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The Shape of the (15,3)-Arc of PG(2,7)

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Abstract: Marcugini et al. proved, by computer-based proof, the unicity of the maximum $(k,3)$-arc in PG(2,7). In this paper, we show how the (15,3)-Arc in PG(2,7) may be described using only geometrical properties. The description we provide, believing it is novel, relies on the union of a conic and a complete external quadrangle.

Keywords: $(k,3)$-arc; NMDS code; complete quadrangle

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1. Introduction and Motivation

A $(k,n)-$arc K in a projective plane PG$(2,q)$ is a set of k points such that some n, but no $n + 1$ of them, are collinear. By writing the homogeneous coordinates of the k points of K as columns of a generator matrix, one obtains a projective linear $[k,3,k-3]_q$ code. Thus, $(k,3)$-arcs in PG$(2,q)$ correspond to NMDS codes of dimension 3. Since its error-correcting capability increases with k, it is natural to determine the largest value of k for which a $(k,3)$-arc exists. Define $m_3(2,q)$ to be the maximum size of a $(k,3)$-arc in PG$(2,q)$. In 1975, J. Thas [1] proved that if $q > 3$, then $m_3(2,q) \leq 2q + 1$. The projective plane of order seven is the dominant focus of this work. Our reason for deciding to conduct a detailed investigation of this special case is that PG(2,7) is the smallest projective plane of prime power such that the maximum $(k,3)$-arc, i.e., the (15,3)-Arc, is unique, up to projectivity, cf. [2,3], and the awareness that the unique (15,3)-Arc reveals interesting geometric descriptions, cf. [4–8]. The object of this paper is to show that the (15,3)-Arc of PG(2,7) may be described by means of geometrical properties only. An easy geometric description of the (15,3)-Arc in PG(2,7) is provided by considering the union of the vertices of a complete quadrangle with the conic for which the nine lines of the complete quadrangle are external lines.

2. The Description of the (15,3)-Arc in PG(2,7)

Let K be a (15,3)-Arc in PG(2,7). For each integer i such that $0 \leq I \leq 3$, let us denote by $t_i = t_i(K)$ the number of lines of PG(2,7) meeting K in exactly i points. The numbers t_i are called the *characters* of K with respect to the lines, see [9]. By double counting the number of lines, the number of pairs (P,r), where $P \in K$ and r is a line through P, and the number of pairs $((P, Q),r)$, where $\{P, Q\} \subset K$ and r is the line through P and Q, we get the following equations:

$$\begin{cases} t_0 + t_1 + t_2 + t_3 = 57 \\ t_1 + 2t_2 + 3t_3 = 120 \\ 2t_2 + 6t_3 = 210 \end{cases}$$

Solving these equations, we obtain the characters of K with respect to the lines:

Therefore, we have six character vectors:

$$\begin{cases} t_0 = 42 - t_3 \\ t_1 = 3t_3 - 90 \\ t_2 = 105 - 3t_3 \end{cases},$$
$$7 \le t_0 \le 12,\ 0 \le t_1 \le 15,\ 0 \le t_2 \le 15,\ 30 \le t_3 \le 35.$$

$$(t_0,t_1,t_2,t_3) \in \{(7,15,0,35),(8,12,3,34),(9,9,6,33),(10,6,9,32),(11,3,12,31),(12,0,15,30)\}.$$

Let P be a point of K. For each integer i such that $0 \le i \le 3$, let us denote by $v_i = v_i(P)$ the number of lines through P meeting K in exactly i points. The numbers v_i are called the *characters* of P with respect to the lines, cf. [9]. By double counting the number of lines through P, the number of pairs (Q,r), where $Q \in K - P$ and r is a line through P and Q, we get the following equations:

$$\begin{cases} v_1 + v_2 + v_3 = 8 \\ v_2 + 2v_3 = 14 \end{cases}$$

Solving these equations we obtain the characters of P with respect to the lines:

$$\begin{cases} v_1 = v_3 - 6 \\ v_2 = 14 - 2v_3 \end{cases}$$

Since $v_1 \ge 0$ and $v_2 \ge 0$ we get $v_3 \in \{6,7\}$. Therefore, we have two types of inner points:

$$(v_1,v_2,v_3) \in \{(0,2,6),(1,0,7)\}.$$

Let Q be a point nonbelonging to K. For each integer i such that $0 \le i \le 3$, let us denote by $u_i = u_i(Q)$ the number of lines through Q meeting K in exactly i points. The numbers u_i are called the *characters* of Q with respect to the lines, cf. [9]. By double counting the number of lines through Q, we get

$$\begin{cases} u_0 + u_1 + u_2 + u_3 = 8 \\ u_1 + 2u_2 + 3u_3 = 15 \end{cases}$$

Solving these equations we obtain the characters of Q with respect to the lines:

$$\begin{cases} u_1 = 1 + u_3 - 2u_0 \\ u_2 = 7 - 2u_3 + u_0 \end{cases}$$

Therefore, we have eleven types of outer points:

$$(u_0,u_1,u_2,u_3) \in \{(0,1,7,0),(0,2,5,1),(0,3,3,2),(0,4,1,3),(1,0,6,1), (1,1,4,2),(1,2,2,3),(1,3,0,4),(2,0,3,3),(2,1,1,4),(3,0,0,5)\}.$$

In order to prove the unicity of the (15,3)-Arc in PG(2,7), we firstly prove that the character vector of a (15,3)-Arc in PG(2,7) is (12,0,15,30). We divide the proof into five steps.

Step 1. The nonexistence of a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (7,15,0,35).

Proof. Let K be a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (7,15,0,35). Since $t_2 = 0$, we have that $u_2 = v_2 = 0$, so the inner points are of type (1,0,7). The outer points are of types $(u_0,u_1,u_2,u_3) \in \{(1,3,0,4),(3,0,0,5)\}$. Let l denote a $0-$line. Since the other six $0-$lines meet l in outer points of type (3,0,0,5), we get that there are exactly 35 outer points of type (1,3,0,4) and exactly 7 points of type (3,0,0,5). The seven outer points of type (3,0,0,5) with the seven $0-$lines has the structure of a Steiner triple system S(2,3,7), i.e., a Fano subplane of order two, a contradiction because PG(2,7) contains no Fano subplanes, cf. Lemma 7.2, page 154, of [9]. □

Step 2. The nonexistence of a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (8,12,3,34).

Proof. Let K be a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (8,12,3,34). Since t_2 = 3, we get that any two 2−lines meet in one inner point of type (0,2,6). Since t_2 = 3, we have that $0 \leq u_2 \leq 1$, the outer points are of types (u_0,u_1,u_2,u_3) ∈ {(0,4,1,3),(1,3,0,4),(2,1,1,4),(3,0,0,5)}. Let x, y and z denote the number of outer points of type (0,4,1,3), (1,3,0,4) and (2,1,1,4), respectively, of a 1−line l_1. By double counting the number of pairs (Q,r), where $Q \in l_1$ and r is a i−line, I = 0,2,3, through Q, we get:

$$\left\{\begin{array}{c} y+2z=8 \\ x+z=3 \\ 3x+4y+4z=27 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x=1 \\ y=4 \\ z=2 \end{array}\right.$$

So, there are exactly 3 outer points of type (0,4,1,3). □

Moreover, let x and y denote the number of outer points of type (0,4,1,3) and (2,1,1,4), respectively, of a 2−line l_2. By double counting the number of pairs (Q,r), where $Q \in l_2$ and r is a i−line, I = 0,1,3, through Q, we get:

$$\left\{\begin{array}{c} 2y=8 \\ 4x+y=12 \\ 3x+4y=22 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x=2 \\ y=4 \end{array}\right.$$

Since t_2 = 3 and any two 2−lines meet in one inner point of type (0,2,6), we get that there are exactly 6 outer points of type (0,4,1,3), a contradiction.

Step 3. The nonexistence of a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (9,9,6,33).

