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Existence and Concentration Behavior of Solutions of the Critical Schrödinger–Poisson Equation in $\mathbb{R}^3$

Jichao Wang * and Ting Yu

College of Science, China University of Petroleum, Qingdao 266580, China; S20090010@s.upc.edu.cn
* Correspondence: wangjc@upc.edu.cn; Tel.: +86-1333-503-9103

Abstract: In this paper, we study the singularly perturbed problem for the Schrödinger–Poisson equation with critical growth. When the perturbed coefficient is small, we establish the relationship between the number of solutions and the profiles of the coefficients. Furthermore, without any restriction on the perturbed coefficient, we obtain a different concentration phenomenon. Besides, we obtain an existence result.

Keywords: Schrödinger–Poisson equation; critical growth; variational method

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1. Introduction

In this paper, we consider the Schrödinger–Poisson equation with critical growth:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \phi u = h(x)f(u) + g(x)u^5, & \text{in} \quad \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1}$$

The Schrödinger–Poisson equation arises while looking for standing wave solutions of a Schrödinger equation interacting with an electrostatic field. In recent years, many researchers are interested in semiclassical states of

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + K(x)\phi u = f(x,u), & \text{in} \quad \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi = K(x)u^2, & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{2}$$

which can be used to describe the transition from quantum to classical mechanics.

When $K = 0$, problem (2) reduces to the singularly perturbed problem

$$-\varepsilon^2\Delta u + V(x)u = g(u) \ \text{ in } \ \mathbb{R}^N. \tag{3}$$

In the past decade, there is a lot of results on problem (3). By using variational methods, Rabinowitz [1] first obtained the existence of solutions of (3) under the assumption

$$\liminf_{x\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) > 0. \tag{4}$$

In [2], Wang proved the concentration behavior of solutions of (3) as $\varepsilon \to 0$. In [3], del Pino and Felmer introduced a penalization approach and obtained a localized version of the results in [1,2]. In [4], Jeanjean and Tanaka extended the results of [3] to a more general case. For other related results, see [5–8] and the reference therein.

When $K \neq 0$, a lot of research focus on the case $K \equiv 1$, $f = u^p$ $(1 < p < 5)$. By using the Lyapunov–Schmidt reduction method, the authors in [9,10] obtained positive bound state solutions and multi-bump solutions concentrating around a local minimum of the potential V. In [11,12], the authors proved the existence of radically symmetric solutions concentrating on the spheres. It should be pointed it out that, the Lyapunov–Schmidt reduction method is based on the uniqueness or non-degeneracy of solutions of the corresponding limiting equation. Recently, by using variational methods, he [13] considered the subcritical problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = f(u), & \text{in} \quad \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{5}$$

Under the assumption (4), he related the number of solutions with the topology of the set where V attains its minimum and obtained the multiplicity of positive solutions. Subsequently, the authors in [14] studied the problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \lambda \phi u = b(x)f(u), & \text{in} \quad \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{6}$$

Under suitable assumptions on λ, V, b and f, they proved the existence and concentration behavior of positive ground state solutions. For the critical case, He and Zou [15] studied the Schrödinger–Poisson equation

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = f(u) + u^5, & \text{in} \quad \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{7}$$

By using (4), they obtained the ground state solution concentrating around the global minimum of the potential V. Furthermore, in [16,17], the authors considered the existence, multiplicity and concentration behavior of the critical Schrödinger–Poisson equation

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u + \phi u = f(u) + u^5, & \text{in} \quad \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{8}$$

Motivated by the above results, in this paper, we study the multiplicity and concentration behavior of positive solutions of (1). Before stating the results, we introduce the following conditions:

(h_1) $h(x) \in C(\mathbb{R}^3, \mathbb{R})$, $h(x) \geq 0$ and $\lim_{|x|\to\infty} h(x) = h_\infty > 0$. Moreover, there exist k points $x^1, x^2, \ldots x^k$ in $\mathbb{R}^3$ such that each point is the strict global maximum of h.
(V_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) := V_0 > 0$.
(V_2) $V(x^i) = V_0$, $i = 1, 2, \ldots k$ and $\lim_{|x|\to\infty} V(x) = V_\infty > 0$.
(g_1) $g(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $0 \leq g(x) \leq 1$. Moreover, $g(x^i) = 1$, $i = 1, 2, \ldots k$ and $\lim_{|x|\to\infty} g(x) = g_\infty > 0$.
(f_1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$ and $\lim_{u\to 0+} \frac{f(u)}{u^3} = \lim_{u\to+\infty} \frac{f(u)}{u^5} = 0$. Moreover, $\frac{f(u)}{u^3}$ is increasing for $u > 0$ and $\lim_{u\to+\infty} \frac{f(u)}{u^3} = +\infty$.

Theorem 1. *Assume that (h_1), (V_1)-(V_2), (g_1) and (f_1) hold. Then there exists $\varepsilon^* > 0$ such that problem (1) has at least k different positive solutions w^i_ε, $i = 1, 2, \ldots k$ for $\varepsilon \in (0, \varepsilon^*)$. Moreover, w^i_ε possesses a maximum point $y^i_\varepsilon \in \mathbb{R}^3$ satisfying $h(y^i_\varepsilon) \to \sup_{x\in\mathbb{R}^3} h(x)$ as $\varepsilon \to 0$. Besides, there exist $C^i, c^i > 0$ such that for $\varepsilon \in (0, \varepsilon^*)$,*

$$w^i_\varepsilon(x) \leq C^i \exp\left(-c^i \frac{|x - y^i_\varepsilon|}{\varepsilon}\right), \quad x \in \mathbb{R}^3.$$

Remark 1. *In Theorem 1, we obtain the existence of spikes (multiple solutions concentrating at a single point) on the strict global maximum of h. The behavior of solutions of (1) describes the transition between quantum mechanics and classical mechanics in some sense.*

Remark 2. *Some ideas to prove Theorem 1 come from [18], where the authors studied the subcritical problem*

$$-\Delta u + \mu u = Q(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

In [18], the authors imposed the condition

$$\sup_{x\in\mathbb{R}^N} Q(x) > \limsup_{|x|\to\infty} Q(x) > 0, \tag{9}$$

which plays an important role in proving the compactness of the Palais–Smale sequences. This type condition is first introduced by Rabinowitz in [1]. We pointed out that, when we seek multiplicity of solutions, it is crucial to prove the compactness of the Palais–Smale sequence. Many authors solved the problem by imposing the Rabinowitz type assumption, which is restrictive. Similar results can be found in [13,15,16] and the reference therein. In this paper, by estimating the Palais–Smale sequences delicately, we remove this technical condition. In fact, we use a different argument. Compared with the existing results, in this paper, we also need to study the influence of the variable coefficient of the critical term on the problem.

