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Properties of the Global Total k-Domination Number

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Abstract: A nonempty subset $D \subset V$ of vertices of a graph $G = (V, E)$ is a dominating set if every vertex of this graph is adjacent to at least one vertex from this set except the vertices which belong to this set itself. $D \subseteq V$ is a total k-dominating set if there are at least k vertices in set D adjacent to every vertex $v \in V$, and it is a global total k-dominating set if D is a total k-dominating set of both G and $\overline{G}$. The global total k-domination number of G, denoted by $\gamma^g_{kt}(G)$, is the minimum cardinality of a global total k-dominating set of G, GTkD-set. Here we derive upper and lower bounds of $\gamma^g_{kt}(G)$, and develop a method that generates a GTkD-set from a GT$(k-1)$D-set for the successively increasing values of k. Based on this method, we establish a relationship between $\gamma^g_{(k-1)t}(G)$ and $\gamma^g_{kt}(G)$, which, in turn, provides another upper bound on $\gamma^g_{kt}(G)$.

Keywords: global total domination; total k-domination number

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1. Introduction

We start by introducing the basic notation. Suppose we are given a simple graph $G = (V, E)$ with $|V| = n$ (n is called the order of graph G) and $|E| = m$ (m is called the size of graph G). Given $D \subseteq V$ ($D \neq \emptyset$) and vertex $v \in V$, let $N_D(v)$ be the set of all vertices from set D, adjacent to vertex v (also called the neighbors of vertex v from set D); we will use $\overline{N}_D(v)$ for the set of vertices in set D which are not neighbors of vertex v ($\overline{N}_D[v] = \overline{N}_D(v) \cup \{v\}$). We let $N_D[v] = N_D(v) \cup \{v\}$, and we call $\delta_D(v) = |N_D(v)|$ the degree of vertex v in set D. We denote by $\overline{\delta}_D(v)$ the cardinality of set $\overline{N}_D(v)$ ($\overline{\delta}_D(v) = |\overline{N}_D(v)|$). We will use more compact notation $N(v)$, $N[v]$, $\delta(v)$, $\overline{N}(v)$ and $\overline{N}[v]$ instead of $N_G(v)$, $N_G[v]$, $\delta_G(v)$, $\overline{N}_G(v)$ and $\overline{N}_G[v]$, respectively, when this will cause no confusion. The minimum (the maximum, respectively) degree in graph G is traditionally denoted by δ (Δ, respectively). $G[S]$ and $\overline{G}$, respectively, will stand for the subgraph of graph G induced by $S \subseteq V$ and the complement of graph G, respectively.

Let X and Y be subsets of set V. We denote by $E(X, Y)$ the set of all the edges in graph G joining a vertex $x \in X$ with a vertex $y \in Y$. Let u and v be vertices from set V. Then the distance between these two vertices $d(u, v)$ is the length (the number of edges) of a minimum $u - v$-path. The length of the longest $u - v$ path, for any u and v, is called the diameter of graph G, denoted by $diam(G)$. The girth of graph G is the length of the shortest cycle in that graph and is denoted by $g(G)$.

Let $D \subseteq V$ be a nonempty subset of set V. Then D is called a total k-dominating set for graph G if there are at least k vertices in set D adjacent to every vertex $v \in V$ (we will also say that vertex v is totally k-dominated by set D). The cardinality of a total k-dominating set in graph G with the minimum cardinality is called the total k-domination number of graph G and is denoted by $\gamma_{kt}(G)$. We will refer to a total k-dominating set with cardinality $\gamma_{kt}(G)$ as a $\gamma_{kt}(G)$-set. A total 1-dominating set is normally referred to as a total dominating set, and the total 1-domination number is referred to as the total domination number, denoted by $\gamma_t(G)$. We refer the reader to [1–9] for more detail on these definitions.

Given again a non-empty set $D \subseteq V$, D is called a global total k-dominating set of graph G (GTkD set for short) if D is a total k-dominating set of both graphs G and $\overline{G}$. The global total k-domination number of G, denoted by $\gamma^g_{kt}(G)$, is the cardinality of a global total k-dominating set with the minimum cardinality. A global total k-dominating set of cardinality $\gamma^g_{kt}(G)$ will be referred to as a $\gamma^g_{kt}(G)$-set. Again, if $k = 1$, a global total 1-dominating set is a global total dominating set (see [10,11]).

As it is well-known and also easily be seen,

$$2k + 1 \le \gamma^g_{kt}(G) \le n,$$

for any graph G with order n. Here we shall exclusively deal with the connected graphs due to a known fact that if $G_1, G_2, \ldots, G_r$ ($r \ge 2$) are the connected components in graph G, then

$$\gamma^g_{kt}(G) = \sum_{i=1}^{r} \gamma_{kt}(G_i)$$

(see [12]).

The main goal of this paper is to complete the current study of global total k-domination number in graphs. First, we give upper and lower bounds on $\gamma^g_{kt}(G)$, and then we develop a method that generates a GTkD-set from a GT$(k-1)$D-set for the successively increasing values of k. Based on this method, we establish a relationship between $\gamma^g_{(k-1)t}(G)$ and $\gamma^g_{kt}(G)$, which, in turn, provides another upper bound on $\gamma^g_{kt}(G)$.

The rest of the paper is organized as follows. In the next section, we present known results and give some remarks. In Sections 3 and 4, we derive upper and lower bounds, respectively, for global total k-domination number. In the Section 5, we provide our method that obtains a global total $(k+1)$-dominating set from a global total k-dominating set.

2. Relations between $\gamma^g_{kt}(G)$ and $\gamma_{kt}(G)$

Clearly, the definition of a GTkD set gives us an implicit lower bound for the parameter $\gamma^g_{kt}(G)$:

Observation 1. *Let G be a graph; then $\gamma^g_{kt}(G) \ge \max\{\gamma_{kt}(G), \gamma_{kt}(\overline{G})\}$.*

The above lower bound is not necessarily attainable, as we illustrate in the following figure: we depict graph G and its complement $\overline{G}$, and the corresponding minimum total 2-dominating set in both graphs (black vertices); see Figure 1.