Proof. Let K be a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (9,9,6,33). Since a 2−line contains inner points of type (0,2,6) and t_2 = 6, the six 2−lines with the six inner points of type (0,2,6) form either two disjoint triangles or one hexagon. If they form two triangles, P_1, P_3, P_5 and P_2, P_4, P_6 the lines $P_{2h-1}P_{2k}$, with h = 1,2,3, and k = 1,2,3 are nine 3−lines no three of which form a triangle with vertices P_i, I = 1,2, … , 6. If they form one hexagon with consecutive vertices, P_1, P_2, P_3, P_4, P_5 and P_6 the lines $P_{2h-1}P_{2k-1}$, and $P_{2h}P_{2k}$ with h = 1,2,3, k = 1,2,3 and $h \neq k$ are 3−lines no three of which form a triangle with vertices P_i, I = 1,2, … , 6. Moreover, the outer points are of types

(u_0,u_1,u_2,u_3)∈{(0,3,3,2),(0,4,1,3),(1,2,2,3),(1,3,0,4),(2,0,3,3),(2,1,1,4),(3,0,0,5)}.

□

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (1,2,2,3), (1,3,0,4), (2,0,3,3), (2,1,1,4), (3,0,0,5) respectively, of a 0−line l_0. By double counting the number of pairs (Q,r), where $Q \in l_0 - K$ and $r \neq l_0$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\left\{\begin{array}{c} x_{20}+x_{21}+2x_{30}=8 \\ 2x_{12}+3x_{13}+x_{21}=9 \\ 2x_{12}+3x_{20}+x_{21}=6 \\ 3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=33 \end{array}\right. \Rightarrow \left\{\begin{array}{c} x_{20}=x_{13}-1 \\ x_{21}=9-2x_{12}-3x_{13} \\ x_{30}=x_{12}+x_{13} \end{array}\right.$$

We get $0 \leq x_{12} \leq 3$ and $1 \leq x_{13} \leq \frac{9-2x_{12}}{3}$.
Thus, $(x_{12}, x_{13}) \in \{(0,1),(0,2),(0,3),(1,1),(1,2),(2,1),(3,1)\}$.
If $(x_{12}, x_{13}) = (0,1)$ we get $\left\{\begin{array}{l} x_{20}=0 \\ x_{21}=6 \\ x_{30}=1 \end{array}\right. \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (0,1,0,6,1)$.

If $(x_{12}, x_{13}) = (0,2)$ we get $\begin{cases} x_{20} = 1 \\ x_{21} = 3 \\ x_{30} = 2 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (0,2,1,3,2).$

If $(x_{12}, x_{13}) = (0,3)$ we get $\begin{cases} x_{20} = 2 \\ x_{21} = 0 \\ x_{30} = 3 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (0,3,2,0,3).$

If $(x_{12}, x_{13}) = (1,1)$ we get $\begin{cases} x_{20} = 0 \\ x_{21} = 4 \\ x_{30} = 2 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (1,1,0,4,2).$

If $(x_{12}, x_{13}) = (1,2)$ we get $\begin{cases} x_{20} = 1 \\ x_{21} = 1 \\ x_{30} = 3 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (1,2,1,1,3).$

If $(x_{12}, x_{13}) = (2,1)$ we get $\begin{cases} x_{20} = 0 \\ x_{21} = 2 \\ x_{30} = 3 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (2,1,0,2,3).$

If $(x_{12}, x_{13}) = (3,1)$ we get $\begin{cases} x_{20} = 0 \\ x_{21} = 0 \\ x_{30} = 4 \end{cases} \Rightarrow (x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (3,1,0,0,4).$

Therefore, the 0−lines can be of type

$$(x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,1,0,6,1), (0,2,1,3,2), (0,3,2,0,3), (1,1,0,4,2), (1,2,1,1,3), \\ (2,1,0,2,3), (3,1,0,0,4)\}$$

Let x_{ij}, $I = u_0$, $j = u_1$, denote the number of outer points of type (0,3,3,2), (0,4,1,3), (1,2,2,3), (1,3,0,4), (2,1,1,4), respectively, of a 1−line l_1. By double counting the number of pairs (Q,r), where $Q \in l_1 - K$ and $r \neq l_1$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{12} + x_{13} + 2x_{21} = 933 \\ 2x_{03} + 3x_{04} + x_{12} + 2x_{13} = 8 \\ 3x_{03} + x_{04} + 2x_{12} + x_{21} = 6 \\ 2x_{03} + 3x_{04} + 3x_{12} + 4x_{13} + 4x_{21} = 26 \end{cases} \Rightarrow \begin{cases} x_{12} = 2 - 2x_{03} - x_{04} \\ x_{13} = 3 - x_{04} \\ x_{21} = 2 + x_{03} + x_{04} \end{cases}$$

By the first equation, we get $0 \leq 2x_{03} + x_4 \leq 2$. Thus, $(x_{03}, x_{04}) \in \{(0,0), (0,1), (0,2), (1,0)\}$.

If $(x_{03}, x_{04}) = (0,0)$ we get $\begin{cases} x_{12} = 2 \\ x_{13} = 3 \\ x_{21} = 2 \end{cases} \Rightarrow (x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) = (0,0,2,3,2).$

If $(x_{03}, x_{04}) = (0,1)$ we get $\begin{cases} x_{12} = 1 \\ x_{13} = 2 \\ x_{21} = 3 \end{cases} \Rightarrow (x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) = (0,1,1,2,3).$

If $(x_{03}, x_{04}) = (0,2)$ we get $\begin{cases} x_{12} = 0 \\ x_{13} = 1 \\ x_{21} = 4 \end{cases} \Rightarrow (x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) = (0,2,0,1,4).$

If $(x_{03}, x_{04}) = (1,0)$ we get $\begin{cases} x_{12} = 0 \\ x_{13} = 3 \\ x_{21} = 3 \end{cases} \Rightarrow (x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) = (1,0,0,3,3).$

Therefore, the 1−lines can be of type

$$(x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) \in \{(0,0,2,3,2), (0,1,1,2,3), (0,2,0,1,4), (1,0,0,3,3)\}$$

Now, let x_{ij}, $I = u_0$, $j = u_1$, denote the number of outer points of type (0,3,3,2), (0,4,1,3), (1,2,2,3), (2,0,3,3), (2,1,1,4), respectively, of a 2−line l_2. By double counting the number of pairs (Q,r), where $Q \in l_2 - K$ and $r \neq l_2$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{12} + 2x_{20} + 2x_{21} = 9 \\ 3x_{03} + 4x_{04} + 2x_{12} + x_{21} = 9 \\ 2x_{03} + x_{12} + 2x_{20} = 3 \\ 2x_{03} + 3x_{04} + 3x_{12} + 3x_{20} + 4x_{21} = 21 \end{cases} \Rightarrow \begin{cases} x_{03} = x_{21} - 3 \\ x_{04} = x_{20} \\ x_{12} = 9 - 2x_{20} - 2x_{21} \end{cases}$$

Therefore, the $2-$lines can be of type $(x_{03}, x_{04}, x_{12}, x_{20}, x_{21}) \in \{(0,0,3,0,3), (0,1,1,1,3), (1,0,1,0,4)\}$.

We note that $x_{12} \neq 0$.

Two $1-$lines meet in an outer point.

We have two possibility: an outer point of type (0,3,3,2) exists or not.