Inspired by Theorem 1, a natural question is whether (1) has multiple solutions without any restriction on ε. In this paper, by using the Lusternik–Schnirelman category, we obtain a new result. We assume the following conditions:

(h_2) $h(x) \in C(\mathbb{R}^3, \mathbb{R})$ is positive and $\sup_{x\in\mathbb{R}^3} h(x) := h_M < +\infty$.

(Vh) $\lim_{R\to+\infty} \sup_{|x|\geq R} \frac{h(x)}{V(x)} = 0$.

(g_2) $g(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $0 \leq g(x) \leq 1$. Moreover, there exists $\rho_0 > 0$ such that $g(x) = 1$ for $\rho_0 < |x| < 2\rho_0$.

Theorem 2. *Let $\varepsilon > 0$. Assume that (h_2), (V_1), (Vh), (g_2) and (f_1) hold with $g(0) < 1$. Then there exists $h_0 > 0$ such that for $\|h\|_\infty < h_0$, problem (1) has two positive solutions $u_{i,h}$, $i = 1, 2$.*

When $\varepsilon = 1$, there are an enormous amount of papers studying problem (1) or the more general form

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u), & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & \text{in} \quad \mathbb{R}^3. \end{cases} \tag{10}$$

Many papers, see for example, [19–27], focus on the case V and K being positive constant or radially symmetric functions, $f = |u|^{p-2}u$ or $f = f(u)$. If V is non-radial, $K \equiv 1$, $f = |u|^{p-2}u$, the authors in [20,28] obtained the existence of ground state solutions of (10) for $p \in (3,6)$. If $V \equiv 1$, $f = a(x)|u|^{p-2}u$ is non-radial, by requiring suitable assumptions on K and a, the authors in [29] obtained ground state and bound state solutions. For other related results, see [30–33] and the reference therein. Usually, in order to ensure the boundedness of the Palais–Smale sequences, the Ambrosetti-Rabinowitz condition or some monotonicity condition on f is needed. It is natural to ask whether we can prove the boundedness of the Palais–Smale sequences without the above restrict conditions. When we study (1), we solve the problem under mild conditions and get an interesting result. Instead of (g_2) and (f_1), we assume the following conditions.

(g_2') $g(x) \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} g(x) > 0$. Moreover, $g(x) \leq 1$ and there exists $\rho_0 > 0$ such that $g(x) = 1$ for $\rho_0 < |x| < 2\rho_0$.

(f_1') $f \in C(\mathbb{R}^+, \mathbb{R})$ and $f(u) \geq 0$ for $u \geq 0$. Moreover, $\lim_{u\to 0+} \frac{f(u)}{u^3} = \lim_{u\to+\infty} \frac{f(u)}{u^5} = 0$ and $\lim_{u\to+\infty} \frac{f(u)}{u^3} = +\infty$.

Theorem 3. *Let $\varepsilon > 0$. Assume that (h_2), (V_1), (Vh), (g_2') and (f_1') hold. Then problem (1) has a positive solution.*

The outline of this paper is as follows: in Section 2, we give some important lemmas; in Section 3, we prove Theorems 1; in Section 4, we prove Theorem 2 and 3; in Section 5, we make the conclusions.

Notations:

- $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{\frac{1}{s}}, 1 \le s \le +\infty$.
- $H^1 = H^1(\mathbb{R}^3)$ denotes the Hilbert space equipped with the inner product $\langle u, v\rangle = \int_{\mathbb{R}^3}(\nabla u \nabla vs. + uv)\mathrm{d}x$ and the norm $\|u\|^2_{H^1} = \int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)\mathrm{d}x$, $D^{1,2} = D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ denotes the Sobolev space equipped with the norm $\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x$.
- $S = \inf_{u \in D^{1,2}\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}}}$ denotes the best Sobolev constant.
- C denotes a positive constant (possibly different).

2. Preliminary Lemmas

Since we look for positive solutions, we assume $f(u) = 0$ for $u \le 0$. Make the change of variable $x \to \varepsilon x$, problem (1) becomes

$$-\Delta u + V(\varepsilon x)u + \phi_u u = h(\varepsilon x) f(u) + g(\varepsilon x)u^5 \quad \text{in } \mathbb{R}^3. \tag{11}$$

For any $\varepsilon > 0$, define $H_\varepsilon = \left\{u \in H^1 : \int_{\mathbb{R}^3} V(\varepsilon x)|u|^2 \mathrm{d}x < +\infty\right\}$ the Hilbert space with the inner product $\langle u, v\rangle_\varepsilon = \int_{\mathbb{R}^3}(\nabla u \nabla vs. + V(\varepsilon x)uv)\mathrm{d}x$ and the norm $\|u\|_\varepsilon = (\int_{\mathbb{R}^3} |\nabla u|^2 + V(\varepsilon x)|u|^2 \mathrm{d}x)^{\frac{1}{2}}$. By the Lax–Milgram theorem, for any $u \in H_\varepsilon$, there exists a unique $\phi_u \in D^{1,2}$ satisfying $-\Delta\phi_u = u^2$. Moreover, by [20,26,28], we have the following results.

Lemma 1.

(i) $\phi_u \ge 0$ *and* $\phi_{tu} = t^2\phi_u$ *for any* $t \in \mathbb{R}$.

(ii) *If* $y \in \mathbb{R}^3$ *and* $\tilde{u}(x) = u(x + y)$, *then* $\phi_{\tilde{u}}(x) = \phi_u(x + y)$ *and* $\int_{\mathbb{R}^3} \phi_{\tilde{u}}\tilde{u}^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x$.

(iii) *If* $u_n \rightharpoonup u$ *weakly in* H^1, *then* $\phi_{u_n} \rightharpoonup \phi_u$ *weakly in* $D^{1,2}$. *Moreover, let* $v_n = u_n - u$, *then*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \mathrm{d}x + o_n(1),$$
$$\left|\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)\varphi \mathrm{d}x - \int_{\mathbb{R}^3} \phi_{v_n} v_n \varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_{H^1}, \quad \forall\, \varphi \in H^1.$$

(iv) $\|\phi_u\|^2_{D^{1,2}} = \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x \le \frac{\|u\|^4_{\frac{12}{5}}}{S}$.

The functional associated with (11) is

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} h(\varepsilon x)F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g(\varepsilon x)|u|^6 \mathrm{d}x, \tag{12}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$. Obviously, $I_\varepsilon : H_\varepsilon \mapsto \mathbb{R}$ is of class C^1 and critical points of I_ε are weak solutions of (11). Let $m_\varepsilon = \inf\{I_\varepsilon(u) : u \in M_\varepsilon\}$, where $M_\varepsilon = \{u \in H_\varepsilon \setminus \{0\} : (I'_\varepsilon(u), u) = 0\}$.

