2n-1}) = f(f(x_1^n), x_{n+1}^{2n-1})$ for any $x_1, \dots, x_{2n-1} \in S$. Many notations above can also be found in [21–24].

For $1 \leq i < j \leq n$, the sequence $A_i, A_{i+1}, \dots, A_j$ of nonempty sets of S is denoted by the symbol A_i^j . For nonempty subsets A_1^n of S , we denote

$$f(A_1^n) = \{f(a_1^n) \mid a_i \in A_i \text{ where } 1 \leq i \leq n\}.$$

If $A_1 = A_2 = \dots = A_n = A$, we write $f(\overset{n}{A})$ instead of $f(A_1^n)$. If $A_1 = \{a_1\}$, then we write $f(a_1, A_2^n)$ instead of $f(\{a_1\}, A_2^n)$, and similarly in other cases. For $j < i$, we set the notations A_j^i and $\overset{0}{A}$ to be the empty symbols as a similar way of the notations x_j^i and $\overset{0}{x}$.

An ordered semigroup is an algebraic structure $(S, \cdot, \leq)$ such that $(S, \cdot)$ is a semigroup and $(S, \leq)$ is a poset satisfying the compatibility property, i.e., if $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ for all $a, b, c \in S$. To generalize the notion of n -ary semigroups and ordered (binary) semigroups, the notion of an ordered n -ary semigroup is defined as follows.

Definition 1. An ordered n -ary semigroup is a system $(S, f, \leq)$ such that (S, f) is an n -ary semigroup and $(S, \leq)$ is a partially ordered set satisfying the following property. For any $a, b, x_1, \dots, x_n \in S$, if $a \leq b$, then

$$f(x_1^{i-1}, a, x_{i+1}^n) \leq f(x_1^{i-1}, b, x_{i+1}^n)$$

for all $1 \leq i \leq n$.

Throughout this paper, we write S for an ordered n -ary semigroup, unless specified otherwise.

For any $\emptyset \neq A \subseteq S$, we denote

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Now, we give some basic properties of the operator $(\]$ on an ordered n -ary semigroup as follows.

Lemma 1. Let $A, B, A_1, \dots, A_n$ be nonempty subsets of S . Then the following statements hold:

- (i) $A \subseteq [A]$;
- (ii) $[A] = ([A])$;
- (iii) $A \subseteq B$ implies $[A] \subseteq [B]$;
- (iv) $f([A_1], [A_2], \dots, [A_n]) \subseteq f([A_1^n])$;
- (v) $[A \cup B] = [A] \cup [B]$;
- (vi) $[A \cap B] \subseteq [A] \cap [B]$.

The following remark is directly obtained by the notion of f_k where $k = m$ and $k = m + 1$.

Remark 1. Let A be a nonempty subset of S , $1 \leq i \leq n$ and $m \in \mathbb{N}$. If $y \in f_m(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S})$, then $f(\overset{i-1}{S}, \{y\}, \overset{n-i}{S}) \subseteq f_{m+1}(\overset{(m+1)(i-1)}{S}, A, \overset{(m+1)(n-i)}{S})$.

Lemma 2. Let A be a nonempty subset of S . Then $f_n(\overset{n(i-1)}{S}, A, \overset{n(n-i)}{S}) \subseteq f(\overset{i-1}{S}, A, \overset{n-i}{S})$ for all positive integers $i \leq n$.

Proof. Let $y \in f_n(\overset{n(i-1)}{S}, A, \overset{n(n-i)}{S})$.