If an outer point Q of type (0,3,3,2) exists, then the two $3-$lines through Q contain six inner points of type (1,0,7). An outer point R of type (0,4,1,3) does not exist, otherwise the line QR would be a $3-$line with an inner point of type (0,2,6), a contradiction. Let l_i and m_i, i = 1,2,3, denote the $1-$lines of type $(1,0,0,3,3)$ and the $2-$lines of type $(1,0,1,0,4)$ through Q, respectively. Let us denote by M_1, M_2 and M_3 the points of type (1,2,2,3) on m_1, m_2 and m_3, respectively. Let us denote by n_1, n_2 and n_3 the other three $2-$lines on M_1, M_2 and M_3, respectively. The $2-$lines n_1, n_2 and n_3 are of type $(1,0,1,0,4)$, because they meet l_1, l_2 and l_3 in nine points of type (2,1,1,4). Thus, there is another outer point R of type (0,3,3,2). Let r_h denote the $1-$lines through R, h = 1,2,3. The line QR is a $3-$line with three inner points P_1, P_2 and P_3 of type (1,0,7). Let p_k denote the $1-$line through P_k, k = 1,2,3. Let us denote by L_{ij} the points of type (1,3,0,4) on the $1-$lines l_i, j = 1,2,3. The eleven points Q, R, L_{ij} with the nine $1-$lines are contained in a dual affine plane structure DAG(2,3) embedded in PG(2,7). It follows that the three $1-$lines p_k meet in a unique point, a contradiction because they pairwise meet in the points M_1, M_2 and M_3.

Therefore, an outer point of type (0,3,3,2) does not exist.

Thus, the $1-$lines can be of type $(x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) \in \{(0,0,2,3,2), (0,1,1,2,3), (0,2,0,1,4)\}$, and the $2-$lines can be of type $(x_{03}, x_{04}, x_{12}, x_{20}, x_{21}) \in \{(0,0,3,0,3), (0,1,1,1,3)\}$. Since every $2-$line contains exactly three points of type (2,1,1,4), we have exactly 18 points of type (2,1,1,4). Let x, y and z be the number of $1-$lines of type $(0,0,2,3,2)$, $(0,1,1,2,3)$ and $(0,2,0,1,4)$, respectively. By double counting the number of $1-$lines and the number of the pairs (Q,l) where Q is a point of type (2,1,1,4), l is a $1-$line and $Q \in l$, we get $\begin{cases} x + y + z = 9 \\ 2x + 3y + 4z = 18 \end{cases}$. Multiply the first equation by -2 and add: $y + 2z = 0$. It follows that $y = z = 0$. Thus, the $1-$lines are of type $(x_{03}, x_{04}, x_{12}, x_{13}, x_{21}) = (0,0,2,3,2)$, an outer point of type (0,4,1,3) does not exist, and the $2-$lines are of type $(x_{03}, x_{04}, x_{12}, x_{20}, x_{21}) = (0,0,3,0,3)$. Thus, an outer point of type (2,0,3,3) does not exist.

Therefore, the $0-$lines can be of type

$$(x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,1,0,6,1), (1,1,0,4,2), (2,1,0,2,3), (3,1,0,0,4)\}$$

Since every $0-$line contains exactly one point of type (1,3,0,4), we have exactly 9 points of type (1,3,0,4). Since every $1-$line contains exactly two points of type (1,2,2,3) and through any point of type (1,2,2,3) there pass two $1-$lines, we have exactly 9 points of type (1,2,2,3). Thus, the number of points of type (3,0,0,5) is 6. By the 9 points of type (1,3,0,4) with the 9 $1-$lines we find an affine plane of order three AG(2,3) in which the incidences between the 9 points and the lines of one parallel class has been removed. It follows that the missing parallelism class is formed by three $3-$lines of type $(x_{03}, x_{04}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) = (0,0,0,3,0,0,2)$ which form a triangle with vertices points of type (0,2,6), a contradiction.

Step 4. The nonexistence of a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (10,6,9,32).

Proof. Let K be a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (10,6,9,32). We have eleven types of outer points:

$$(u_0,u_1,u_2,u_3)\in\{(0,1,7,0),(0,2,5,1),(0,3,3,2),(0,4,1,3),(1,0,6,1),\\ (1,1,4,2),(1,2,2,3),(1,3,0,4),(2,0,3,3),(2,1,1,4),(3,0,0,5)\}.$$

Since t_2 = 9 and through any inner point of type (0,2,6) there are two 2−lines, we get that there are no outer points of type $(u_0,u_1,u_2,u_3) \in \{(0,1,7,0),(0,2,5,1),(1,0,6,1)\}$. □

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (1,1,4,2), (1,2,2,3), (1,3,0,4), (2,0,3,3), (2,1,1,4), (3,0,0,5), respectively, of a 0−line l_0. By double counting the number of pairs (Q,r), where $Q \in l_0$ and $r \neq l_0$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{20}+x_{21}+2x_{30}=9 \\ x_{11}+2x_{12}+3x_{13}+x_{21}=6 \\ 4x_{11}+2x_{12}+3x_{20}+x_{21}=9 \\ 2x_{11}+3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=32 \end{cases} \Rightarrow \begin{cases} x_{20}=1-x_{11}+x_{13} \\ x_{21}=6-x_{11}-2x_{12}-3x_{13} \\ x_{30}=1+x_{11}+x_{12}+x_{13} \end{cases}$$

By the third equation, we have that $0 \le x_{11} \le 2$.

If $x_{11} = 0$ we get $\begin{cases} x_{20}+x_{21}+2x_{30}=9 \\ 2x_{12}+3x_{13}+x_{21}=6 \\ 2x_{12}+3x_{20}+x_{21}=9 \\ 3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=32 \end{cases} \Rightarrow$
$\begin{cases} x_{20}=1+x_{13} \\ x_{21}=6-2x_{12}-3x_{13} \\ x_{30}=1+x_{12}+x_{13} \end{cases}$

Thus, $0 \le x_{13} \le 2$, $0 \le x_{12} \le \frac{6-3x_{13}}{2}$. Therefore,

$$(x_{11},x_{12},x_{13},x_{20},x_{21},x_{30}) \in \\ \{(0,0,0,1,6,1),(0,0,2,3,0,3),(0,1,0,1,4,2),(0,2,0,1,2,3),(0,3,0,1,0,4),\ (0,0,1,2,3,2),(0,1,1,2,1,3)\}$$

If $x_{11} = 1$ we get $\begin{cases} x_{20}+x_{21}+2x_{30}=9 \\ 2x_{12}+3x_{13}+x_{21}=5 \\ 2x_{12}+3x_{20}+x_{21}=5 \\ 3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=30 \end{cases} \Rightarrow$
$\begin{cases} x_{20}=x_{13} \\ x_{21}=5-2x_{12}-3x_{13} \\ x_{30}=2+x_{12}+x_{13} \end{cases}$

Thus, $0 \le x_{13} \le 1$, $0 \le x_{12} \le \frac{5-3x_{13}}{2}$. Therefore, $(x_{11},x_{12},x_{13},x_{20},x_{21},x_{30}) \in \{(1,0,0,0,5,2),(1,1,0,0,3,3),(1,2,0,0,1,4),\ (1,0,1,1,2,3),(1,1,1,1,0,4)\}$.

If $x_{11} = 2$ we get $\begin{cases} x_{20}+x_{21}+2x_{30}=9 \\ 2x_{12}+3x_{13}+x_{21}=4 \\ 2x_{12}+3x_{20}+x_{21}=1 \\ 3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=28 \end{cases} \Rightarrow$
$\begin{cases} x_{20}=x_{13}-1 \\ x_{21}=4-2x_{12}-3x_{13} \\ x_{30}=3+x_{12}+x_{13} \end{cases}$,

Thus, $x_{13}=1$ and $x_{12}=0$. Therefore, $(x_{11},x_{12},x_{13},x_{20},x_{21},x_{30})=(2,0,1,0,1,4)$.