From [34], we know S is attained by $\frac{\delta^{\frac{1}{4}}}{\left(\delta+|x|^2\right)^{\frac{1}{2}}}$, where $\delta > 0$. Let $u_{\delta,z}(x) = \frac{\psi(x)\delta^{\frac{1}{4}}}{\left(\delta+|x-z|^2\right)^{\frac{1}{2}}}$, where $\psi \in C_0^\infty(B_{2r}(z))$ such that $\psi(x) = 1$ on $B_r(z)$, $0 \le \psi(x) \le 1$ and $|\nabla\psi| \le 2$. By the direct calculation, we have the following results.

Lemma 2. *For $\delta > 0$ small,*

$$\int_{\mathbb{R}^3} |\nabla u_{\delta,z}|^2 \mathrm{d}x = K_1 + O(\delta^{\frac{1}{2}}), \quad \int_{\mathbb{R}^3} |u_{\delta,z}|^6 \mathrm{d}x = K_2 + O(\delta^{\frac{3}{2}}),$$
$$\int_{\mathbb{R}^3} |u_{\delta,z}|^t \mathrm{d}x = O\left(\delta^{\frac{t}{4}}\right), \qquad t \in (1,3),$$

where $S = \frac{K_1}{K_2^{\frac{1}{3}}}$.

3. Proof of Theorem 1

By (f_1), we derive that

$$\frac{1}{4} f(u)u \geq F(u) \geq 0, \ \ f'(u)u^2 - 3f(u)u \geq 0, \ \ \forall\, u \in \mathbb{R}. \tag{13}$$

By (h_1), we have $h(x^i) = h_M$, $i = 1, 2, \dots k$. We consider the equation:

$$-\Delta u + V_0 u + \phi_u u = h_M f(u) + u^5 \ \text{ in } \ \mathbb{R}^3. \tag{14}$$

Let $\|u\|_{V_0} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V_0|u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. Then the functional of (14) is

$$\hat{I}(u) = \frac{1}{2}\|u\|_{V_0}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} h_M F(u) \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x.$$

Let $\hat{m} = \inf\{\hat{I}(u) : u \in \hat{N}\}$, where $\hat{N} = \{u \in H^1 \setminus \{0\} : (\hat{I}'(u), u) = 0\}$. It is well known that $\hat{m}$ is attained by w. Moreover, $\hat{m} \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $\hat{I}'(w) = 0$. Let $\|u\|_{V_\infty} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V_\infty |u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. Define the functional on H^1 by

$$\check{I}(u) = \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} h_\infty F(u) \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g_\infty |u|^6 \mathrm{d}x.$$

Let $m_\infty = \inf\{\check{I}(u) : u \in N_\infty\}$, where $N_\infty = \{u \in H^1 \setminus \{0\} : (\check{I}'(u), u) = 0\}$. For any $v \in N_\infty$, by (f_1), we have $\check{I}(v) = \sup_{t \geq 0} \check{I}(tv)$. Moreover, there exists a unique $t_v > 0$ satisfying $t_v vs. \in \hat{N}$. Then by $V_\infty \geq V_0$, $h_\infty \leq h_M$, $g_\infty \leq 1$, we get

$$\check{I}(v) = \sup_{t \geq 0} \check{I}(tv) \geq \check{I}(t_v v) \geq \hat{I}(t_v v) \geq \hat{m}.$$

So $m_\infty \geq \hat{m}$. Similarly, we have $m_\varepsilon \geq \hat{m}$.

For $\eta > 0$, denote $C_\eta(x^i)$ the hypercube $\Pi_{j=1}^3 (x_j^i - \eta, x_j^i + \eta)$ centered at $x^i = (x_1^i, x_2^i, x_3^i)$, $i = 1, 2, \dots k$. Denote $\overline{C_\eta(x^i)}$ and $\partial C_\eta(x^i)$ the closure and the boundary of $C_\eta(x^i)$, respectively. By (h_1), we have $h(x^i) = h_M$. Moreover, there exist $\eta, L_0 > 0$ such that $\overline{C_\eta(x^i)}$, $i = 1$, $2, \dots k$ are disjoint, $h(x) < h(x^i)$ for $x \in \overline{C_\eta(x^i)} \setminus x^i$ and $\overline{C_\eta(x^i)} \subset \Pi_{j=1}^3(-L_0, L_0)$.

Let $\varepsilon \in (0,1)$. Define $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^3)$ such that $\varphi_\varepsilon(x) = 1$ for $|x| < \frac{1}{\sqrt{\varepsilon}} - 1$, $\varphi_\varepsilon(x) = 0$ for $|x| > \frac{1}{\sqrt{\varepsilon}}$, $0 \leq \varphi_\varepsilon \leq 1$ and $|\nabla \varphi_\varepsilon| \leq 2$. Let $\zeta_\varepsilon^i(x) = w\left(x - \frac{x^i}{\varepsilon}\right)\varphi_\varepsilon\left(x - \frac{x^i}{\varepsilon}\right)$, $i = 1, 2, \dots k$. Let $\rho > \max\{|x^1| + \eta, |x^2| + \eta, \dots, |x^k| + \eta\}$. Define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\chi(x) = x$ for $|x| < \rho$, $\chi(x) = \frac{\rho x}{|x|}$ for $|x| \geq \rho$. For $u \in H^1 \setminus \{0\}$, define $\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)|u|^6 \mathrm{d}x}{\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x}$. Then β_ε is continuous in $H^1 \setminus \{0\}$.

For simplicity, denote $C_\eta(x^i) = C_\eta^i$. Set $M_\varepsilon^i = \left\{u \in M_\varepsilon : u \geq 0, \ \beta_\epsilon(u) \in C_\eta^i\right\}$, $\partial M_\varepsilon^i = \left\{u \in M_\varepsilon : u \geq 0, \ \beta_\varepsilon(u) \in \partial C_\eta^i\right\}$, $i = 1, 2, \dots k$. Let $\gamma_\varepsilon^i = \inf_{u \in M_\varepsilon^i} I_\varepsilon(u)$, $\tilde{\gamma}_\varepsilon^i = \inf_{u \in \partial M_\varepsilon^i} I_\varepsilon(u)$, $i = 1, 2, \dots k$.