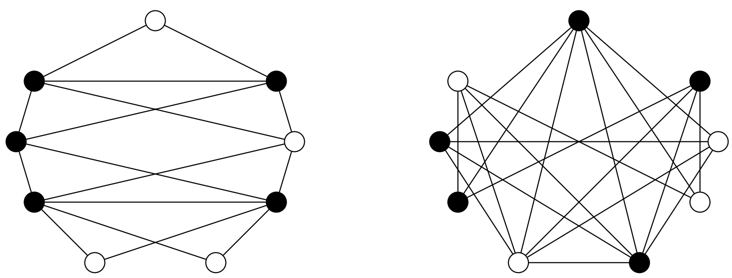

Figure 1. Graph G and its complement $\overline{G}$, which satisfy $\gamma_{2t}(G) = 5$, $\gamma_{2t}(\overline{G}) = 5$ and $\gamma^g_{2t}(G) = 6$.

The following proposition was proved in [12].

Proposition 1. *Let G be a graph,*

(i) *If $\gamma_{kt}(G) > \Delta(G) + k$, then $\gamma^g_{kt}(G) = \gamma_{kt}(G)$.*

(ii) *If $\gamma_{kt}(G) \le \Delta(G) + k$, then $\gamma^g_{kt}(G) \le \Delta(G) + k + 1$.*

Corollary 1. *Let G be a graph with maximum degree Δ. Then, $\gamma^g_{kt}(G) \leq \max\{\gamma_{kt}(G), \Delta + k + 1\}$.*

Proposition 2. *Let G be a graph with order n and maximum degree Δ. If $n > \frac{\Delta(\Delta+k)}{k}$, then $\gamma^g_{kt}(G) = \gamma_{kt}(G)$.*

Proof. If $n > \frac{\Delta(\Delta+k)}{k}$, then $\Delta + k < \frac{kn}{\Delta} \leq \gamma_{kt}(G)$; consequently, $\Delta + k + 1 \leq \gamma_{kt}(G)$. By Corollary 1 we have $\gamma^g_{kt}(G) = \gamma_{kt}(G)$. □

Theorem 1. *For any graph G, $\gamma^g_{kt}(G) = \gamma_{kt}(G)$ if and only if there exists a minimum total k-dominating set D such that any subset D' of D with $|D| - k + 1$ vertices is not included in any star in the graph—that is, and only if there is not a vertex $v \in V$ such that $D' \subseteq N[v]$.*

Proof. Let D be a minimum total k-dominating set which is also a global total k-dominating set, and let D' be a subset of D with cardinality $|D| - k + 1$. If there exists a vertex $v \in V$ such that $D' \subseteq N[v]$, then $v \in D'$ and it is adjacent to $|D| - k$ vertices in D', so v has less than k non-adjacent vertices in D, or $v \notin D'$, and it is adjacent to $|D| - k + 1$ vertices in D', so v has less than k non-adjacent vertices in D. In both cases we have a contradiction with the fact that D is a global total k-dominating set.

On the other hand, we take a minimum total k-dominating set D such that for any subset D' of D with $|D| - k + 1$ vertices and every vertex $v \in V$, we have $D' \not\subseteq N[v]$. Then, for any vertex $v \in D$ we have $|N(v)| < |D| - k$, so v has, at least, k non-neighbors in D. If $v \in V \setminus D$ we have $|N(v)| < |D| - k + 1$, so v has, at least, k non-neighbors in D. Therefore, D is a global total k-dominating set. □

3. Upper Bounds for the Global Total k-Domination Number

In this section, we obtain some upper bounds for the global total k-domination number in a graph. Bermudo et al. in [12] showed a characterization when the global total k-domination number is equal to the order of the graph, but we give here that characterization in a more specific way. To do that, in the following proposition we give a condition to guarantee that the global total k-domination number is less than n.

Proposition 3. *Let G be a graph with order n, minimum degree δ and maximum degree Δ. If $k < \min\{\delta, n - \Delta - 1\}$, then $\gamma^g_{kt}(G) \leq n - 1$.*

Proof. Let us see that, for any $v \in V$, the set $D = V \setminus \{v\}$ is a GTkD set of G. We have that $\delta_D(v) = \delta(v) \geq \delta > k$ and $\overline{\delta}_D(v) = n - 1 - \delta(v) \geq n - 1 - \Delta > k$. For every $u \in D$ we have $\delta_D(u) \geq \delta(u) - 1 \geq \delta - 1 \geq k$ and $\overline{\delta}_D(u) \geq n - 1 - \delta(u) - 1 \geq n - 2 - \Delta \geq k$. Therefore, D is a GTkD set of G. □

Proposition 3 is not an equivalence, as we can see if we consider a triangle and we add a leaf to every vertex of the triangle. In such a case $\gamma^g_{1t}(G) \leq n - 1 = 5$ and $k = 1 = \min\{\delta, n - \Delta - 1\}$.

Now, in order to present the characterization of all graphs having a global total k-domination number equal to the number of vertices, we need to define the following set. Given a graph G and an integer i, let $T_i(G) = \{v \in V(G) : \delta(v) = i\}$ (i.e., the set of vertices in graph G with the degree i).

Theorem 2. *Given graph G with order n and the minimum and the maximum degrees δ and Δ, $\gamma^g_{kt}(G) = n$ if and only if one of the conditions (a)–(c) below hold*

(a) $k = \delta < n - \Delta - 1$ *and* $V = \bigcup_{v \in T_\delta(G)} N(v)$.

(b) $k = n - \Delta - 1 < \delta$ *and* $V = \bigcup_{w \in T_\Delta(G)} (V \setminus N[w])$.

(c) $k = \delta = n - \Delta - 1$ *and* $V = \left(\bigcup_{v \in T_\delta(G)} N(v) \right) \cup \left(\bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \right)$.

Proof. (a) If $k = \delta < n - \Delta - 1$ and $V = \bigcup_{v \in T_\delta(G)} N(v)$, we consider $D = V \setminus \{u\}$ for any $u \in V$. We note that there exists $v \in N(u)$ such that $\delta(v) = k$; this implies that $\delta_D(v) < k$. Thus, D is not a GTkD set of G. Hence, $\gamma^g_{kt}(G) = n$.