Then $y = f_n(s_1^{n(i-1)}, a, s_{n(i-1)+2}^{n(n-1)+1})$ for some $s_1^{i-1}, s_{i+1}^{n(n-1)+1} \in S, a \in A$.

$$\begin{aligned} y &= f_n(s_1^{n(i-1)}, a, s_{n(i-1)+2}^{n(n-1)+1}) \\ &= f(f(s_1^n), f(s_{n+1}^{2n}), \dots, f(s_{n(i-1)-n+1}^{n(i-1)}), a, f(s_{n(i-1)+2}^{ni+1}), \dots, f(s_{n(n-2)+2}^{n(n-1)+1})) \\ &\in f(\overset{i-1}{S}, A, \overset{n-i}{S}). \end{aligned}$$

Hence, $f_n(\overset{n(i-1)}{S}, A, \overset{n(n-i)}{S}) \subseteq f(\overset{i-1}{S}, A, \overset{n-i}{S})$ for all positive integers $i \leq n$. $\square$

3. Ideals of Ordered n -ary Semigroups

In this section, we study the concept of i -ideals of an ordered n -ary semigroup and give a construction of i -ideal of an ordered n -ary semigroup generated by its nonempty subset.

Definition 2. Let $1 \leq i \leq n$. A nonempty subset I of S is called an i -ideal of S if $f(\overset{i-1}{S}, I, \overset{n-i}{S}) \subseteq I$ and $I = (I]$. A nonempty subset I of S is called an ideal of S if I is an i -ideal for every $1 \leq i \leq n$.

Next, we define the notion of a Λ -ideal of S whenever $\emptyset \neq \Lambda \subseteq \{1, \dots, n\}$ as follows.

Definition 3. Let $\emptyset \neq \Lambda \subseteq \{1, \dots, n\}$. A nonempty subset I of S is called a Λ -ideal of S if $f(\overset{\lambda-1}{S}, I, \overset{n-\lambda}{S}) \subseteq I$ for all $\lambda \in \Lambda$ and $I = (I]$.

Lemma 3. Let $A_1, A_2, \dots, A_n$ be nonempty subsets of S . If A_1, A_n are $\{1, n\}$ -ideals of S , then $(f(A_1^n))$ is a $\{1, n\}$ -ideal of S .

Proof. Let A_1, A_n be $\{1, n\}$ -ideals of S . Then $f((f(A_1^n)), \overset{n-1}{S}) \subseteq (f(A_1^{n-1}), f(A_n, \overset{n-1}{S})) \subseteq (f(A_1^{n-1}), A_n) = (f(A_1^n))$ and $((f(A_1^n))) = (f(A_1^n))$. So, $(f(A_1^n))$ is a 1-ideal of S . Similarly, we can show that $(f(A_1^n))$ is an n -ideal of S . Hence, $(f(A_1^n))$ is a $\{1, n\}$ -ideal of S . $\square$

Lemma 4. The following statements are equivalent.

- (i) $(f(\overset{n}{A})) = A$ for every ideal A of S .
- (ii) $\bigcap_{i=1}^n A_i = (f(A_1^n))$ for all ideals A_i of S .

Proof. (i) $\Rightarrow$ (ii). Let $A_1, \dots, A_n$ be ideals of S .

We consider $f(\overset{i-1}{S}, \bigcap_{i=1}^n A_i, \overset{n-i}{S}) \subseteq \bigcap_{i=1}^n f(\overset{i-1}{S}, A_i, \overset{n-i}{S}) \subseteq \bigcap_{i=1}^n A_i$ and $(\bigcap_{i=1}^n A_i) \subseteq \bigcap_{i=1}^n (A_i) = \bigcap_{i=1}^n A_i \subseteq (\bigcap_{i=1}^n A_i]$, i.e., $(\bigcap_{i=1}^n A_i) = \bigcap_{i=1}^n A_i$. So, $\bigcap_{i=1}^n A_i$ is an i -ideal of S for all $i = 1, \dots, n$. That is $\bigcap_{i=1}^n A_i$ is an ideal of S . Let $\bigcap_{i=1}^n A_i = B$. By assumption, $B = (f(\overset{n}{B})) \subseteq (f(A_1^n))$. It is clear that $(f(A_1^n)) \subseteq \bigcap_{i=1}^n A_i$.

(ii) $\Leftarrow$ (i). Let A be an ideal of S . Then $A = \bigcap_{i=1}^n A = (f(\overset{n}{A}))$. $\square$

Corollary 1. The following statements are equivalent.

- (i) $(f(\overset{n}{A})) = A$ for every $\{1, n\}$ -ideal A of S .
- (ii) $\bigcap_{i=1}^n A_i \subseteq (f(A_1^n))$ for all $\{1, n\}$ -ideals A_i of S .