Lemma 3. *Let $i = 1, 2, \dots k$. Then for any $\nu \in (0, \hat{m})$, there exists $\varepsilon_\nu > 0$ such that $m_\varepsilon \le \gamma_\varepsilon^i < \hat{m} + \nu$ for $\varepsilon \in (0, \varepsilon_\nu)$.*

Proof of Lemma 3. It is clear that $\gamma_\varepsilon^i \ge m_\varepsilon$. Let $l_\varepsilon(t) = I_\varepsilon(t\zeta_\varepsilon^i)$, where $t > 0$. By (f_1), we derive that $l_\varepsilon(t)$ admits a unique critical point $t_\varepsilon^i > 0$ corresponding to its maximum. Then $l_\varepsilon(t_\varepsilon^i) = \sup_{t\ge 0} l_\varepsilon(t\zeta_\varepsilon^i)$, $l'_\varepsilon(t_\varepsilon^i) = 0$. So $t_\varepsilon^i \zeta_\varepsilon^i \in M_\varepsilon$. Note that $\lim_{\varepsilon\to 0} \|w\varphi_\varepsilon - w\|_{V_0} = 0$. By the Lebesgue dominated convergence theorem,

$$\lim_{\varepsilon\to 0} \beta_\varepsilon\left(t_\varepsilon^i \zeta_\varepsilon^i\right) = \lim_{\varepsilon\to 0} \beta_\varepsilon\left(\zeta_\varepsilon^i\right) = x^i.$$

Then $\beta_\varepsilon(t_\varepsilon^i\zeta_\varepsilon^i) \in C_\eta^i$ for $\varepsilon > 0$ small. Since $t_\varepsilon^i\zeta_\varepsilon^i \in M_\varepsilon$, by the definition of γ_ε^i, we get $\gamma_\varepsilon^i \le I_\varepsilon(t_\varepsilon^i\zeta_\varepsilon^i) = \sup_{t\ge 0} I_\varepsilon(t\zeta_\varepsilon^i)$. So we just prove $I_\varepsilon(t_\varepsilon^i\zeta_\varepsilon^i) < \hat{m} + \nu$ for $\varepsilon > 0$ small. By $\lim_{\varepsilon\to 0} \|w\varphi_\varepsilon - w\|_{V_0} = 0$ and Lemma 1 (ii), we derive that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\nabla \zeta_\varepsilon^i|^2 \mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} |\nabla (w\varphi_\varepsilon)|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |\nabla w|^2 \mathrm{d}x, \\ &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi_{\zeta_\varepsilon^i} (\zeta_\varepsilon^i)^2 \mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi_{w\varphi_\varepsilon} (w\varphi_\varepsilon)^2 \mathrm{d}x = \int_{\mathbb{R}^3} \phi_w w^2 \mathrm{d}x. \end{aligned} \tag{15}$$

By $V(x^i) = V_0$, $h(x^i) = h_M$, $g(x^i) = 1$ and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} V(\varepsilon x) |\zeta_\varepsilon^i|^2 \mathrm{d}x = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} V(\varepsilon x + x^i) |w\varphi_\varepsilon|^2 \mathrm{d}x = \int_{\mathbb{R}^3} V_0 |w|^2 \mathrm{d}x, \\ &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} g(\varepsilon x) |\zeta_\varepsilon^i|^6 \mathrm{d}x = \int_{\mathbb{R}^3} |w|^6 \mathrm{d}x, \ \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} h(\varepsilon x) F(\zeta_\varepsilon^i) \mathrm{d}x = \int_{\mathbb{R}^3} h_M F(w) \mathrm{d}x, \\ &\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} h(\varepsilon x) f(\zeta_\varepsilon^i) \zeta_\varepsilon^i \mathrm{d}x = \int_{\mathbb{R}^3} h_M f(w) w \mathrm{d}x. \end{aligned} \tag{16}$$

By $t_\varepsilon^i\zeta_\varepsilon^i \in M_\varepsilon^i$, we have

$$\begin{aligned} &(t_\varepsilon^i)^2 \|\zeta_\varepsilon^i\|_\varepsilon^2 + (t_\varepsilon^i)^4 \int_{\mathbb{R}^3} \phi_{\zeta_\varepsilon^i} (\zeta_\varepsilon^i)^2 \mathrm{d}x \\ &= \int_{\mathbb{R}^3} h(\varepsilon x) f(t_\varepsilon^i\zeta_\varepsilon^i)(t_\varepsilon^i\zeta_\varepsilon^i) \mathrm{d}x + (t_\varepsilon^i)^6 \int_{\mathbb{R}^3} g(\varepsilon x) |\zeta_\varepsilon^i|^6 \mathrm{d}x \ge (t_\varepsilon^i)^6 \int_{\mathbb{R}^3} g(\varepsilon x) |\zeta_\varepsilon^i|^6 \mathrm{d}x. \end{aligned} \tag{17}$$

From (15)–(17), we obtain that t_ε^i is bounded. Furthermore, by (17), (f_1), (h_1), (g_1), for $\eta = \frac{V_0}{2}$, there exists $C_\eta = C_{\frac{V_0}{2}} > 0$ satisfying

$$(t_\varepsilon^i)^2 \|\zeta_\varepsilon^i\|_{V_0}^2 \le \frac{V_0 (t_\varepsilon^i)^2}{2} \int_{\mathbb{R}^3} |\zeta_\varepsilon^i|^2 \mathrm{d}x + C_{\frac{V_0}{2}} (t_\varepsilon^i)^6 \int_{\mathbb{R}^3} |\zeta_\varepsilon^i|^6 \mathrm{d}x.$$

Then $t_\varepsilon^i \to t^i > 0$ in view of (15) and (16). By (15)–(17),

$$\|t^i w\|_{V_0}^2 = \int_{\mathbb{R}^3} \left(h_M f(t^i w) t^i w + |t^i w|^6\right) \mathrm{d}x,$$

that is, $t^i w \in \hat{N}$. By (f_1), there exists a unique $t > 0$ satisfying $tw \in \hat{N}$. Since $w \in \hat{N}$, we have $t^i = 1$. Then by (15) and (16),

$$I_\varepsilon(t_\varepsilon^i\zeta_\varepsilon^i) = I_\varepsilon(\zeta_\varepsilon^i) + O(\varepsilon) = \hat{I}(w) + O(\varepsilon) = \hat{m} + O(\varepsilon).$$

So there exists $\varepsilon_\nu > 0$ such that $\gamma_\varepsilon^i < \hat{m} + \nu$ for $\varepsilon \in (0, \varepsilon_\nu)$. $\square$

Lemma 4. *Let $i = 1, 2, \dots k$. Then there exist $\delta, \varepsilon_\delta > 0$ such that $\widetilde{\gamma}_\varepsilon^i > \hat{m} + \delta$ for $\varepsilon \in (0, \varepsilon_\delta)$.*