(b) If $k = n - \Delta - 1 < \delta$ and $V = \bigcup_{w \in T_\Delta(G)} (V \setminus N[w])$, for any $u \in V$ there exists $w \in V$ such that $\delta(w) = \Delta$ and $u \notin N[w]$. If we consider $D = V \setminus \{u\}$, then $\overline{\delta}_D(w) \leq n - \Delta - 2 < k$; thus, D is not a GTkD set of G. Therefore, $\gamma^g_{kt}(G) = n$.

(c) If $k = \delta = n - \Delta - 1$ and $V = \left(\bigcup_{v \in T_\delta(G)} N(v) \right) \cup \left(\bigcup_{w \in T_\Delta(G)} (V \setminus N[w]) \right)$, using (a) or (b), we obtain that $V \setminus \{u\}$ is not a GTkD set of G, for any $u \in V$. Consequently, $\gamma^g_{kt}(G) = n$.

Finally, if we assume that $\gamma^g_{kt}(G) = n$, by Proposition 3 we have that $k \in \{\delta, n - \Delta - 1\}$. For every vertex $v \in V$, we note that $D = V \setminus \{v\}$ is not a GTkD set of G, so there exists $u \in D$ such that $\delta_D(u) < k$ or $\overline{\delta}_D(u) < k$. If $k = \delta < n - \Delta - 1$, since $\overline{\delta}_D(u) \geq n - 2 - \delta(u) \geq n - 2 - \Delta \geq k$, then we have that $\delta_D(u) < k = \delta$; this implies that $u \in T_\delta(G)$ and $v \in N(u)$. If $k = n - \Delta - 1 < \delta$, since $\delta_D(u) \geq \delta(u) - 1 \geq \delta - 1 \geq k$, then we have that $n - 2 - \delta(u) \leq \overline{\delta}_D(u) < k = n - \Delta - 1$; that is, $n - 2 - \delta(u) = \overline{\delta}_D(u) = n - \Delta - 2$, so $u \in T_\Delta(G)$ and $v \in V \setminus N[u]$. If $k = \delta = n - \Delta - 1$, since $\delta_D(u) < k$ or $\overline{\delta}_D(u) < k$, we have that $u \in T_\delta(G)$ and $v \in N(u)$, or $u \in T_\Delta(G)$ and $v \in V \setminus N[u]$. □

The following corollary was directly obtained from Theorem 2.

Corollary 2. *Let G be a graph with minimum degree δ, maximum degree Δ and order $n \neq \Delta + \delta + 1$. Then $\gamma^g_{kt}(G) = n$ if and only if one of the following condition holds.*

(a) $k = \delta < n - \Delta - 1$ *and* $\gamma_{kt}(G) = n$.

(b) $k = n - \Delta - 1 < \delta$ *and* $\gamma_{kt}(\overline{G}) = n$.

Corollary 3. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. If one of the following conditions holds:*

(a) $k = \delta < n - \Delta - 1$ *and* $|T_\delta(G)| \geq n - \delta$.

(b) $k = n - \Delta - 1 < \delta$ *and* $|T_\Delta(G)| \geq \Delta + 1$

(c) $k = \delta = n - \Delta - 1$ *and* $|T_\delta(G)| \geq n - \delta$ *or* $|T_\Delta(G)| \geq \Delta + 1$,

then $\gamma^g_{kt}(G) = n$.

Proof. Since $\gamma^g_{kt}(G) = \gamma^g_{kt}(\overline{G})$, $\overline{\Delta} = n - \delta - 1$, $T_{\overline{\Delta}}(\overline{G}) = T_\delta(G)$ and $V \setminus N_{\overline{G}}[w] = N(w)$, it is enough to check that $|T_\Delta(G)| \geq \Delta + 1$ implies $V = \bigcup_{w \in T_\Delta(G)} (V \setminus N[w])$. However, for any vertex $v \in V$, if $|T_\Delta(G)| \geq \Delta + 1$, then there exists a vertex $w \in T_\Delta(G)$ which is not a neighbor of v, so $v \in \bigcup_{w \in T_\Delta(G)} (V \setminus N[w])$. □

It was proved in [12] that $\gamma^g_{kt}(G) \leq \min\{\gamma_{kt}(G) + \Delta, \gamma_{kt}(G) + \gamma_{kt}(\overline{G})\}$. It would be convenient to characterize the graphs G such that $\gamma^g_{kt}(G) = \gamma_{kt}(G) + \Delta$, and the graphs G such that $\gamma^g_{kt}(G) = \gamma_{kt}(G) + \gamma_{kt}(\overline{G})$. On the other hand, the invariants of a graph are important when characterizing them; below we use some of them such as diameter and girth. The following proofs use the ideas showed in [11].

Theorem 3. *If G is a graph such that $diam(G) \geq 5$, every total k-dominating set is a GTkD set of G.*

Proof. Let D be a total k-dominating set and $u, v \in V$ such that $d(u,v) \geq 5$. Since $\delta_D(u) \geq k$ and $\delta_D(v) \geq k$, there exist $\{u_1, \ldots, u_k\} \subseteq D \cap N(u)$ and $\{v_1, \ldots, v_k\} \subseteq D \cap N(v)$. For any vertex $w \in V$ we know that $\delta_D(w) \geq k$. If $u_i \in N(w)$ for some $i \in \{1, \ldots, k\}$, then $w \notin \bigcup_{i=1}^{k} N[v_i]$; that means, $\overline{\delta}_D(w) \geq k$. Therefore, D is a GTkD set of G. □

Corollary 4. *If G is a graph such that $diam(G) \geq 5$, then $\gamma^g_{kt}(G) = \gamma_{kt}(G)$.*

According to the idea given in [11], we obtain the following result.