Therefore, the 0−lines can be of type

$$(x_{11},x_{12},x_{13},x_{20},x_{21},x_{30}) \in \\ \{(0,0,0,1,6,1),\ (0,0,1,2,3,2),(0,0,2,3,0,3),(0,1,0,1,4,2),(0,1,1,2,1,3),(0,2,0,1,2,3),(0,3,0,1,0,4), \\ (1,0,0,0,5,2),(1,0,1,1,2,3),(1,1,0,0,3,3),(1,1,1,1,0,4),(1,2,0,0,1,4),\ (2,0,1,0,1,4)\}$$

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (0,3,3,2), (0,4,1,3), (1,1,4,2), (1,2,2,3), (1,3,0,4), (2,1,1,4), respectively, of a $1-$line l_1. By double counting the number of pairs (Q,r), where $Q \in l_1$ and $r \neq l_1$ is a $i-$line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{11}+x_{12}+x_{13}+2x_{21}=10 \\ 2x_{03}+3x_{04}+x_{12}+2x_{13}=5 \\ 3x_{03}+x_{04}+4x_{11}+2x_{12}+x_{21}=9 \\ 2x_{03}+3x_{04}+2x_{11}+3x_{12}+4x_{13}+4x_{21}=25 \end{cases} \Rightarrow$$

The system of linear equation is equivalent to

$$\begin{cases} 3x_{03}+x_{04}+4x_{11}+2x_{12}+x_{21}=9 \\ 2x_{03}+3x_{04}+2x_{11}+3x_{12}+4x_{13}+4x_{21}=25 \\ x_{11}+x_{12}+x_{13}+2x_{21}=10 \end{cases} \Rightarrow \begin{cases} x_{12}=3-2x_{03}-x_{04}-2x_{11} \\ x_{13}=1-x_{04}+x_{11} \\ x_{21}=3+x_{03}+x_{04} \end{cases}$$

Thus, $0 \leq x_{03} \leq 1$, $0 \leq x_{11} \leq \frac{3-2x_{03}}{2}$, $0 \leq x_{04} \leq \min\{1+x_{11}, 3-2x_{03}-2x_{11}\}$. Therefore, $(x_{03}, x_{04}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(0,0,0,3,1,3), (0,1,0,2,0,4), (0,0,1,1,2,3), (0,1,1,0,1,4), (1,0,0,1,1,4), (1,1,0,0,0,5)\}$ Therefore, the $1-$lines can be of type

$$\begin{aligned} (x_{03}, x_{04}, x_{11}, x_{12}, x_{13}, x_{21}) \in &\{(0,0,0,3,1,3), (0,1,0,2,0,4), (0,0,1,1,2,3) \\ &(0,1,1,0,1,4),\ (1,0,0,1,1,4), (1,1,0,0,0,5)\}. \end{aligned}$$

Two $1-$lines meet in an outer point.

We have two possibilities: an outer point of type (0,3,3,2) exists or not.

If an outer point Q of type (0,3,3,2) exists, then through Q there pass three $1-$lines of type either $(1,0,0,1,1,4)$ or $(1,1,0,0,0,5)$. A $1-$line of type (1,1,0,0,0,5) does not exist because the other three $1-$lines not through Q pass through the outer point R of type (0,4,1,3) and meet another $1-$line in at most two outer points, a contradiction. Therefore, an outer point of type (0,4,1,3) does not exist and through Q there pass three $1-$lines, l_1, l_2 and l_3 of type $(1,0,0,1,1,4)$. The other three $1-$lines m_1, m_2 and m_3 meet l_1, l_2 and l_3 in points of type either (1,3,0,4) or (1,2,2,3). Thus, m_1, m_2 and m_3 must contain two outer points of type (1,3,0,4) and one outer point of type (1,2,2,3), and they are of type $(0,0,1,1,2,3)$. Now, we count the outer point types by the $1-$line types. The pointset $l_1 \cup l_2 \cup l_3$ consists of one point of type (0,3,3,2), the point Q, 12 points of type (2,1,1,4), 3 points of type (1,3,0,4) and 3 of type (1,2,2,3). The pointset $(m_1 \cup m_2 \cup m_3)-(l_1 \cup l_2 \cup l_3)$ consists of 9 points of type (2,1,1,4) and 3 points of type (1,1,4,2). The types of the 11 outer points contained in no $1-$lines remain to be determined. Since an outer point contained in no $1-$lines is of type $(u_0,u_1,u_2,u_3) \in \{(2,0,3,3),(3,0,0,5)\}$, we have through it at least a pair of $0-$lines. Each of the $\begin{pmatrix} 10 \\ 2 \end{pmatrix} = 45$ $0-$lines pairs meet on points of type $(u_0,u_1,u_2,u_3) \in$ {(2,0,3,3), (2,1,1,4), (3,0,0,5)}. Exactly 21 $0-$lines pairs meet on points of type (2,1,1,4). So, 24 $0-$lines pairs meet on points of type $(u_0,u_1,u_2,u_3) \in$ {(2,0,3,3), (3,0,0,5)}. Let x and y denote the number of outer points of type (2,0,3,3) and (3,0,0,5), respectively. We get $\begin{cases} x+y=11 \\ x+3y=24 \end{cases} \Rightarrow \begin{cases} x=\frac{9}{2} \\ y=\frac{13}{2} \end{cases}$, a contradiction.

Therefore, an outer point of type (0,3,3,2) does not exist.

Therefore, the $1-$lines can be of type

$$(x_{03}, x_{04}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(0,0,0,3,1,3), (0,1,0,2,0,4), (0,0,1,1,2,3), (0,1,1,0,1,4)\}$$

Now, let x_{ij}, $I = u_0$, $j = u_1$, denote the number of outer points of type (0,4,1,3), (1,1,4,2), (1,2,2,3), (2,0,3,3), (2,1,1,4), respectively, of a $2-$line l_2. By double counting the number of pairs (Q,r), where $Q \in l_2 - K$ and $r \neq l_2$ is a $i-$line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{11} + x_{12} + 2x_{20} + 2x_{21} = 10 \\ 4x_{04} + x_{11} + 2x_{12} + x_{21} = 6 \\ 3x_{11} + x_{12} + 2x_{20} = 6 \\ 3x_{04} + 2x_{11} + 3x_{12} + 3x_{20} + 4x_{21} = 20 \end{cases}$$

Taking into account the second equation, since $0 \leq 4x_{0.4} \leq 6 \Rightarrow x_{04} \in \{0,1\}$.
The system of linear equation is equivalent to

$$\begin{cases} x_{11} = x_{21} - 2 \\ x_{12} = 4 - x_{21} - 2x_{04} \\ x_{20} = 4 - x_{21} + x_{04} \end{cases}$$

If $x_{04} = 0$ we get

$$\begin{cases} x_{11} = x_{21} - 2 \\ x_{12} = 4 - x_{21} \\ x_{20} = 4 - x_{21} \end{cases}$$

Thus, $2 \leq x_{21} \leq 4$ and $(x_{04}, x_{11}, x_{12}, x_{20}, x_{21}) \in \{(0,0,2,2,2),(0,1,1,1,3),(0,2,0,0,4)\}$.
If $x_{04} = 1$ we get

$$\begin{cases} x_{11} = x_{21} - 2 \\ x_{12} = 2 - x_{21} \\ x_{20} = 5 - x_{21} \end{cases} \Rightarrow \begin{cases} x_{11} = 0 \\ x_{12} = 0 \\ x_{20} = 3 \\ x_{21} = 2 \end{cases} \text{ and } (x_{04}, x_{11}, x_{12}, x_{20}, x_{21}) = (0,1,0,0,3,2).$$

Therefore, the $2-$lines can be of type
$(x_{04}, x_{11}, x_{12}, x_{20}, x_{21}) \in \{(0,0,2,2,2),(0,1,1,1,3),(0,2,0,0,4),(1,0,0,3,2)\}$.
Two $1-$lines meet in an outer point.
We have two possibilities: an outer point of type (0,4,1,3) exists or not.