Proof of Lemma 4. Assume to the contrary that there exists $\varepsilon_n \downarrow 0$ such that $\widetilde{\gamma}^i_{\varepsilon_n} \to c \le \hat{m}$. Then there exists $\{u_n\} \subset \partial M^i_{\varepsilon_n}$ satisfying $I_{\varepsilon_n}(u_n) \to c \le \hat{m}$. By (f_1), there exists a unique $t_n > 0$ such that $t_n u_n \in \hat{N}$. Then

$$\begin{aligned}\hat{m} \ge I_{\varepsilon_n}(u_n) + o_n(1) =& \sup_{t\ge 0} I_{\varepsilon_n}(tu_n) + o_n(1)\\ \ge& I_{\varepsilon_n}(t_n u_n) + o_n(1) \ge \hat{I}(t_n u_n) + o_n(1) \ge \hat{m} + o_n(1),\end{aligned}$$

from which we get $\hat{I}(t_n u_n) = \hat{m} + o_n(1)$. Moreover,

$$\int_{\mathbb{R}^3} (h(\varepsilon_n x) - h_M) F(t_n u_n) \mathrm{d}x = o_n(1). \tag{18}$$

Since $t_n u_n \in \hat{N}$, by the Ekeland's variational principle, there exist $\{v_n\} \subset \hat{N}$, $\mu_n \in \mathbb{R}$ satisfying $\|v_n - t_n u_n\|_{V_0} = o_n(1)$, $\hat{I}(v_n) = \hat{m} + o_n(1)$, $\hat{I}'(v_n) - \mu_n \hat{G}'(v_n) = o_n(1)$, where $\hat{G}(v_n) = (\hat{I}'(v_n), v_n)$. By the standard argument, we derive that $\mu_n \to 0$ as $n \to \infty$. Then

$$\hat{I}(v_n) = \hat{m} + o_n(1), \quad \hat{I}'(v_n) = o_n(1). \tag{19}$$

By (13), we get $\|v_n\|_{V_0}$ is bounded. By the Lions Lemma, $\int_{\mathbb{R}^3} |v_n|^t \mathrm{d}x \to 0$ for any $t \in (2,6)$, or there exists $y_n \in \mathbb{R}^3$ such that $w_n = v_n(. + y_n) \rightharpoonup w \ne 0$ weakly in $H^1(\mathbb{R}^3)$. If $\int_{\mathbb{R}^3} |v_n|^t \mathrm{d}x \to 0$ for any $t \in (2,6)$, then $\int_{\mathbb{R}^3} F(v_n)\mathrm{d}x = \int_{\mathbb{R}^3} f(v_n)v_n \mathrm{d}x = o_n(1)$. Let $\int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x \to l$. Then $\lim_{n\to\infty} \|v_n\|^2_{V_0} \le l$. Since $\hat{m} > 0$, we have $l > 0$. By $S \le \frac{\|v_n\|^2_{V_0}}{(\int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x)^{\frac{1}{3}}}$, we get $l \ge S^{\frac{3}{2}}$. Thus, $\hat{m} = \hat{I}(v_n) - \frac{1}{4}(\hat{I}'(v_n), v_n) + o_n(1) \ge \frac{1}{3} S^{\frac{3}{2}} + o_n(1)$, a contradiction with $\hat{m} < \frac{1}{3} S^{\frac{3}{2}}$. So $w_n \rightharpoonup w \ne 0$ weakly in H^1. By (19), we have $\hat{I}(w_n) = \hat{m} + o_n(1)$, $\hat{I}'(w_n) = o_n(1)$. Then $\hat{I}'(w) = 0$. Moreover,

$$\begin{aligned}\hat{m} + o_n(1) =& \hat{I}(w_n) - \frac{1}{4}(\hat{I}'(w_n), w_n)\\ =& \frac{1}{4}\|w_n\|^2_{V_0} + \frac{1}{12}\int_{\mathbb{R}^3} |w_n|^6 \mathrm{d}x + \int_{\mathbb{R}^3} h_M \left(\frac{1}{4} f(w_n)w_n - F(w_n)\right)\mathrm{d}x.\end{aligned}$$

By Fatou's Lemma,

$$\begin{aligned}\hat{m} \ge& \frac{1}{4}\|w\|^2_{V_0} + \frac{1}{12}\int_{\mathbb{R}^3} |w|^6 \mathrm{d}x + \int_{\mathbb{R}^3} h_M \left(\frac{1}{4} f(w)w - F(w)\right)\mathrm{d}x\\ =& \hat{I}(w) - \frac{1}{4}(\hat{I}'(w), w) = \hat{I}(w) \ge \hat{m}.\end{aligned}$$

So $w_n \to w$ in H^1. Note that $\beta_{\varepsilon_n}(t_n u_n) = \beta_{\varepsilon_n}(u_n) \in \partial C^i_\eta$. By $\|v_n - t_n u_n\|_{V_0} \to 0$, we get $\beta_{\varepsilon_n}(v_n) \to z_0 \in \partial C^i_\eta$. Thus,

$$z_0 = \lim_{n\to\infty} \frac{\int_{\mathbb{R}^3} \chi(\varepsilon_n x)|v_n|^6 \mathrm{d}x}{\int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x} = \lim_{n\to\infty} \frac{\int_{\mathbb{R}^3} \chi(\varepsilon_n x + \varepsilon_n y_n)|w_n|^6 \mathrm{d}x}{\int_{\mathbb{R}^3} |w_n|^6 \mathrm{d}x}. \tag{20}$$

If $|\varepsilon_n y_n| \to \infty$, by $w_n \to w$ in H^1, we get $|z_0| = \rho$, a contradiction with $z_0 \in \partial C^i_\eta$ and $\rho > |x^i| + \eta$. So $|\varepsilon_n y_n|$ is bounded. Assume that $\varepsilon_n y_n \to y_0$ as $n \to \infty$. If $|y_0| \ge \rho$, by (20), we get $|z_0| = \rho$, a contradiction. If $|y_0| < \rho$, we have $y_0 = z_0 \in \partial C^i_\eta$. Then $h(y_0) < h_M$. By $\|v_n - t_n u_n\|_{V_0} \to 0$, $w_n = v_n(. + y_n) \to w$ in H^1,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} h_M F(t_n u_n)\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^3} h_M F(v_n)\mathrm{d}x = \int_{\mathbb{R}^3} h_M F(w)\mathrm{d}x.$$

Furthermore,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} h(\varepsilon_n x)F(t_n u_n)\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^3} h(\varepsilon_n x+\varepsilon_n y_n)F(v_n(.+y_n))\mathrm{d}x$$
$$= \int_{\mathbb{R}^3} h(y_0)F(w)\mathrm{d}x.$$