Proposition 4. *If G is a graph such that $diam(G) = 4$ and there exist $\{u, v_1, \ldots, v_k\} \subseteq V$ such that $dist(u, v_j) = 4$ for every $j \in \{1, \ldots, k\}$, then $\gamma^g_{kt}(G) \leq \gamma_{kt}(G) + k$.*

Proof. Let D be a minimum total k-dominating set of a graph; then there exists the vertex set $\{u_1, \ldots, u_k\} \subseteq D$ such that $\{u_1, \ldots, u_k\} \subseteq N(u)$. For any vertex $w \in V$ and $i \in \{1, \ldots, k\}$, w cannot be adjacent to both u_i and v_i, so $D \cup \{v_1, \ldots, v_k\}$ is a global total k-dominating set. □

In Figure 2 we can see an example where the equality in Proposition 4 for $k = 2$ is attained. Taking into account that any neighbor of a vertex of degree 2 must belong to any total 2-dominating set (grey vertices), we show in that figure the minimum total 2-dominating set (b) and the minimum global total 2-dominating set (c).

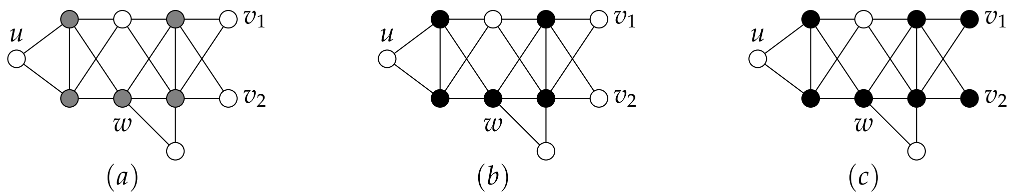

Figure 2. (**a**) Grey vertices are neighbors of vertices of degree 2. (**b**) Minimum total 2-dominating set and (**c**) minimum global total 2-dominating set.

For a graph G, we let $\delta^*(G) = min\{\delta(G), \delta(\overline{G})\}$.

Proposition 5. *Let G be a graph of order n and minimum degree δ; then $\gamma^g_{kt}(G) \leq n - \delta^*(G) + k$.*

Proof. Let us see that every set $D \subseteq V$ such that $|D| \geq n - \delta^*(G) + k$ is a global total k-dominating set. Since $|D| \geq n - \delta + k$, every vertex v satisfies $\delta_{V\setminus D}(v) \leq \delta - k$, $\delta_D(v) \geq k$. Since $|D| \geq n - \overline{\delta} + k$, every vertex v satisfies $\overline{\delta}_{V\setminus D}(v) \leq \overline{\delta} - k$, so $\overline{\delta}_D(v) \geq k$. □

4. Lower Bounds for the Global Total k-Domination Number

We know that any graph G satisfies $\gamma^g_{kt}(G) \geq 2k + 1$, and a characterization for graphs satisfying the equality was given in [12]. Additionally, in that work the authors showed the following inequality.

Remark 1. *Let G be a graph with order n, minimum degree δ and maximum degree Δ. Then,*

$$\gamma^g_{kt}(G) \geq \max\left\{\frac{kn}{\Delta}, \frac{kn}{n - \delta - 1}\right\}$$

For example, the lower bound given above can be reached in the graph shown in Figure 3.

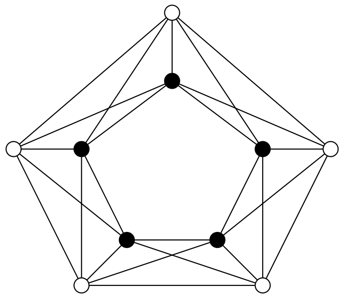

Figure 3. A graph G with order $n = 10$, $\delta = 5$ and $\gamma^g_{2t}(G) = \frac{2n}{n-\delta-1}$.

Theorem 4. *Let G be a graph of order n, maximum degree Δ and size m. Then*

$$\gamma^g_{kt}(G) \geq \frac{2m + n(2k-\Delta) + (2k+1)^2}{n+2k}.$$

Proof. Let D be a $\gamma_{kt}(G)$-set. Since every vertex in $V \setminus D$ cannot have more that $|D| - k$ neighbors in D, we have $E(D, V \setminus D) \leq (n - |D|)(|D| - k)$, so

$$\begin{aligned} m &= E(D,D) + E(D, V\setminus D) + E(V\setminus D, V\setminus D) \\ &\leq \frac{|D|\Delta(G) - E(D, V\setminus D)}{2} + E(D, V\setminus D) + \frac{(\Delta-k)(n-|D|)}{2} \\ &\leq \frac{|D|\Delta + (n-|D|)(|D|-k) + (\Delta-k)(n-|D|)}{2} \\ &= \frac{|D|\Delta + (n-|D|)(|D|-2k+\Delta)}{2} \\ &= \frac{-|D|^2 + (n+2k)|D| + n\Delta - 2kn}{2}, \end{aligned}$$

which implies that

$$(2k+1)^2 + 2m \leq |D|^2 + 2m \leq (n+2k)|D| + n\Delta - 2kn,$$

then

$$|D| \geq \frac{2m + n(2k-\Delta) + (2k+1)^2}{n+2k}.$$

□

Theorem 5. *Let G be a graph with order n, maximum degree Δ and size m. Then,*

$$\gamma^g_{kt}(G) \geq \frac{2m + n(\Delta - 2k)}{n+k-\Delta}.$$

Proof. We suppose that D is a $\gamma_{kt}(G)$-set and $|D| \geq 2r+1$ for some $r \geq 2$, and $|D| \geq 2k+2$. Since D is minimal, for any vertex $v_1 \in D$ there exists a vertex w_{v_1} such that one of the following conditions holds.

(1) $w_{v_1} \in D$, $v_1 \in N(w_{v_1})$ and $\delta_D(w_{v_1}) = k$,

(2) $w_{v_1} \in D$, $v_1 \notin N(w_{v_1})$ and $\delta_D(w_{v_1}) = |D| - k - 1$,

(3) $w_{v_1} \in V \setminus D$, $v_1 \in N(w_{v_1})$ and $\delta_D(w_{v_1}) = k$,

(4) $w_{v_1} \in V \setminus D$, $v_1 \notin N(w_{v_1})$ and $\delta_D(w_{v_1}) = |D| - k$.