Let A be a nonempty subset of S . We call the intersection of all Λ -ideals of S containing A to be the Λ -ideal of S generated by A . Then, we denote the notation $J_\Lambda(A)$ to mean the Λ -ideal of S generated by A . In a particular case $\Lambda = \{i\}$, we write $J_i(A)$ instead of $J_{\{i\}}(A)$.

Next, we give the constructions of the Λ -ideals of S generated by a nonempty subset whenever $\Lambda = \{i\}$ and $\Lambda = \{1, n\}$ as follows.

Theorem 1. Let A be a nonempty subset of S . Then

$$J_i(A) = \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \cup A \right).$$

Proof. Let A be a nonempty subset of S and $\mathcal{I} = \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \cup A \right) = \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \right] \cup (A]$. First, we show that $\mathcal{I}$ is an i -ideal of S . Let $y \in \mathcal{I}$. Then we consider the following two cases.

- Case 1: $y \in (A]$. Then $f \left(\overset{i-1}{S}, y, \overset{n-i}{S} \right) \subseteq f \left(\overset{i-1}{S}, (A], \overset{n-i}{S} \right) \subseteq \left(f \left(\overset{i-1}{S}, A, \overset{n-i}{S} \right) \right] \subseteq \mathcal{I}$.
- Case 2: $y \in \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \right]$. Using Remark 1 and Lemma 2, we have

$$\begin{aligned} f \left(\overset{i-i}{S}, \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \right], \overset{n-i}{S} \right) &\subseteq \left(f \left(\overset{i-i}{S}, \bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right), \overset{n-i}{S} \right) \right] \\ &\subseteq \left(\bigcup_{m=2}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \cup f_n \left(\overset{n(i-1)}{S}, A, \overset{n(n-i)}{S} \right) \right] \\ &\subseteq \left(\bigcup_{m=2}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \cup f \left(\overset{(i-1)}{S}, A, \overset{(n-i)}{S} \right) \right] \\ &= \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \right] \\ &\subseteq \mathcal{I}. \end{aligned}$$

So, $\mathcal{I}$ is an i -ideal of S . Next, we show that $\mathcal{I}$ is the smallest i -ideal of S containing A .

Let I be any i -ideal of S containing A . We consider $\mathcal{I} = \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, A, \overset{m(n-i)}{S} \right) \cup A \right) \subseteq \left(\bigcup_{m=1}^{n-1} f_m \left(\overset{m(i-1)}{S}, I, \overset{m(n-i)}{S} \right) \cup I \right) \subseteq \left(f \left(\overset{i-1}{S}, I, \overset{n-i}{S} \right) \cup I \right) \subseteq (I) = I$. Hence, $\mathcal{I}$ is the smallest i -ideal of S containing A . $\square$

Corollary 2. Let A be a nonempty subset of S . Then the following statements hold:

- $J_1(A) = (f(A, \overset{n-1}{S}) \cup A)$ and $J_n(A) = (f(\overset{n-1}{S}, A) \cup A)$.
- If $1 < i < n$ and $i = \frac{n+1}{2} \in \mathbb{N}$, then

$$J_i(A) = (f(\overset{i-1}{S}, A, \overset{n-i}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup A).$$

Proposition 1. Let A be a nonempty subset of S . Then

$$J_{\{1,n\}}(A) = (f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A).$$

Proof. Let A be a nonempty subset of S .