Then by (18), we get $\int_{\mathbb{R}^3} h_M F(w)\mathrm{d}x = \int_{\mathbb{R}^3} h(y_0)F(w)\mathrm{d}x$, a contradiction with $h(y_0) < h_M$. $\square$

By Lemmas 3 and 4, there exists $\varepsilon_1 > 0$ such that $\hat{m} \le \gamma_\varepsilon^i < \tilde{\gamma}_\varepsilon^i$ for $\varepsilon \in (0,\varepsilon_1)$. Furthermore, $\gamma_\varepsilon^i \to \hat{m}$ as $\varepsilon \to 0$. By (f_1), for $\xi = \frac{V_\infty}{2h_\infty}$, there exists $C_\xi = C_{\frac{V_\infty}{2h_\infty}} > 0$ such that

$$\max\{|F(u)|, |f(u)u|\} \le \frac{V_\infty}{2h_\infty}|u|^2 + C_{\frac{V_\infty}{2h_\infty}}|u|^6. \tag{21}$$

Note that $\hat{m} < \frac{1}{3}S^{\frac{3}{2}}$. We choose $\varepsilon_0 \in (0,\varepsilon_1)$ small such that for $\varepsilon \in (0,\varepsilon_0)$,

$$\gamma_\varepsilon^i < \min\left\{\frac{1}{3}S^{\frac{3}{2}}, \frac{S^{\frac{3}{2}}}{4\sqrt{2\left(h_\infty C_{\frac{V_\infty}{2h_\infty}}+1\right)}} + \hat{m}\right\}. \tag{22}$$

Following the ideas of [18,35], we can use the implicit function theorem to get the following result. Since the proof is standard, we omit it here.

Lemma 5. *Let $\varepsilon \in (0,\varepsilon_0)$, $i = 1,2,\dots k$. Then for any $u \in M_\varepsilon^i$, there exist $\rho > 0$ and a differential function $s(w) > 0$, where $w \in H_\varepsilon$ and $\|w\|_\varepsilon < \rho$, satisfying $s(0) = 1$, $s(w)(u+w) \in M_\varepsilon^i$. Moreover, for any $\varphi \in H_\varepsilon$,*

$$(s'(0),\varphi)$$
$$= \frac{-2\langle u,\varphi\rangle_\varepsilon - 4\int_{\mathbb{R}^3}\phi_u u\varphi\mathrm{d}x + \int_{\mathbb{R}^3}(h(\varepsilon x)f(u) + h(\varepsilon x)f'(u)u + 6g(\varepsilon x)u^5)\varphi\mathrm{d}x}{2\|u\|_\varepsilon^2 + 4\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \int_{\mathbb{R}^3} h(\varepsilon x)(f(u)u + f'(u)u^2) + 6g(\varepsilon x)|u|^6\mathrm{d}x},$$

that is, $(s'(0),\varphi) = \frac{-(G_\varepsilon'(u),\varphi)}{(G_\varepsilon'(u),u)}$, where $G_\varepsilon(u) = (I_\varepsilon'(u),u)$.

Lemma 6. *Let $\varepsilon \in (0,\varepsilon_0)$, $i = 1,2,\dots k$. Then there exists $\{u_n\} \subset M_\varepsilon^i$ satisfying $I_\varepsilon(u_n) \to \gamma_\varepsilon^i$, $I_\varepsilon'(u_n) \to 0$. Moreover, u_n converges strongly in H_ε up to a subsequence.*

Proof of Lemma 6. By the definition of γ_ε^i, there is $u_n \in M_\varepsilon^i$ satisfying $I_\varepsilon(u_n) \to \gamma_\varepsilon^i$. Then $\|u_n\|_\varepsilon$ is bounded in view of (13). By the Ekeland's variational principle, $I_\varepsilon(u_n) \le \gamma_\varepsilon^i + \frac{1}{n}$, $I_\varepsilon(v) \ge I_\varepsilon(u_n) - \frac{1}{n}\|u_n - v\|_\varepsilon$ for any $v \in M_\varepsilon^i$. By Lemma 5, there exist $\rho_n \downarrow 0$ and $s_n(w) > 0$ satisfying $s_n(w)(u_n+w) \in M_\varepsilon^i$ for any $w \in H_\varepsilon$ with $\|w\|_\varepsilon < \rho_n$. Let $w = t\phi$, where $\phi \in H_\varepsilon$, $t > 0$. For $t > 0$ small,

$$\begin{aligned}
&\frac{1}{n}[ts_n(t\phi)\|\phi\|_\varepsilon + |s_n(t\phi)-1|\|u_n\|_\varepsilon] \\
&\ge \frac{1}{n}\|u_n - s_n(t\phi)(u_n+t\phi)\|_\varepsilon \\
&\ge I_\varepsilon(u_n) - I_\varepsilon(s_n(t\phi)(u_n+t\phi)) \\
&= [I_\varepsilon(u_n) - I_\varepsilon(u_n+t\phi)] \\
&\quad + (1-s_n(t\phi))\big(I_\varepsilon'(\theta_n(u_n+t\phi) + (1-\theta_n)(s_n(t\phi)(u_n+t\phi))), u_n+t\phi\big),
\end{aligned}$$

where $\theta_n \in (0,1)$. Dividing by t and let $t \to 0$, $\frac{1}{n}[|(s_n'(0),\phi)|\|u_n\|_\varepsilon + \|\phi\|_\varepsilon] \geq -(I_\varepsilon'(u_n),\phi)$. By Lemma 5, $(s_n'(0),\phi) = \frac{-(G_\varepsilon'(u_n),\phi)}{(G_\varepsilon'(u_n),u_n)}$. By (13),

$$\begin{aligned}(G_\varepsilon'(u_n),u_n) &= (G_\varepsilon'(u_n),u_n) - 4(I_\varepsilon'(u_n),u_n)\\ &\leq -2\|u_n\|_\varepsilon^2 - 2\int_{\mathbb{R}^3} g(\varepsilon x)|u_n|^6 \mathrm{d}x < 0.\end{aligned}$$

So $\lim_{n\to\infty}(G_\varepsilon'(u_n),u_n) \leq 0$. If $\lim_{n\to\infty}(G_\varepsilon'(u_n),u_n) = 0$, we get $\|u_n\|_\varepsilon \to 0$, a contradiction with $I(u_n) \to \gamma_\varepsilon^i > 0$. Then $\lim_{n\to\infty}(G_\varepsilon'(u_n),u_n) < 0$. Since $\|u_n\|_\varepsilon$ is bounded, we know $|(s_n'(0),\phi)|$ is bounded. So $I_\varepsilon'(u_n) \to 0$. Assume $u_n \rightharpoonup u$ weakly in H_ε.

Case 1. $u_n \rightharpoonup 0$ weakly in H_ε.