Now, in cases (1) and (3), we take $v_2 \in D \setminus N(w_{v_1})$, and in cases (2) and (4), we take $v_2 \in D \cap N(w_{v_1})$, and we know that there exists a vertex $w_{v_2} \neq w_{v_1}$ such that one of the above conditions holds. Since $|D| \geq 2r+1$ we can obtain $w_{v_1}, \ldots, w_{v_r}$ vertices satisfying one of the conditions above. We suppose that there exist i, $j-i$, s and $r-j-s$ vertices satisfying (1), (2), (3) and (4), respectively. Then,

$$\begin{aligned}
E(D,D) &\leq \frac{ik+(j-i)(|D|-k-1)+(|D|-j)(|D|-k-1)}{2} \\
&= \frac{ik-i(|D|-k-1)+|D|(|D|-k-1)}{2} \\
&= \frac{i(2k-|D|+1)+|D|(|D|-k-1)}{2},
\end{aligned}$$

$$\begin{aligned}
E(D,V\setminus D) &\leq \frac{sk+(r-j-s)(|D|-k)+(n-|D|-r+j)(|D|-k)}{2} \\
&= \frac{sk-s(|D|-k)+(n-|D|)(|D|-k)}{2} \\
&= \frac{(n-|D|)(|D|-k)+s(2k-|D|)}{2},
\end{aligned}$$

and

$$\begin{aligned}
E(V\setminus D,V\setminus D) &\leq \frac{s(\Delta-k)+(r-j-s)(\Delta-|D|+k)}{2} \\
&+ \frac{(n-|D|-r+j)(\Delta-k)}{2} \\
&= \frac{s(\Delta-k)+(r-j-s)(\Delta-k-|D|+2k)}{2} \\
&+ \frac{(n-|D|-r+j)(\Delta-k)}{2} \\
&= \frac{(\Delta-k)(n-|D|)+(r-j-s)(2k-|D|)}{2};
\end{aligned}$$

therefore,

$$\begin{aligned}
m &\leq E(D,D)+E(D,V\setminus D)+E(V\setminus D,V\setminus D) \\
&\leq \frac{i(2k-|D|+1)+|D|(|D|-k-1)}{2}+\frac{(n-|D|)(|D|-k)+s(2k-|D|)}{2} \\
&\quad +\frac{(\Delta-k)(n-|D|)+(r-j-s)(2k-|D|)}{2} \\
&= \frac{i(2k-|D|+1)+|D|(|D|-k-1)}{2} \\
&+ \frac{(n-|D|)(|D|-2k+\Delta)+(r-j)(2k-|D|)}{2} \\
&= \frac{|D|(n+k-\Delta)+n(-2k+\Delta)+(i+r-j)(2k-|D|)+i}{2} \\
&\leq \frac{|D|(n+k-\Delta)+n(-2k+\Delta)}{2};
\end{aligned}$$

then

$$|D| \geq \frac{2m+n(\Delta-2k)}{n+k-\Delta}.$$

□

Let us see another lower bound using the algebraic connectivity. Given a graph G, its adjacency matrix A and the diagonal matrix D whose entries are the degrees of all vertices in the graph, the Laplacian matrix is defined as $L = A - D$. The algebraic connectivity of G, denoted by μ is the second smallest eigenvalue of the Laplacian matrix.

The algebraic connectivity of $G = (V, E)$ with order n satisfies the following equality given by Fielder [13].

$$\mu = 2n \min\left\{ \frac{\sum_{v_i v_j \in E}(w_i - w_j)^2}{\sum_{v_i \in V}\sum_{v_j \in V}(w_i - w_j)^2} : w \neq \alpha \mathbf{j} \text{ for } \alpha \in \mathbb{R} \right\},$$

where $\mathbf{j} = (1, 1, \ldots, 1)$ and $w \in \mathbb{R}^n$.

Theorem 6. *Let G be a graph with order n and algebraic connectivity μ. Then,*

$$\gamma^g_{kt}(G) \geq \frac{kn}{n - \mu}.$$

Proof. Let D be a $\gamma_{kt}(G)$-set. It can be found that if we take

$$w = \begin{cases} 1 \text{ if } v \in D \\ 0 \text{ if } v \notin D \end{cases}$$

in the set given above, since μ is the minimum, we have

$$\mu \leq \frac{n\sum_{v \in D}\delta_{\overline{D}}(v)}{|D|(n - |D|)} \leq \frac{n(n - |D|)(|D| - k)}{|D|(n - |D|)} = \frac{n(|D| - k)}{|D|};$$

therefore, $|D| \geq \frac{kn}{n-\mu}$. □

Theorem 7. *Let G be a graph of order n and maximum degree Δ. If $k \geq \min\left\{\frac{\Delta}{2}, \frac{n-\delta-1}{2}\right\}$, then*

$$\gamma^g_{kt}(G) \geq \frac{\sqrt{4kn+1}+1}{2}.$$

Proof. Let D be a $\gamma_{kt}(G)$-set. For every $v \in D$, if we suppose that $k \geq \frac{\Delta}{2}$, we have $\delta_D(v) \geq \delta_{\overline{D}}(v)$, then

$$|D|(|D| - k - 1) \quad \geq \quad \sum_{v \in D}\delta_D(v) \geq \sum_{v \in D}\delta_{\overline{D}}(v) \geq (n - |D|)k,$$

which implies that $|D|^2 - |D| \geq kn$, or equivalently, that $\left(|D| - \frac{1}{2}\right)^2 \geq kn + \frac{1}{4}$; that is, $|D| \geq \frac{\sqrt{4kn+1}+1}{2}$.

If $\frac{n-\delta-1}{2} \leq k < \frac{\Delta}{2}$, since $\gamma^g_{kt}(G) = \gamma^g_{kt}(\overline{G})$ and $\overline{\Delta} = n - \delta - 1$, we can obtain the same result. □

The lower bound given in Theorem 7 is attained, for instance, in the graph given in Figure 4.

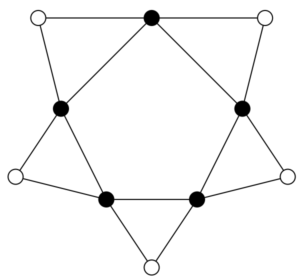

Figure 4. A graph G such that $\gamma^g_{2t}(G) \geq \frac{\sqrt{8n+1}+1}{2}$.