If an outer point Q of type (0,4,1,3) exists, then through Q there pass four $1-$lines: three, l_1, l_2 and l_3, of type $(0,1,0,2,0,4)$ and one, l, of type $(0,1,1,0,1,4)$. The other two $1-$lines not through Q, m_1 and m_2, are of type $(0,0,0,3,1,3)$. Now, we count the outer point types by the $1-$line types. The pointset $l \cup l_1 \cup l_2 \cup l_3$ consists of one point of type (0,4,1,3), the point Q, 16 points of type (2,1,1,4), 1 point of type (1,3,0,4), 1 point of type (1,1,4,2) and 6 of type (1,2,2,3). The pointset $(m_1 \cup m_2) - (l \cup l_1 \cup l_2 \cup l_3)$ consists of 6 points of type (2,1,1,4). The types of the 11 outer points contained in no $1-$lines remain to be determined. Since an outer point contained in no $1-$lines is of type $(u_0,u_1,u_2,u_3) \in \{(2,0,3,3),(3,0,0,5)\}$, we have through it at least a pair of $0-$lines. Each of the $\binom{10}{2} = 45$ $0-$lines pairs meet on points of type $(u_0,u_1,u_2,u_3) \in \{(2,0,3,3), (2,1,1,4), (3,0,0,5)\}$. Exactly 22 $0-$lines pairs meet on points of type (2,1,1,4). So, 23 $0-$lines pairs meet on points of type $(u_0,u_1,u_2,u_3) \in \{(2,0,3,3), (3,0,0,5)\}$. Let x and y denote the number of outer points of type (2,0,3,3) and (3,0,0,5), respectively. We get $\begin{cases} x + y = 11 \\ x + 3y = 23 \end{cases} \Rightarrow \begin{cases} x = 5 \\ y = 6 \end{cases}$. Thus, we have 5 outer points of type (2,0,3,3) and 6 outer points of type (3,0,0,5).

Therefore, the nine $2-$lines can be of type

$$(x_{04}, x_{11}, x_{12}, x_{20}, x_{21}) \in \{(0,0,2,2,2),\ (0,1,1,1,3),\ (1,0,0,3,2)\}$$

Since there is exactly one point of type (0,4,1,3), we have exactly one $2-$line of type $(1,0,0,3,2)$. Since there is exactly one point of type (1,1,4,2), we have exactly four $2-$lines of type $(0,1,1,1,3)$. It follows that the other four $2-$lines are all of type $(0,0,2,2,2)$.

Therefore, the ten $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,0,1,2,3,2),(0,0,2,3,0,3),(0,1,0,1,4,2),(0,1,1,2,1,3),\\ (0,2,0,1,2,3),(1,0,0,0,5,2),(1,0,1,1,2,3),(1,1,0,0,3,3),(1,2,0,0,1,4),\ (1,1,1,1,0,4)\}$$

Since there is exactly one point of type (1,1,4,2), there is exactly one $0-$line of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(1,0,0,0,5,2),(1,1,0,0,3,3),(1,2,0,0,1,4),\ (1,0,1,1,2,3),(1,1,1,1,0,4)\}$$

Since there is exactly one point of type (1,3,0,4), a $0-$line of type $(0,0,2,3,0,3)$ does not exist, and there is exactly one $0-$line of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,1,2,3,2),(0,1,1,2,1,3),\ (1,0,1,1,2,3),(1,1,1,1,0,4)\}$$

If there is exactly one $0-$line of type $(1,0,0,0,5,2)$ and one of type $(0,0,1,2,3,2)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly 6 points of type (1,2,2,3) and another 14 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=6 \\ 6x+4y+2z=28 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,0,0,0,5,2)$ and one of type $(0,1,1,2,1,3)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly five points of type (1,2,2,3) and another 16 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=5 \\ 6x+4y+2z=32 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type (1,1,0,0,3,3) and one of type $(0,0,1,2,3,2)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, (0,1,0,1,4,2) and $(0,2,0,1,2,3)$, respectively. Since there are exactly five points of type (1,2,2,3), and another 16 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=5 \\ 6x+4y+2z=32 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,1,0,0,3,3)$ and one of type $(0,1,1,2,1,3)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and (0,2,0,1,2,3), respectively. Since there are exactly four points of type (1,2,2,3), and another 18 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=4 \\ 6x+4y+2z=36 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,2,0,0,1,4)$ and one of type $(0,0,1,2,3,2)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly four points of type (1,2,2,3), and another 18 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=4 \\ 6x+4y+2z=36 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,2,0,0,1,4)$ and one of type $(0,1,1,2,1,3)$, then the other eight $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly three points of type (1,2,2,3), and another 20 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=3 \\ 6x+4y+2z=40 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,0,1,1,2,3)$, then the other nine $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly six points of type (1,2,2,3), and another 20 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=6 \\ 6x+4y+2z=40 \end{cases}$ which has no solutions, a contradiction.

If there is exactly one $0-$line of type $(1,1,1,1,0,4)$, then the other nine $0-$lines can be of type

$$(x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3)\}$$

Let us denote by x, y and z the number of $0-$lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$ and $(0,2,0,1,2,3)$, respectively. Since there are exactly five points of type (1,2,2,3), and 22 points of type (2,1,1,4), we get $\begin{cases} x+y+z=8 \\ y+2z=5 \\ 6x+4y+2z=44 \end{cases}$ which has no solutions, a contradiction.

Therefore, an outer point of type (0,4,1,3) does not exist, and the $1-$lines can be of type

$$(x_{03}, x_{04}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(0,0,0,3,1,3),(0,0,1,1,2,3)\}$$

and the $2-$lines can be of type

$$(x_{04}, x_{11}, x_{12}, x_{20}, x_{21}) \in \{(0,0,2,2,2),(0,1,1,1,3),(0,2,0,0,4)\}$$

The number of outer points of type (2,1,1,4) is 18 because every 1−line contains exactly 3 outer points of type (2,1,1,4). Let x, y and z denote the numbers of 2−lines of type $(0,0,2,2,2)$, $(0,1,1,1,3)$ and $(0,2,0,0,4)$, respectively. We get $\begin{cases} x+y+z=9 \\ 2x+3y+4z=18 \end{cases}$ $\Rightarrow$ we get $\begin{cases} x=9 \\ y=0 \\ z=0 \end{cases}$. Thus, the 2−lines are of type $(0,0,2,2,2)$. It follows that there are no outer points of type (1,1,4,2) and that the 1−lines are of type $(0,0,0,3,1,3)$. There are exactly two outer points, P and Q, of type (1,3,0,4). The 3 1−lines through P meet the 3 1−lines through Q in different points of type (1,2,2,3). Therefore, there are exactly 9 outer points of type (1,2,2,3). There are exactly 6 outer points of type (2,0,3,3) and, so, 7 outer points of type (3,0,0,5).

Therefore, the 0−lines can be of type

$$(x_{11},x_{12},x_{13},x_{20},x_{21},x_{30}) \in \{(0,0,0,1,6,1),(0,1,0,1,4,2),(0,2,0,1,2,3),(0,3,0,1,0,4),\ (0,0,1,2,3,2)\ (0,1,1,2,1,3)\}$$

Let x_1, x_2, x_3, x_4, x_5 and x_6 denote the numbers of 0−lines of type $(0,0,0,1,6,1)$, $(0,1,0,1,4,2)$, $(0,2,0,1,2,3)$, $(0,3,0,1,0,4)$, $(0,0,1,2,3,2)$, and $(0,1,1,2,1,3)$, respectively. We get $\begin{cases} x_1+x_2+x_3+x_4+x_5+x_6=10 \\ x_2+2x_3+3x_4+x_6=9 \\ x_5+x_6=2 \\ x_1+2x_2+3x_3+4x_4+2x_5+3x_6=35 \end{cases}$ which has no solutions, a contradiction.

Step 5. The nonexistence of a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (11,3,12,31).