Define the functional on H^1 by

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_{V_\infty}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} h_\infty F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g(\varepsilon x)|u|^6\mathrm{d}x.$$

By $\lim_{|x|\to\infty} V(x) = V_\infty$, $\lim_{|x|\to\infty} h(x) = h_\infty$, we get $\gamma_\varepsilon^i = J_\varepsilon(u_n) + o_n(1)$, $J_\varepsilon'(u_n) = o_n(1)$. By the Lions Lemma, $\int_{\mathbb{R}^3}|u_n|^t\mathrm{d}x \to 0$ for any $t \in (2,6)$, or there exists $y_n \in \mathbb{R}^3$ with $|y_n| \to \infty$ satisfying $v_n = u_n(.+y_n) \rightharpoonup vs. \neq 0$ weakly in H^1. If $\int_{\mathbb{R}^3}|u_n|^t\mathrm{d}x \to 0$ for any $t \in (2,6)$, by Lemma 1 (iv), we get $\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\mathrm{d}x = o_n(1)$. Moreover, $\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3}f(u_n)u_n\mathrm{d}x = o_n(1)$. Then

$$\begin{aligned}&\gamma_\varepsilon^i + o_n(1) = \frac{1}{2}\|u_n\|_{V_\infty}^2 - \frac{1}{6}\int_{\mathbb{R}^3} g(\varepsilon x)|u_n|^6\mathrm{d}x,\\ &\|u_n\|_{V_\infty}^2 - \int_{\mathbb{R}^3} g(\varepsilon x)|u_n|^6\mathrm{d}x = o_n(1).\end{aligned}$$

Since $\gamma_\varepsilon^i > 0$, we assume $\lim_{n\to\infty}\|u_n\|_{V_\infty}^2 > 0$. By $S \leq \frac{\|u_n\|_{V_\infty}^2}{\left(\int_{\mathbb{R}^3} g(\varepsilon x)|u_n|^6\mathrm{d}x\right)^{\frac{1}{3}}}$, we get $\lim_{n\to\infty}\|u_n\|_{V_\infty}^2 \geq S^{\frac{3}{2}}$. So $\gamma_\varepsilon^i \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction. Thus, $v_n = u_n(.+y_n) \rightharpoonup vs. \neq 0$ weakly in H^1 with $|y_n| \to \infty$. Define

$$\begin{aligned}L_\varepsilon(v_n) =& \frac{1}{2}\|v_n\|_{V_\infty}^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\mathrm{d}x - \int_{\mathbb{R}^3}h_\infty F(v_n)\mathrm{d}x\\ &- \frac{1}{6}\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)|v_n|^6\mathrm{d}x.\end{aligned}$$

Then

$$|y_n| \to \infty,\ \ \gamma_\varepsilon^i = L_\varepsilon(v_n) + o_n(1),\ \ L_\varepsilon'(v_n) = o_n(1). \tag{23}$$

By $v_n \rightharpoonup vs. \neq 0$ weakly in H^1 and $\lim_{|x|\to\infty} g(x) = g_\infty$, we get

$$\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)v_n^5\varphi\mathrm{d}x \to \int_{\mathbb{R}^3} g_\infty v^5\varphi\mathrm{d}x,\ \forall\ \varphi \in H^1.$$

Then $\check{I}'(v) = 0$. Let $w_n = v_n - v$. By Lemma 1.3 in [36], we have

$$\int_{\mathbb{R}^3}F(v_n)\mathrm{d}x - \int_{\mathbb{R}^3}F(v)\mathrm{d}x = \int_{\mathbb{R}^3}F(w_n)\mathrm{d}x + o_n(1). \tag{24}$$

By the Brezis–Lieb Lemma in [34], $\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)\big||v_n|^6 - |v|^6 - |w_n|^6\big|\mathrm{d}x = o_n(1)$. By the Lebesgue dominated convergence theorem, $\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)|v|^6\mathrm{d}x \to \int_{\mathbb{R}^3} g_\infty|v|^6\mathrm{d}x$. Then

$$\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)|v_n|^6\mathrm{d}x = \int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)|w_n|^6\mathrm{d}x + \int_{\mathbb{R}^3} g_\infty|v|^6\mathrm{d}x + o_n(1). \tag{25}$$

Combining (24) and (25) and Lemma 1 (iii), we get

$$\gamma_\varepsilon^i = L_\varepsilon(v_n) + o_n(1) = L_\varepsilon(w_n) + \check{I}(v) + o_n(1). \tag{26}$$

If $w_n \to 0$ in H^1, that is, $v_n \to v$ in H^1, by $|y_n| \to \infty$,

$$|\beta_\varepsilon(u_n)| = \left|\frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)|u_n|^6 \mathrm{d}x}{\int_{\mathbb{R}^3} |u_n|^6 \mathrm{d}x}\right| = \left|\frac{\int_{\mathbb{R}^3} \chi(\varepsilon x + \varepsilon y_n)|v_n|^6 \mathrm{d}x}{\int_{\mathbb{R}^3} |v_n|^6 \mathrm{d}x}\right| \to \rho,$$

a contradiction with $u_n \in C_\eta(x^i)$. So w_n converges weakly and not strongly to 0 in H^1. By Lemma 8.9 in [34], $\left|\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)(v_n^5 - v^5 - w_n^5)\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_{V_\infty}$ for any $\varphi \in H^1$. By the Lebesgue dominated convergence theorem, $\left|\int_{\mathbb{R}^3} (g(\varepsilon x + \varepsilon y_n) - g_\infty)v^5 \varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_{V_\infty}$ for any $\varphi \in H^1$. Then

$$\left|\int_{\mathbb{R}^3} \left(g(\varepsilon x + \varepsilon y_n)v_n^5 - g_\infty v^5 - g(\varepsilon x + \varepsilon y_n)w_n^5\right)\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_{V_\infty}, \forall\, \varphi \in H^1.$$

Similar to Lemma 8.1 in [34], we derive that

$$\left|\int_{\mathbb{R}^3} (f(v_n) - f(v) - f(w_n))\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_{V_\infty}, \quad \forall\, \varphi \in H^1.$$

Together with Lemma 1 (iii), $L_\varepsilon'(v_n) = o_n(1)$, $\check{I}'(v) = 0$, we get $L_\varepsilon'(w_n) = o_n(1)$. Thus,

$$\begin{aligned} o_n(1) =& (L_\varepsilon'(w_n), w_n) \\ =& \|w_n\|_{V_\infty}^2 + \int_{\mathbb{R}^3} \phi_{w_n} w_n^2 \mathrm{d}x - \int_{\mathbb{R}^3} h_\infty f(w_n) w_n \mathrm{d}x - \int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon y_n)|w_n|^6 \mathrm{d}x. \end{aligned}$$