In graph theory, it is common to analyze graphs obtained by some transformation from an originally given graph. An example of such a transformation is the elimination of one or more edges of the graph. Given a graph G, it is natural to think about what happens if you add or delete edges on the graph. We note that removing an edge in G is equivalent to adding an edge to graph $\overline{G}$. Therefore, it suffices to study just one of these cases.

Proposition 6. *Let G be a graph with order n, minimum degree δ and maximum degree Δ, and let $k < min\{\delta, n - \Delta - 1\}$. Then the following inequalities are satisfied (for an edge e):*

$$\gamma^g_{kt}(G - e) \le \gamma^g_{kt}(G) + 2,$$
$$\gamma^g_{kt}(G + e) \le \gamma^g_{kt}(G) + 2.$$

Proof. Let G be a graph and D be a $\gamma^g_{kt}(G)$-set, and we consider $e \in E$. Notice that $e \in E(V \setminus D, V \setminus D)$, $e \in E(D, V \setminus D)$ or $e \in E(D, D)$; we will divide the proof into three cases and we denote $G' = G - e$.

Case 1: If $e \in E(V \setminus D, V \setminus D)$. Note that every vertex in $V(G')$ has at least k neighbors and k non-neighbors in D. Therefore, $\gamma^g_{kt}(G') \le |D| = \gamma^g_{kt}(G) < \gamma^g_{kt}(G) + 2$.

Case 2: If $e \in E(D, V \setminus D)$. Let $e = uv$, where $u \in D$ and $v \in V \setminus D$. We note that for every $w \in V(G) - \{v\}$, $\delta_D(w) \ge k$ and $\overline{\delta}_D(w) \ge k$. On the other hand, note that $\overline{\delta}_D(v) > k$ in G', and if $\delta_D(v) \ge k$ in G', then $\gamma^g_{kt}(G') \le |D| = \gamma^g_{kt}(G) < \gamma^g_{kt}(G) + 2$. Now, if $\delta_D(v) = k - 1$ in G', then there exists $w \in V(G') \setminus D$ such that $w \in N_{G'}(v)$. Therefore, $D \cup \{w\}$ is a GTkD set of G', so $\gamma^g_{kt}(G') \le |D \cup \{w\}| = \gamma^g_{kt}(G) + 1 < \gamma^g_{kt}(G) + 2$.

Case 3: If $e \in E(D, D)$. Let $e = uv$ where $u, v \in D$. We note that for every $w \in V(G) - \{u, v\}$, $\delta_D(w) \ge k$ and $\overline{\delta}_D(w) \ge k$. In the worst case $\delta_D(u) < k$ and $\delta_D(v) < k$; the others cases are solved as the above; there exists $w, p \in V(G') \setminus D$ such that $w \in N_{G'}(u)$ and $p \in N_{G'}(v)$. Now, if $w = p$ then $D \cup \{w\}$ is a GTkD set of G' and $\gamma^g_{kt}(G') \le |D \cup \{w\}| = \gamma^g_{kt}(G) + 1 < \gamma^g_{kt}(G) + 2$; otherwise, $w \ne p$ and then $D \cup \{w, p\}$ is a GTkD set of G'; hence $\gamma^g_{kt}(G') \le |D \cup \{w, p\}| = \gamma^g_{kt}(G) + 2$.

Thus, the first inequality is satisfied: $\gamma^g_{kt}(G - e) \le \gamma^g_{kt}(G) + 2$. Now, as we say above for this problem, removing an edge in G is analogous to adding an edge in $\overline{G}$. Since $G - e$ and $\overline{G} + e$ are complementary graphs and it is known that $\gamma^g_{kt}(\overline{G}) = \gamma^g_{kt}(G)$, it is verified that $\gamma^g_{kt}(G - e) = \gamma^g_{kt}(\overline{G} + e)$. Hence, by the first inequality $\gamma^g_{kt}(\overline{G} + e) = \gamma^g_{kt}(G - e) \le \gamma^g_{kt}(G) + 2 = \gamma^g_{kt}(\overline{G}) + 2$. So, $\gamma^g_{kt}(G + e) \le \gamma^g_{kt}(G) + 2$. □

Let S be a subset of set V such that the maximum degree of the subgraph induced by the vertices from set S is no more than $k - 1$. Then set S will be referred to as a k-independent set of vertices. The cardinality of a k-independent set of the maximum cardinality will be referred to as the k-independence number in graph G and will be denoted by $\beta_k(G)$. The lower k-independence number $i_k(G)$ is the minimum cardinality of a maximal k-independent set in graph G.

Proposition 7. *Let D be a global total k-dominating set in G and let $V \setminus D$ be a maximum $(\Delta - k)$-independent. Then,*

$$n - \beta_{\Delta-k}(G) \le |D| \le \min\{n - \gamma(G), n - i_{\Delta-k}(G)\}.$$

Proof. Since $V \setminus D$ is a maximal $(\Delta - k)$-independent set, $V \setminus D$ is a dominating set; thus, $n - |D| \ge \gamma(G)$. Moreover, $i_{\Delta-k}(G) \le n - |D| \le \beta_{\Delta-k}(G)$. □

5. Deriving Upper Bounds for $\gamma^g_{(k+1)t}(G)$ from $\gamma^g_{kt}(G)$

It is intuitively clear that the greater k is, the more difficult is to find a global total k-dominating set of graph $G = (V, E)$ with the minimum cardinality. In particular, the following relationship is easy to see: $\gamma^g_{1t}(G) \le \gamma^g_{2t}(G) \le \gamma^g_{3t}(G) \le \ldots \le \gamma^g_{kt}(G)$, for every

$k \leq min\{\delta, n-\Delta-1\}$. Ideally, one would wish to have a method that obtains a GT$(k+1)$D set of minimum cardinality from a GTkD set with the minimum cardinality. It is clear that this is not an easy task. In this next section we develop a method that generates a GT$(k+1)$D set from a GTkD, based on which we establish a relationship between minimum cardinality GTkD and GT$(k+1)$D sets—more precisely, between $\gamma^g_{kt}(G)$ and $\gamma^g_{(k+1)t}(G)$, which, in turn, provides upper bounds for $\gamma^g_{(k+1)t}(G)$.