Proof. Let K be a (15,3)-Arc in PG(2,7) with character vector (t_0,t_1,t_2,t_3) = (11,3,12,31). We have ten types of outer points:

$$(u_0,u_1,u_2,u_3)\in\{(0,1,7,0),(0,2,5,1),(0,3,3,2),(1,0,6,1),(1,1,4,2),(1,2,2,3),(1,3,0,4),(2,0,3,3),(2,1,1,4),(3,0,0,5)\}.$$

□

We firstly prove that an outer point Q of type (0,1,7,0) does not exist. Suppose, on the contrary, that an outer point Q of type (0,1,7,0) exists. Since t_1 = 3, there are exactly three inner points P_1, P_2 and P_3 of type (1,0,7). The lines QP_1, QP_2 and QP_3 are three 1−lines, a contradiction.

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (1,0,6,1), (1,1,4,2), (1,2,2,3), (1,3,0,4), (2,0,3,3), (2,1,1,4), (3,0,0,5), respectively, of a 0−line l_0. By double counting the number of pairs (Q,r), where $Q \in l_0$ and $r \neq l_0$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{20}+x_{21}+2x_{30}=10 \\ x_{11}+2x_{12}+3x_{13}+x_{21}=3 \\ 6x_{10}+4x_{11}+2x_{12}+3x_{20}+x_{21}=12 \\ x_{10}+2x_{11}+3x_{12}+4x_{13}+3x_{20}+4x_{21}+5x_{30}=31 \end{cases} \Rightarrow \begin{cases} x_{20}=3-2x_{10}-x_{11}+x_{13} \\ x_{21}=3-x_{11}-2x_{12}-3x_{13} \\ x_{30}=2+x_{10}+x_{11}+x_{12}+x_{13} \end{cases}$$

By the second equation, we have that $0 \leq x_{13} \leq 1$ and $0 \leq x_{12} \leq \frac{3-3x_{13}}{2}$.

If $(x_{12},x_{13})=(0,0)$, we get $\begin{cases} x_{20}=3-2x_{10}-x_{11} \\ x_{21}=3-x_{11} \\ x_{30}=2+x_{10}+x_{11} \end{cases}$

Thus, $0 \leq x_{10} \leq 1$, $0 \leq x_{11} \leq 3-2x_{10}$. Therefore, $(x_{10},x_{11},x_{12},x_{13},x_{20},x_{21},x_{30}) \in \{(0,0,0,0,3,3,2),(0,1,0,0,2,2,3),(0,2,0,0,1,1,4),(0,3,0,0,0,0,5),\ (1,0,0,0,1,3,3),\ (1,1,0,0,0,2,4)\}$.

If $(x_{12}, x_{13}) = (1,0)$, we get $\begin{cases} x_{20} = 3 - 2x_{10} - x_{11} \\ x_{21} = 1 - x_{11} \\ x_{30} = 3 + x_{10} + x_{11} \end{cases}$

Thus, $0 \le x_{10} \le 1$ and $0 \le x_{11} \le 1$. Therefore, $(x_{10}, x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,1,0,3,1,3), (0,1,1,0,2,0,4), (1,0,1,0,1,1,4), (1,1,1,0,0,0,5)\}$

If $x_{13} = 1 \Rightarrow x_{11} = 0,\ x_{12} = 0,\ x_{21} = 0 \Rightarrow \begin{cases} x_{20} = 4 - 2x_{10} \\ 0 = 0 \\ x_{30} = 3 + x_{10} \end{cases} \Rightarrow 0 \le x_{10} \le 2$. Thus, $(x_{10}, x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,1,4,0,3), (1,0,0,1,2,0,4), (2,0,0,1,0,0,5)\}$

Therefore, the $0-$lines can be of type

$$\begin{gathered}(x_{10}, x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,0,3,3,2), (0,1,0,0,2,2,3), (0,2,0,0,1,1,4) \\ (0,3,0,0,0,0,5),\ (1,0,0,0,1,3,3), (1,1,0,0,0,2,4), (0,0,1,0,3,1,3), (0,1,1,0,2,0,4), (1,0,1,0,1,1,4) \\ (1,1,1,0,0,0,5), (0,0,0,1,4,0,3), (1,0,0,1,2,0,4), (2,0,0,1,0,0,5)\}\end{gathered}$$

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (0,2,5,1), (0,3,3,2), (1,1,4,2), (1,2,2,3), (1,3,0,4), (2,1,1,4), respectively, of a $1-$line l_1. By double counting the number of pairs (Q,r), where $Q \in l_1$ and $r \neq l_1$ is a $i-$line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{11} + x_{12} + x_{13} + 2x_{21} = 11 \\ x_{02} + 2x_{03} + x_{12} + 2x_{13} = 2 \\ 5x_{02} + 3x_{03} + 4x_{11} + 2x_{12} + x_{21} = 12 \\ x_{02} + 2x_{03} + 2x_{11} + 3x_{12} + 4x_{13} + 4x_{21} = 24 \end{cases}$$

The system of linear equation is equivalent to

$$\begin{cases} x_{02} + 2x_{03} + x_{12} + 2x_{13} = 2 \\ x_{11} + x_{12} + x_{13} + 2x_{21} = 11 \\ x_{02} - x_{12} - 2x_{13} - 2x_{21} = -10 \end{cases} \Rightarrow \begin{cases} x_{02} = x_{12} + 2x_{13} + 2x_{21} - 10 \\ x_{03} = 6 - x_{12} - 2x_{13} - x_{21} \\ x_{11} = 11 - x_{12} - x_{13} - 2x_{21} \end{cases}$$

By the first equation, we have that $0 \le x_{03} \le 1$ and $0 \le x_{13} \le \frac{2-2x_{03}}{2}$.

If $(x_{03}, x_{13}) = (0,0)$, we get $\begin{cases} x_{02} = x_{21} - 4 \\ x_{12} = 6 - x_{21} \\ x_{11} = 5 - x_{21} \end{cases}$.

Thus, $4 \le x_{21} \le 5$. Therefore,

$$(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(1,0,0,1,0,5), (0,0,1,2,0,4)\}.$$

If $(x_{03}, x_{13}) = (0,1)$, we get $\begin{cases} x_{02} = x_{21} - 4 \\ x_{12} = 4 - x_{21} \\ x_{11} = 6 - x_{21} \end{cases}$

Thus, $x_{21} = 4$. Therefore, $(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) = (0,0,2,0,1,4)$.

If $(x_{03}, x_{13}) = (1,0)$, we get $\begin{cases} x_{02} = x_{21} - 5 \\ x_{12} = 5 - x_{21} \\ x_{11} = 6 - x_{21} \end{cases}$

Thus, $x_{21} = 5$. Therefore, $(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) = (0,1,1,0,0,5)$.

Therefore, the $1-$lines can be of type

$$(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(1,0,0,1,0,5), (0,0,1,2,0,4), (0,0,2,0,1,4), (0,1,1,0,0,5)\}$$

Two $1-$lines meet in an outer point.

We have two possibilities: an outer point of type (0,3,3,2) exists or not.

If an outer point Q of type (0,3,3,2) exists, then through Q there pass three $1-$lines, l_i, $i \in$ {1,2,3}, of type $(0,1,1,0,0,5)$. Let us denote by L_i the point of type (1,1,4,2) on the $1-$line l_i, $i \in$ {1,2,3}. Let us denote by m_{ij}, $j \in$ {1,2,3,4}, the four $2-$lines through L_i. Let us denote by M_{ih}, $h \in$ {1,2,3,4,5}, the five points of type (2,1,1,4) on the $1-$line l_i, $i \in$ {1,2,3}. Two of the eight 2−lines m_{1j} and m_{2j}, $j \in$ {1,2,3,4}, meet l_3 in the point L_3 and the other six in six different points M_{3h}, $h \in$ {1,2,3,4,5}, a contradiction.

Therefore, an outer point of type (0,3,3,2) does not exist.

Thus, the 1−lines can be of type

$$(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(1,0,0,1,0,5), (0,0,1,2,0,4), (0,0,2,0,1,4)\}$$

We have two possibilities: an outer point of type (1,3,0,4) exists or not.