First, we show that $\mathcal{I} = (f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A)$ is a $\{1, n\}$ -ideal of S . We consider $f((f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A), \overset{n-1}{S}) \subseteq (f(f(A, \overset{n-1}{S}), \overset{n-1}{S}) \cup f(f_2(\overset{n-1}{S}, A, \overset{n-1}{S}), \overset{n-1}{S}) \cup f(f(\overset{n-1}{S}, A), \overset{n-1}{S}) \cup f(A, \overset{n-1}{S})) \subseteq ((f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A))$. So, $\mathcal{I}$ is a 1-ideal of S . Similarly, we can show that $\mathcal{I}$ is an n -ideal of

S. Hence, $\mathcal{I}$ is a $\{1, n\}$ -ideal of S . Next, we show that $\mathcal{I}$ is the smallest $\{1, n\}$ -ideal of S containing A . Let I be a $\{1, n\}$ -ideal of S containing A . We consider $(f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A) \subseteq (f(I, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, I, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, I) \cup I) \subseteq (I) = I$. Hence, $\mathcal{I}$ is the smallest $\{1, n\}$ -ideal of S containing A . $\square$

4. Prime, Semiprime, Weakly Prime and Weakly Semiprime of Ordered n -ary Semigroups

In this section, we define the notions of prime, weakly prime, semiprime and weakly semiprime of an ordered n -ary semigroup and study prime properties by using the sets that are 1-ideals and n -ideals of an ordered n -ary semigroup.

Definition 4. Let P be a nonempty subset of S . For any nonempty subset A_i of S for $i = 1, \dots, n$, P is called prime if $f(A_1^n) \subseteq P$ implies $A_i \subseteq P$ for some $i = 1, \dots, n$.

Remark 2. A nonempty subset P of S is prime if and only if for all $a_1, \dots, a_n \in S$, $f(a_1^n) \in P$ implies $a_i \in P$ for some $i = 1, \dots, n$.

Definition 5. Let P be a Λ -ideal of S . Then P is called weakly prime if for all Λ -ideals $A_1, \dots, A_n$ of S such that $f(A_1^n) \subseteq P$ implies $A_i \subseteq P$ for some $i = 1, \dots, n$.

Definition 6. Let P be a nonempty subset of S . Then P is called semiprime if for all nonempty subsets A of S , $f(\overset{n}{A}) \subseteq P$ implies $A \subseteq P$.

Remark 3. A nonempty subset P of S is semiprime if and only if for all $a \in S$, $f(\overset{n}{a}) \in P$ implies $a \in P$.

Definition 7. Let P be a Λ -ideal of S . Then P is called weakly semiprime if for all Λ -ideals A of S such that $f(\overset{n}{A}) \subseteq P$, we have that $A \subseteq P$.

Remark 4. Let S be an ordered n -ary semigroup.

- (i) If P is prime, then P is weakly prime.
- (ii) If P is prime, then P is semiprime.
- (iii) If P is weakly prime, then P is weakly semiprime.
- (iv) If P is semiprime, then P is weakly semiprime.

It is clear that every prime ideal of an ordered n -ary semigroup is weakly prime. The converse is not true as shown by the following example.

Example 1. Let $S = \{a, b, c, d, e\}$. Define a binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	d	a
c	a	e	c	c	e
d	a	b	d	d	b
e	a	e	a	c	a

Define a binary relation $\leq$ on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (a, d), (a, e), (b, d), (e, c)\}.$$

Then $(S, \cdot, \leq)$ is an ordered semigroup. All ideals of S are as follows: $\{a\}$ and S . We have that $\{a\}$ is a weakly prime ideal. However, it is not a prime ideal and not a semiprime ideal since $e \cdot e = a \in \{a\}$ but $e \notin \{a\}$. Moreover, $\{a\}$ is a weakly semiprime ideal.

Example 2. Let $\mathbb{N}$ be the set of all natural numbers. Define an n -ary operation $f : \mathbb{N}^n \rightarrow \mathbb{N}$ by

$$f(x_1^n) = x_1 \cdot x_2 \cdot \dots \cdot x_n$$

for all $x_1, x_2, \dots, x_n \in \mathbb{N}$ where $\cdot$ is the usual multiplication on $\mathbb{N}$. It is easy to see that $(\mathbb{N}, f, =)$ is an ordered n -ary semigroup with partially relation $=$. We have that $6\mathbb{N}$ is a semiprime ideal of $(\mathbb{N}, f, =)$. However, $f(2\mathbb{N}, 3\mathbb{N}) \subseteq 6\mathbb{N}$ but $2\mathbb{N} \not\subseteq 6\mathbb{N}$ and $3\mathbb{N} \not\subseteq 6\mathbb{N}$. This shows that $6\mathbb{N}$ is not a prime ideal and not a weakly prime ideal. Moreover, $6\mathbb{N}$ is a weakly semiprime ideal.