We first need to introduce some necessary definitions. Given $D \subseteq V$, a subset of the set of vertices V, let $N(D)$ be the set of vertices from $V \setminus D$ having at least one neighbor in D; that is, $N(D) = \{x \in V \setminus D \mid \exists\, y \in D$ such that $x \in N_G(y)\}$. Similarly, we denote by $\overline{N}(D)$ the set of vertices from $V \setminus D$ having at least one non-neighbor in D.

Now let A and B be subsets of set V. We will say that a subset $D \subseteq A$ is a relative dominating set of B from set A if for every $x \in B$ there exists at least one vertex $v \in D$ such that $v \in N(x)$ or $v \in B$. Correspondingly, we call the minimum cardinality of such a relative dominating set the relative domination number of set B from set A and denote it by $\gamma'(A,B)$. We abbreviate by $\gamma'(A,B)$-set a relative dominating set of B from set A of cardinality $\gamma'(A,B)$.

Finally, $\gamma'(\overline{A,B})$ is the relative domination number of B from set A in graph $\overline{G}$ and $\gamma'(\overline{A,B})$-set is a relative dominating set of B from set A in graph $\overline{G}$ with cardinality $\gamma'(\overline{A,B})$; see an example in Figure 5.

Lemma 1. *Let G be a graph with diam(G) = 2 and g(G) = 4, and let S be an induced subgraph isomorphic to C_4. Let $B = V \setminus (N(S) \cup S)$ and $A = N(B)$. Then $\gamma^g_{1t}(G) \leq \gamma'(A,B) + 4$.*

Proof. Let D' be a $\gamma'(A,B)$-set, $D = S \cup D'$ and $v \in V$. Note that since $diam(G) = 2$, $D' \subseteq A \subseteq N(S)$. Thus, we can see that $v \in N(S)$, $v \in B$ or $v \in S$. If $v \in N(S)$, then it has at least one neighbor in S and hence also in D. On the other hand, if $v \in B$, then v must have at least one neighbor in D' and hence also in D. If $v \in S$, then v has at least one neighbor in S, and hence also in D. Therefore, D is a total 1-dominating set of G.

If $v \in S$, then there exists one non-neighbor vertex of v in S, and hence also in D. If $v \in B$, then the four vertices in S are non-neighbors of v, and hence vertex v has at least one non-neighbor in set D. If $v \in N(S)$, since $g(G) = 4$, v it has at most two neighbors in S; thus, it has at least two non-neighbors in S and hence also in D. Therefore, D is a global 1-dominating set of G. Finally, D is a global total 1-dominating set of G, so $\gamma^g_{1t}(G) \leq \gamma'(A,B) + |S| = \gamma'(A,B) + 4$. □

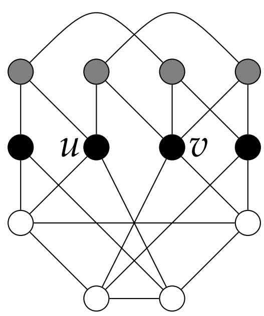

Figure 5. In the depicted graph G, the set S is formed by the white vertices, set A is formed by the black vertices and set B is formed by the gray vertices. Note that $\gamma'(A,B) = 2$ (the set $\{u,v\}$ is a $\gamma'(A,B)$-set) and $\gamma^g_{1t}(G) = 6$.

Corollary 5. *Let G be a graph with diam(G) = 2 and $g(G)$ = 4; let S be an induced subgraph isomorphic to C_4, $B = V \setminus (N(S) \cup S)$ and $A = N(B)$. Then the following conditions hold.*

- *If $B = \emptyset$, then $\gamma^g_{1t}(G) = 4$.*
- *Since $\gamma'(A,B) \leq |B|$, $\gamma^g_{1t}(G) \leq |B| + 4$.*

- *If* $|N(x) \cap S| = 2$, $\forall x \in A$, *then* $\gamma^g_{2t}(G) \leq 2|B| + 4$.

Let k be a positive integer with $1 \leq k < min\{\delta, n - \Delta - 1\}$, and D be a $\gamma^g_{kt}(G)$-set for graph G. Below we define special sets of vertices that will be used in future derivations.

- $H = V(G) \setminus D$.
- $Z = \{x \in H \mid \delta_D(x) \geq k+1$ and $\overline{\delta}_D(x) \geq k+1\}$ are all vertices in H which are global total $(k+1)$-dominated.
- $X = T_k(G[D])$ are all vertices in D with only k neighbors.
- $Y = T_{|D|-k-1}(G[D])$ are all vertices in D with only k non-neighbors.
- $X' = N(X) \cap H$ are all the vertices in H which have at least one neighbor in set X.
- $N = \gamma'(X', X)$-set, a relative dominating set of X from set X'.
- $Y' = \overline{N}(Y) \cap H$ are all the vertices in set H which have at least one non-neighbor in set Y.
- $R = \gamma'(\overline{Y', Y})$-set, a relative dominating set of X from set X' in $\overline{G}$.
- $P = H \setminus Z$ are all the vertices in set H which are not yet global total $(k+1)$-dominated.
- $M = \gamma'(H, P)$-set $\cup\ \gamma'(\overline{H, P})$-set;
- $S = D \cup N \cup R \cup M$;

Now we show that the set S obtained as above is a global total $(k+1)$-dominating set given a $\gamma^g_{kt}(G)$-set D.

Theorem 8. *Let G be a graph and D be an arbitrary $\gamma^g_{kt}(G)$-set. Then the set S obtained as above is a global total $(k+1)$-dominating set of graph G.*

Proof. Let D be an arbitrary $\gamma^g_{kt}(G)$-set, $H = V \setminus D$, $Z = \{x \in H$: $\delta_D(x) \geq k+1$ and $\overline{\delta}_D(x) \geq k+1\}$, $X = T_k(G[D])$ and $Y = T_{|D|-k-1}(G[D])$. Further, let $P = H \setminus Z$, E be a $\gamma'(H, P)$-set, F be a $\gamma'(\overline{H, P})$-set and $M = E \cup F$ (all these sets being constructed as above specified). If $X = \emptyset$ and $Y = \emptyset$, then every vertex from $D \cup Z$ has at least $k+1$ adjacent and $k+1$ non-adjacent vertices in set D. Besides, note that every vertex $v \in P$ has at least $k+1$ adjacent and $k+1$ non-adjacent vertices in set $D \cup M$. Additionally, since $V = D \cup Z \cup P$, $D \cup M$ is a global total $(k+1)$-dominating set of graph G.