If an outer point Q of type (1,3,0,4) exists, then through Q there pass three 1−lines, l_1, l_2 and l_3, of type $(0,0,2,0,1,4)$. Now, we count the outer point types on the 1−lines. The pointset $l_1 \cup l_2 \cup l_3$ consists of one point of type (1,3,0,4), the point Q, 12 points of type (2,1,1,4), 6 points of type (1,1,4,2). Two points of type (1,1,4,2) not on the same 1−line are joined by a 2−line. Thus the 0−lines have $0 \leq x_{11} \leq 1$ and $x_{12} = 0$. The 3−line through a point of type (1,0,6,1) meets l_1, l_2 and l_3 in its inner points. A point of type (1,0,6,1) and Q are joined by a 0−line. Thus, the 0−line through Q contains at most two points of type (1,0,6,1) and there exist no 0−lines with $x_{10} \neq 0$ and $x_{13} = 0$.

Since there are 6 points of type (1,1,4,2), the number of 0−lines of type $(0,1,0,0,2,2,3)$ is 6. The other 5 0−lines are of type

$$(x_{10}, x_{11}, x_{12}, x_{13}, x_{20}, x_{21}, x_{30}) \in \{(0,0,0,0,3,3,2), (0,0,0,1,4,0,3), (1,0,0,1,2,0,4), (2,0,0,1,0,0,5)\}$$

Let x denote the number of 0−lines of type $(0,0,0,0,3,3,2)$. Double counting the pairs (R,l), where R is a point of type (2,1,1,4), l is a 0−line of type $(0,0,0,0,3,3,2)$ and $R \in l$, we get $3x$ = 24, a contradiction.

Therefore, an outer point of type (1,3,0,4) does not exist.

Thus, the 1−lines can be of type $(x_{02}, x_{03}, x_{11}, x_{12}, x_{13}, x_{21}) \in \{(1,0,0,1,0,5), (0,0,1,2,0,4)\}$.

Now, let x_{ij}, $I = u_0$, $j = u_1$, denote the number of outer points of type (0,2,5,1), (1,0,6,1), (1,1,4,2), (1,2,2,3), (2,0,3,3), (2,1,1,4), respectively, of a 2−line l_2. By double counting the number of pairs (Q,r), where $Q \in l_2 - K$ and $r \neq l_2$ is a i−line, I = 0,1,2,3, through Q, we get:

$$\begin{cases} x_{10} + x_{11} + x_{12} + 2x_{20} + 2x_{21} = 11 \\ 2x_{02} + x_{11} + 2x_{12} + x_{21} = 3 \\ 4x_{02} + 5x_{10} + 3x_{11} + x_{12} + 2x_{20} = 9 \\ x_{02} + x_{10} + 2x_{11} + 3x_{12} + 3x_{20} + 4x_{21} = 19 \end{cases}$$

The third equation minus the first equation gives $2x_{02} + 2x_{10} + x_{11} = x_{21} - 1 \Rightarrow x_{21} \geq 1$.

Taking into account the second equation, we get $2x_{02} + x_{11} + 2x_{12} = 3 - x_{21} \leq 2$.

$\Rightarrow (x_{02}, x_{11}, x_{12}, x_{21}) \in \{(0,0,0,3), (0,0,1,1), (0,1,0,2), (0,2,0,1), (1,0,0,1)\}$.

The system of linear equation is equivalent to

$$\begin{cases} x_{02} = x_{20} + x_{21} - 5 \\ x_{10} = x_{12} + x_{21} - 2 \\ x_{11} = 13 - x_{12} - x_{20} - 3x_{21} \end{cases}$$

Taking into account the second equation,

We get $x_{12} + x_{21} \geq 2 \Rightarrow (x_{02}, x_{11}, x_{12}, x_{21}) \in \{(0,0,0,3), (0,0,1,1), (0,1,0,2)\}$.

If $(x_{02}, x_{11}, x_{12}, x_{21}) = (0,0,0,3)$, we get

$$\begin{cases} 0 = x_{20} + 3 - 5 \\ x_{10} = 0 + 3 - 2 \\ 0 = 13 - 0 - x_{20} - 9 \end{cases}, \text{ a contradiction.}$$

If $(x_{02}, x_{11}, x_{12}, x_{21}) = (0,0,1,1)$, we get

$$\begin{cases} 0 = x_{20} + 1 - 5 \\ x_{10} = 1 + 1 - 2 \\ 0 = 13 - 1 - x_{20} - 3 \end{cases}, \text{ a contradiction.}$$

If $(x_{02}, x_{11}, x_{12}, x_{21}) = (0,1,0,2)$, we get
$$\begin{cases} 0 = x_{20} + 2 - 5 \\ x_{10} = 0 + 2 - 2 \\ 1 = 13 - 0 - x_{20} - 6 \end{cases}, \text{ a contradiction.}$$
Thus, the character vector of a (15,3)-Arc in PG(2,7) is (12,0,15,30). Since $t_1 = 0$, the inner points are of type (0,2,6) and the outer points are of type $(u_0,u_1,u_2,u_3) \in \{(1,0,6,1),(2,0,3,3),(3,0,0,5)\}$.

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (1,0,6,1), (2,0,3,3), (3,0,0,5), respectively, of a $0-$line l_0. By double counting the number of pairs (Q,r), where $Q \in l_0$ and $r \neq l_0$ is a $i-$line, $i = 0,2,3$, through Q, we get:

$$\begin{cases} x_{20} + 2x_{30} = 11 \\ 6x_{10} + 3x_{20} = 15 \\ x_{10} + 3x_{20} + 5x_{30} = 30 \end{cases} \Rightarrow \begin{cases} x_{20} = 5 - 2x_{10} \\ x_{30} = x_{10} + 3 \end{cases}$$

We have that $0 \le x_{10} \le 2$.
If $x_{10} = 0$, we get $\begin{cases} x_{20} = 5 \\ x_{30} = 3 \end{cases}$ and $(x_{10}, x_{20}, x_{30}) = (0,5,3)$.
If $x_{10} = 1$, we get $\begin{cases} x_{20} = 3 \\ x_{30} = 4 \end{cases}$ and $(x_{10}, x_{20}, x_{30}) = (1,3,4)$.
If $x_{10} = 2$, we get $\begin{cases} x_{20} = 1 \\ x_{30} = 5 \end{cases}$ and $(x_{10}, x_{20}, x_{30}) = (2,1,5)$.
Thus, the $0-$lines can be of type $(x_{10}, x_{20}, x_{30}) \in \{(0,5,3),(1,3,4),(2,1,5)\}$.

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (1,0,6,1), (2,0,3,3), respectively, of a $2-$line l_2. By double counting the number of pairs (Q,r), where $Q \in l_2$ and $r \neq l_2$ is a $i-$line, $i = 0,2,3$, through Q, we get:

$$\begin{cases} x_{10} + 2x_{20} = 12 \\ 5x_{10} + 2x_{20} = 12 \\ x_{10} + 3x_{20} = 18 \end{cases} \Rightarrow \begin{cases} x_{10} = 0 \\ x_{20} = 6 \end{cases}$$

Thus, the $2-$lines are of type $(x_{10}, x_{20}) = (0,6)$, an outer point of type (1,0,6,1) does not exist, and the $0-$lines are of type $(x_{10}, x_{20}, x_{30}) = (0,5,3)$.

Let x_{ij}, $i = u_0$, $j = u_1$, denote the number of outer points of type (2,0,3,3), (3,0,0,5) respectively, of a $3-$line l_3. By double counting the number of pairs (Q,r), where $Q \in l_3$ and $r \neq l_3$ is a $i-$line, $i = 0,2,3$, through Q, we get:

$$\begin{cases} 2x_{20} + 3x_{30} = 12 \\ 3x_{20} = 9 \\ 2x_{20} + 4x_{30} = 14 \end{cases} \Rightarrow \begin{cases} x_{20} = 3 \\ x_{30} = 2 \end{cases}$$

Thus, the $3-$lines are of type $(x_{20}, x_{30}) = (3,2)$.