Now, we use prime, weakly prime, semiprime and weakly semiprime properties of $\{1, n\}$ -ideals to generalize results of [26] to an ordered n -ary semigroup.

Theorem 2. The following statements are equivalent.

- (i) Every $\{1, n\}$ -ideal of S is weakly semiprime.
- (ii) $(f(I))^n = I$ for every $\{1, n\}$ -ideal I of S .

Proof. Assume that every $\{1, n\}$ -ideal of S is weakly semiprime. Let I be a $\{1, n\}$ -ideal of S . By Lemma 3, $(f(I))^n$ is a $\{1, n\}$ -ideal of S . So, $(f(I))^n$ is weakly semiprime. Since $f(I)^n \subseteq (f(I))^n$, $I \subseteq (f(I))^n$. It is clear that $(f(I))^n \subseteq (I) = I$. Conversely, assume that $(f(I))^n = I$ for every $\{1, n\}$ -ideal I of S . Let P be a $\{1, n\}$ -ideal of S and I be $\{1, n\}$ -ideal of S such that $f(I)^n \subseteq P$. Then $I = (f(I))^n \subseteq (P) = P$. So, P is weakly semiprime. $\square$

Corollary 3. The following statements are equivalent.

- (i) Every $\{1, n\}$ -ideal of S is weakly semiprime.
- (ii) $(f(I))^n = I$ for every $\{1, n\}$ -ideal I of S .

Theorem 3. The following statements are equivalent.

- (i) Every $\{1, n\}$ -ideal of S is weakly prime.
- (ii) The set of all $\{1, n\}$ -ideals of S forms a chain and $(f(I))^n = I$ for every $\{1, n\}$ -ideal of S .

Proof. Assume that every $\{1, n\}$ -ideal of S is weakly prime. Let A and B be $\{1, n\}$ -ideals of S . By Lemma 3, $(f(B, A))^n$ is a $\{1, n\}$ -ideal of S . So, $B \subseteq (f(B, A))^n$ or $A \subseteq (f(B, A))^n$. If $B \subseteq (f(B, A))^n$, then $B \subseteq (f(B, A))^n \subseteq (A) = A$. If $A \subseteq (f(B, A))^n$, then $A \subseteq (f(B, A))^n \subseteq (B) = B$. Since every weakly prime is weakly semiprime and by Theorem 2, $(f(I))^n = I$ for every $\{1, n\}$ -ideal of S .

Conversely, assume that the set of all $\{1, n\}$ -ideals of S forms a chain and $(f(I))^n = I$ for every $\{1, n\}$ -ideal of S . Let P be a $\{1, n\}$ -ideal of S and $A_1, \dots, A_n$ be $\{1, n\}$ -ideals of S

such that $f(A_1^n) \subseteq P$. By assumption, there exists A_j such that $A_j \subseteq A_i$ for all $i = 1, \dots, n$. Using Corollary 1, we have $A_j = \bigcap_{i=1}^n A_i \subseteq (f(A_1^n)) \subseteq P$. So, P is weakly prime. $\square$

Remark 5. For any $a \in S$, we observe that $(f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$ is a $\{1, n\}$ -ideal of S .

Proof. Let $a \in S$. Then $f(\overset{n-1}{S}, (f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))) \subseteq (f(\overset{n-1}{S}, f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S})))$
 $= (f(f(\overset{n-1}{S}, S), \overset{n-2}{S}, f(f(\overset{n}{a}), \overset{n-1}{S}))) \subseteq (f(S, \overset{n-2}{S}, f(f(\overset{n}{a}), \overset{n-1}{S}))) = (f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$
 and $((f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))) = (f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$. So, $(f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$ is an n -ideal of S . Similarly, we can show that $(f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$ is a 1-ideal of S . $\square$

Definition 8. An element $a \in S$ is called intra-regular, if there exist $x_1, \dots, x_{2n-2} \in S$ such that $a \leq f_2(x_1^{n-1}, f(\overset{n}{a}), x_{2n-2}^{2n-2})$. An ordered n -ary semigroup S is called intra-regular, if each one of its elements is intra-regular.