Assume now that $X \neq \emptyset$ and $Y = \emptyset$, and let $X' = N(X) \cap H$ and N be a $\gamma'(X', X)$-set (notice that by the construction of the set X', there always exists the set N). Observe that every vertex from set $D \cup Z$ has at least $k+1$ adjacent and $k+1$ non-adjacent vertices in set $D \cup N$. Besides, every vertex $v \in P$ has at least $k+1$ adjacent and $k+1$ non-adjacent vertices in set $D \cup M$. Since $V = D \cup Z \cup P$, $D \cup N \cup M$ is a global total $(k+1)$-dominating set of G.

The case $X = \emptyset$ and $Y \neq \emptyset$ is analogous to the above case. We obtain that $D \cup R \cup M$ is a global total $(k+1)$-dominating set of G, where $Y' = \overline{N}(Y) \cap H$ and R is a $\gamma'(\overline{Y', Y})$-set.

Finally, assume that $X \neq \emptyset$ and $Y \neq \emptyset$. Let $X' = N(X) \cap H$, $Y' = \overline{N}(Y) \cap H$, N be a $\gamma'(X', X)$-set and R be a $\gamma'(\overline{Y', Y})$-set. Using a similar arguments as above, we again obtain that S is a global total $(k+1)$-dominating set of graph G. $\square$

In the next proposition we derive an upper bound on the cardinality of the global total $(k+1)$-domination number. In the same lemma, we give a necessary condition when the global total $(k+1)$-domination number is equal to the total $(k+1)$-domination number.

Proposition 8. *Let G be a graph with $\delta \geq k$ and D be a $\gamma^g_{kt}(G)$-set. Then the following conditions hold:*

(a) $\gamma^g_{(k+1)t}(G) \leq \gamma^g_{kt}(G) + |N \cup R \cup M|$.

(b) *If* $|N \cup M| > \Delta + k - \gamma^g_{kt}(G)$, *then* $\gamma^g_{(k+1)t}(G) = \gamma_{(k+1)t}(G)$.

Proof. (a) By Theorem 8, S is a global total $(k+1)$-dominating set of G; hence, the bound trivially holds.

(b) Recall that $|S| = \gamma^g_{kt}(G) + |N \cup R \cup M|$. Additionally, it is easy to see that $S \setminus R$ is a total $(k+1)$-dominating set of G. In [12] it is proved that if $\gamma_{kt}(G) > \Delta + k$, then $\gamma^g_{kt}(G) = \gamma_{kt}(G)$ (see Proposition 2.10). Hence, if $|S| \geq \gamma^g_{kt}(G) + |N \cup M| \geq \gamma_{(k+1)t}(G) > \Delta + k + 1$, then $\gamma^g_{(k+1)t}(G) = \gamma_{(k+1)t}(G)$. Hence, if $|N \cup M| > \Delta + k + 1 - \gamma^g_{kt}(G)$ then $\gamma^g_{(k+1)t}(G) = \gamma_{(k+1)t}(G)$. $\square$

Using the definition of the above introduced sets and Theorem 8 and Proposition 8, we can obtain a global total k-domination set for any $k = 2, \ldots, \min\{\delta, n - \Delta - 1\}$. As a side-result, we also obtain the corresponding upper bounds to a global total k-domination number. Finally, we note that this procedure provides a global total k-dominating set of minimum cardinality, $2 \leq k \leq min\{\delta, n - \Delta - 1\}$, for some graphs; see Figure 6.

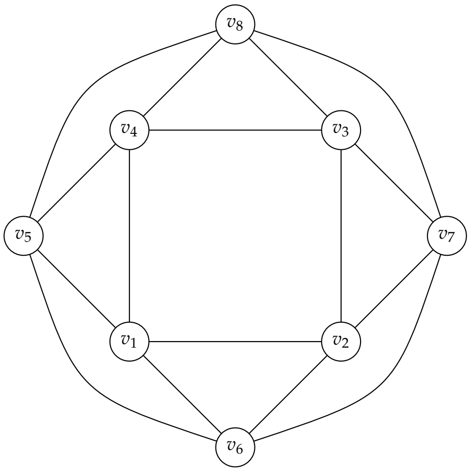

Figure 6. A graph G with $\gamma^g_{1t}(G) = 4$, $\gamma^g_{2t}(G) = 6$ and $\gamma^g_{3t}(G) = 8$. Note that if $D = \{v_1, v_2, v_3, v_4\}$ a $\gamma^g_{1t}(G)$-set, then $S = \{v_1, v_2, v_3, v_4, v_5, v_7\}$ which is a $\gamma^g_{2t}(G)$-set. Likewise, from S we construct $S' = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ which is a $\gamma^g_{3t}(G)$-set.

6. Conclusions

We studied the global total k-domination number in general graphs. In particular, we presented new upper and lower bounds using the algebraic connectivity in graphs. We also established a relationship between the global total k-domination numbers of the originally given graph G and the transformed ones. Then we derived an explicit relationship between a $\gamma^g_{kt}(G)$-set and a $\gamma^g_{(k+1)t}(G)$-set, which allowed us to obtain another upper bound for the global total k-domination number in a recurrent fashion, starting from $k = 1$. We gave an example of a graph G for which a $\gamma^g_{kt}(G)$-set, for every $k = 2, \ldots, \min\{\delta, n - \Delta - 1\}$ is provided. For future work, the global total k-domination number could be studied on unitary operations in graphs, such as edge subdivision, edge contraction, path contraction and removal of a vertex. It would be a challenging task to adopt the proposed method as such and also extend it for a wider class of graphs.

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