Exactly 15 of the $\binom{15}{2} = 105$ $2-$lines pairs meet on points of type (0,2,6), the other 90 on points of type (2,0,3,3). So, we have exactly 30 points of type (2,0,3,3) and 12 points of type (3,0,0,5). This yields a self$-$dual configuration. Indeed, the set of the $2-$lines in the dual plane is a (15,3)-Arc of PG(2,7).

Let X and Y denote two inner points such that the line joining X and Y is a $2-$line l_1. Let l_2 and l_3 denote the other two $2-$lines through X and Y, respectively, different from l_1.

If the point $Z \in l_2 \cap l_3$ does not belong to K then Z is of type (2,0,3,3). There is another $2-$line through Z which meets the line l_1 in a point Q_0. Let P_1 and P_2 denote the two inner points of the $2-$line $Z\,Q_0$. Moreover let Q_j, $j = 1,2,3,4$ denote the other four outer points of the $2-$line $Z\,Q_0$, different from Z and Q_0. The points of K different from X, Y, P_i, $I = 1,2$, are one on each of the lines XZ and YZ, two on the lines P_iQ_j, $i = 1,2$, $j = 1,2,3,4$, and one on the lines XP_i and YP_i, $i = 1,2$. Thus, the size of K is even, a contradiction.

Therefore, the point $Z \in l_2 \cap l_3$ belongs to K and the three 2−lines l_1, l_2 and l_3 form a triangle with vertices X, Y and Z. It follows that there is a partition of the 2−line 15−set into 5 triangles.

Let P_1 and P_2 denote the two other inner points of a 3−line through X. Let P_3 denote the other inner point of the 3−line YP_2. Let P_4 denote the other inner point of the 3−line XP_3. If P_4 is on the 3−line YP_1, then the four 3−lines XP_1P_2, XP_3P_4, YP_2P_3, YP_1P_4 form a quadrangle with diagonal points X, Y and Z. If P_4 is not on the 3−line YP_1, then the 3−line P_2P_4 meets the 3−line P_1P_3 in the point Z, then the four 3−lines XP_1P_2, XP_3P_4, ZP_2P_4, ZP_1P_3 form a quadrangle with diagonal points X, Y and Z. Therefore, there exists a quadrangle of four 3−lines with diagonal points X, Y and Z. Since any quadrangle is projectively equivalent with the reference quadrangle R = {(0,0,1),(0,1,0),(1,0,0),(1,1,1)}, the construction and the uniqueness, up to projective equivalence, of the (15,3)-Arc K in PG(2,7) follows by observing that the other 8 points of K are exactly the complementary set of the union of the nine lines joining the seven points of the quadrangle.

We explicitly note that if one considers as blocks the 3−lines and the 3−sets of vertices of the 2−line triangles, then a Steiner triple system S(2,3,15) is obtained. It follows that the (15,3)-Arc K in PG(2,7) is an embedding of a Steiner triple system S(2,3,15) in PG(2,7).

The automorphism group of K is a group G_{72} of order 72 with 21 elements of order 2, 26 elements of order 3, 18 elements of order 4 and 6 elements of order 6, cf. [3].

For convenience we represent PG(2,7) as a set of orthogonal arrays of AG(2,7) with the intersection point of the members of each parallel class indicated to the right of the row array and at the bottom of the column array. We do this by using the Singer difference set defining PG(2,7) as the line at infinity $l_\infty = \{0, 1, 3, 13, 32, 36, 43, 52\}$, cf. [10]. The remaining lines of the plane are found by adding 1 to each point of the preceding line beginning with l_∞ as l_0 and using addition modulo 57. Any quadrangle together with its three diagonal points is projectively equivalent with the reference quadrangle R = {0,1,2,16,51,52,53}, see Figure 1.

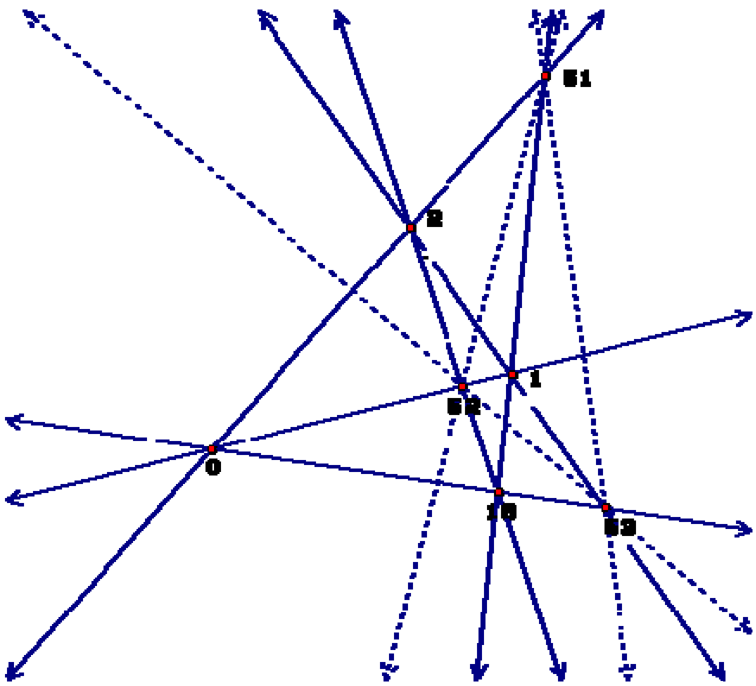

Figure 1. The Reference quadrangle.

One can easily verify by the Singer representation, see Table 1, that the 8-set of points of the plane which are not in the nine lines joining these points is a conic C = {5,9,20,23,39,40,41,49}. Moreover, the set $R \cup C$ is, up to projective equivalence, the (15,3)-Arc of PG(2,7).

Table 1. The Singer representation of PG(2,7).

5	**12**	**21**	**26**	**27**	**29**	**39**		**2**	**5**	**15**	**34**	**38**	**45**	**54**	
6	42	7	38	**9**	**49**	19		6	46	55	35	**39**	**16**	4	
8	35	22	25	17	54	**23**		22	42	48	47	33	26	50	
18	**51**	**16**	28	15	10	47	**1**	28	**23**	7	29	**41**	14	31	3
37	**2**	**53**	4	14	33	44		27	10	25	37	24	56	19	
41	56	34	11	50	55	30		30	**53**	12	**20**	17	**49**	18	
48	31	24	**20**	46	**40**	45		**40**	11	44	**9**	**51**	8	21	
			0								13				
33	35	45	11	7	18	27		48	**49**	**51**	4	**23**	27	34	
34	10	6	31	17	26	44		**2**	50	7	10	**20**	8	**39**	
46	21	25	**49**	47	**2**	**41**		11	24	54	12	6	47	14	
8	30	29	15	4	24	42	32	**16**	**5**	30	**9**	33	31	25	43
12	38	50	37	**40**	**23**	**16**		19	35	26	**41**	15	**53**	**40**	
19	14	**51**	22	**5**	55	**20**		17	28	37	45	21	55	42	
28	48	**53**	**39**	56	**9**	54		29	44	46	22	56	38	18	
			36								**52**				

3. Conclusions

The main contribution of this paper is the geometric construction of the (15,3)-Arc in PG(2,7), relying on the union of a conic and a complete external quadrangle. The construction has a particularly large automorphism group and a nice structure. This is in line with normal experience in all branches of finite geometry. Extremal $(k,3)$-arcs in PG(2,7) are shown to be vastly outnumbered by less interesting objects with no particular group or structure, cf. [3]. This leads us to conclude that except in a very small number of special cases, namely minimum and maximum cases, the question of geometric classification is neither feasible nor of particularly great interest.

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