The following lemma is directly obtained by Definition 8.

Lemma 5. The following statements are equivalent.

- (i) S is intra-regular.
- (ii) $A \subseteq (f_2(\overset{n-1}{S}, f(\overset{n}{A}), \overset{n-1}{S}))$ for any $\emptyset \neq A \subseteq S$.
- (iii) $a \in (f_2(\overset{n-1}{S}, f(\overset{n}{a}), \overset{n-1}{S}))$ for any $a \in S$.

Now, we give an example of an intra-regular ordered n -ary semigroup as the following example.

Example 3. Let $0 \in S$ and $|S| > n$. Define an n -ary operation f on S by

$$f(x_1^n) = \begin{cases} x_1 & \text{if } x_1 = x_2 = \dots = x_n, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x_1, x_2, \dots, x_n \in S$. It is easy to see that $(S, f, =)$ is an ordered n -ary semigroup with partially relation $=$. Let $a \in S$. Case 1: $a = 0$. Since $0 \in \{0\} = (f_2(\overset{n-1}{S}, f(\overset{n}{0}), \overset{n-1}{S}))$, a is intra-regular. Case 2: $a \neq 0$. We have that $a \in \{a\} = (f_2(\overset{n-1}{a}, f(\overset{n}{a}), \overset{n-1}{a}))$. So, S is intra-regular.

Moreover, it is not difficult to verify that all ideals of S are subsets of S containing 0 and they are semiprime. Indeed, let B be an ideal of S and $b \in S$ such that $f(\overset{n}{b}) \in B$. If $b = 0$, then it is clear that $0 \in B$. If $b \neq 0$, then $b = f(\overset{n}{b}) \in B$. However, $\{0\}$ is a semiprime ideal, which is not prime because for $a_1^n \in S \setminus \{0\}$ and $a_i \neq a_j$ for some $i \neq j$, $f(a_1^n) \in \{0\}$ but $a_i \notin \{0\}$ for all $1 \leq i \leq n$.

Proposition 2. Let S be an ordered n -ary semigroup. If S is intra-regular, then $J_{\{1,n\}}(A) = (f_2(\overset{n-1}{S}, A, \overset{n-1}{S}))$.

Proof. Assume that S is intra-regular. Let A be a nonempty subset of S . Then $A \subseteq (f_2(\overset{n-1}{S}, f(\overset{n}{A}), \overset{n-1}{S})) = (f_2(f(\overset{n-1}{S}, A), \overset{n-1}{A}, \overset{n-1}{S})) \subseteq (f_2(S, \overset{n-1}{A}, \overset{n-1}{S})) \subseteq (f_2(\overset{n-1}{S}, A, \overset{n-1}{S}))$. It is clear that $(f_2(\overset{n-1}{S}, A, \overset{n-1}{S}))$ is a $\{1, n\}$ -ideal of S . By Proposition 1, $J_{\{1,n\}}(A) \subseteq (f_2(\overset{n-1}{S}, A, \overset{n-1}{S})) \subseteq (f(A, \overset{n-1}{S}) \cup f_2(\overset{n-1}{S}, A, \overset{n-1}{S}) \cup f(\overset{n-1}{S}, A) \cup A) = J_{\{1,n\}}(A)$. So, $J_{\{1,n\}}(A) = (f_2(\overset{n-1}{S}, A, \overset{n-1}{S}))$. $\square$

Theorem 4. The following statements are equivalent.

- (i) Every $\{1, n\}$ -ideal of S is semiprime.
- (ii) S is intra-regular.

Proof. Assume that every $\{1, n\}$ -ideal of S is semiprime. Let $a \in S$. We have

$$\begin{aligned} f(\underbrace{f(a), \dots, f(a)}_{n \text{ terms}}) &= f(f_2(\underbrace{a, \dots, a}_{n-1 \text{ terms}}, \underbrace{f(a), \dots, f(a)}_{n-2 \text{ terms}})) \\ &= f_2(\underbrace{a, \dots, a}_{n-1 \text{ terms}}, \underbrace{f(a), \dots, f(a)}_{n-2 \text{ terms}}) \\ &\in f_2(\underbrace{S}_{n-1}, \underbrace{f(a), S}_{n-2}) \\ &\subseteq (f_2(\underbrace{S}_{n-1}, \underbrace{f(a), S}_{n-2})). \end{aligned}$$

By Remark 5 and assumption, $f(a) \in (f_2(\underbrace{S}_{n-1}, \underbrace{f(a), S}_{n-2}))$ and so, $a \in (f_2(\underbrace{S}_{n-1}, \underbrace{f(a), S}_{n-2}))$. By Lemma 5, S is intra-regular. Conversely, assume that S is intra-regular. Let P be a $\{1, n\}$ -ideal of S and $a \in S$ such that $f(a) \in P$. By assumption, we have $a \in (f_2(\underbrace{S}_{n-1}, \underbrace{f(a), S}_{n-2})) \subseteq (f_2(\underbrace{S}_{n-1}, \underbrace{P, S}_{n-2})) = (f(\underbrace{S}_{n-1}, \underbrace{P, S}_{n-2})) \subseteq (f(\underbrace{S}_{n-1}, P)) \subseteq (P) = P$. So, P is semiprime. $\square$

Theorem 5. The following statements are equivalent.

- (i) Every i -ideal of S is semiprime.
- (ii) $a \in (\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}))$ for all $a \in S$.

Proof. Assume that every i -ideal of S is semiprime. Let $a \in S$. We have

$$\begin{aligned} f(\underbrace{f(a), \dots, f(a)}_{n \text{ terms}}) &\in f(\underbrace{S}_{i-1}, \underbrace{f(a), S}_{n-i}) \\ &\subseteq \bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}) \\ &\subseteq (\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)})). \end{aligned}$$

Since $(\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}))$ is an i -ideal of S and by assumption,

$f(a) \in (\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}))$ and so $a \in (\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}))$. Conversely,

assume that $a \in (\bigcup_{m=1}^{n-1} f_m(\underbrace{S}_{m(i-1)}, \underbrace{f(a), S}_{m(n-i)}))$ for all $a \in S$.

Let I be an i -ideal of S and $a \in S$ such that $f(a) \in I$. By assumption,

$$\begin{aligned} a &\in (f_k(\underbrace{S}_{k(i-1)}, \underbrace{f(a), S}_{k(n-i)})) \text{ for some } k = 1, \dots, n-1 \\ &\subseteq (f_k(\underbrace{S}_{k(i-1)}, \underbrace{I, S}_{k(n-i)})) \\ &\subseteq (I) \\ &= I. \end{aligned}$$

Hence, I is semiprime. $\square$

Corollary 4. *The following statements are equivalent.*

- (i) Every 1-ideal of S is semiprime if and only if $a \in (f(f(a)^{n-1}), S]$ for all $a \in S$.
- (ii) Every n -ideal of S is semiprime if and only if $a \in (f(S)^{n-1}, f(a)]$ for all $a \in S$.

5. Conclusions and Discussion

In this paper, we study i -ideals of ordered n -ary semigroups. Moreover, we study prime properties of $\{1, n\}$ -ideals of ordered n -ary semigroups, which are generalizations of n -ary semigroups. So, our results hold in n -ary semigroups as well. We introduce the notions of Λ -ideals of ordered n -ary semigroups as a generalization of the notions of i -ideals and investigate prime properties of a Λ -ideal of ordered n -ary semigroups whenever $\Lambda = \{1, n\}$. In ordered semigroups S , ordered n -ary semigroups with $n = 2$, Kehayopulu showed that every ideal of S is prime if and only if S is intra-regular and the set of all ideals of S forms a chain. However, this result still opens for Λ -ideals of (ordered) n -ary semigroups where $n \geq 3$. Moreover, algebraic properties of Λ -ideals in (ordered) n -ary semigroups can be further investigated.

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