By (21), (27) and $g(\varepsilon x + \varepsilon y_n) \le 1$, we derive that

$$\|w_n\|_{V_\infty}^2 \le \frac{V_\infty}{2} \int_{\mathbb{R}^3} |w_n|^2 \mathrm{d}x + \left(h_\infty C_{\frac{V_\infty}{2h_\infty}} + 1\right) \int_{\mathbb{R}^3} |w_n|^6 \mathrm{d}x + o_n(1).$$

So $\frac{1}{2}\|w_n\|_{V_\infty}^2 \le \frac{h_\infty C_{\frac{1}{2h_\infty}} + 1}{S^3}\|w_n\|_{V_\infty}^6 + o_n(1)$. Since w_n converges weakly and not strongly to 0 in H^1, we get $\lim_{n\to\infty} \|w_n\|_{V_\infty}^2 \ge \frac{S^{\frac{3}{2}}}{\sqrt{2\left(h_\infty C_{\frac{V_\infty}{2h_\infty}} + 1\right)}}$. Since $\check{I}'(v) = 0$, we have $\check{I}(v) \ge m_\infty \ge \hat{m}$. Then by (26) and (27),

$$\begin{aligned} \gamma_\varepsilon^i =& L_\varepsilon(w_n) - \frac{1}{4}(L_\varepsilon'(w_n), w_n) + \check{I}(v) + o_n(1) \\ \ge& \frac{1}{4}\|w_n\|_{V_\infty}^2 + \hat{m} + o_n(1) \ge \frac{S^{\frac{3}{2}}}{4\sqrt{2\left(h_\infty C_{\frac{V_\infty}{2h_\infty}} + 1\right)}} + \hat{m} + o_n(1), \end{aligned}$$

a contradiction with (22).

Case 2. $u_n \rightharpoonup u \ne 0$ weakly in H_ε.

By $u_n \rightharpoonup u \ne 0$ weakly in H_ε, we have $I_\varepsilon'(u) = 0$. By Lemma 1.3 in [36],

$$\int_{\mathbb{R}^3} h(\varepsilon x)F(u_n)\mathrm{d}x - \int_{\mathbb{R}^3} h(\varepsilon x)F(u)\mathrm{d}x = \int_{\mathbb{R}^3} h(\varepsilon x)F(\hat{u}_n)\mathrm{d}x + o_n(1),$$

where $\hat{u}_n = u_n - u$. Together with the Brezis–Lieb Lemma in [34] and Lemma 1 (iii), we get $\gamma_\varepsilon^i = I_\varepsilon(u_n) + o_n(1) = I_\varepsilon(\hat{u}_n) + I_\varepsilon(u) + o_n(1)$. By Lemma 8.9 in [34], for any $\varphi \in H_\varepsilon$, $\left|\int_{\mathbb{R}^3} g(\varepsilon x)(u_n^5 - u^5 - \hat{u}_n^5)\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_\varepsilon$. Similar to Lemma 8.1 in [34], for any $\varphi \in H_\varepsilon$,

$\left|\int_{\mathbb{R}^3} h(\varepsilon x)[f(u_n) - f(u) - f(\hat{u}_n)]\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|_\varepsilon$. Together with Lemma 1 (iii), we get $I'_\varepsilon(\hat{u}_n) = o_n(1)$. Thus,

$$\gamma^i_\varepsilon = I_\varepsilon(\hat{u}_n) + I_\varepsilon(u) + o_n(1), \quad I'_\varepsilon(\hat{u}_n) = o_n(1). \tag{27}$$

We claim $\hat{u}_n \to 0$ in H_ε. Otherwise, $\hat{u}_n$ converges weakly and not strongly to 0 in H_ε. By $\lim_{|x|\to\infty} V(x) = V_\infty$, $\lim_{|x|\to\infty} h(x) = h_\infty$, we have

$$\gamma^i_\varepsilon = J_\varepsilon(\hat{u}_n) + I_\varepsilon(u) + o_n(1), \quad J'_\varepsilon(\hat{u}_n) = o_n(1). \tag{28}$$

By (21), $g(\varepsilon x) \leq 1$ and $(J'_\varepsilon(\hat{u}_n), \hat{u}_n) = o_n(1)$,

$$\|\hat{u}_n\|^2_{V_\infty} \leq \frac{V_\infty}{2}\int_{\mathbb{R}^3} |\hat{u}_n|^2 \mathrm{d}x + \left(h_\infty C_{\frac{V_\infty}{2h_\infty}} + 1\right)\int_{\mathbb{R}^3} |\hat{u}_n|^6 \mathrm{d}x + o_n(1).$$

By the Sobolev embedding theorem, we get $\lim_{n\to\infty} \|\hat{u}_n\|^2_{V_\infty} \geq \frac{S^{\frac{3}{2}}}{\sqrt{2\left(h_\infty C_{\frac{V_\infty}{2h_\infty}}+1\right)}}$. Since $I'_\varepsilon(u) = 0$, we have $I_\varepsilon(u) \geq m_\varepsilon \geq \hat{m}$. Then by (28),

$$\begin{aligned}
\gamma^i_\varepsilon =& J_\varepsilon(\hat{u}_n) - \frac{1}{4}(J'_\varepsilon(\hat{u}_n), \hat{u}_n) + I_\varepsilon(u) + o_n(1) \\
\geq& \frac{1}{4}\|\hat{u}_n\|^2_{V_\infty} + \hat{m} + o_n(1) \geq \frac{S^{\frac{3}{2}}}{4\sqrt{2\left(h_\infty C_{\frac{V_\infty}{2h_\infty}} + 1\right)}} + \hat{m} + o_n(1),
\end{aligned}$$

a contradiction with (22). So $\hat{u}_n \to 0$ in H_ε, that is, $u_n \to u$ in H_ε. □

Lemma 7. *Let $\varepsilon \in (0, \varepsilon_0)$, Then problem (11) has at least k different positive solutions u^i_ε, $i = 1, 2, \ldots k$.*

Proof of Lemma 7. Let $i = 1, 2, \ldots k$. By Lemma 6, we have $u^i_n \in M^i_\varepsilon$, $I_\varepsilon(u^i_n) \to \gamma^i_\varepsilon$, $I'_\varepsilon(u^i_n) \to 0$. Moreover, $u^i_n \to u^i_\varepsilon$ in H_ε. Then $u^i_\varepsilon \in \overline{M^i_\varepsilon}$, $I_\varepsilon(u^i_\varepsilon) = \gamma^i_\varepsilon$, $I'_\varepsilon(u^i_\varepsilon) = 0$. By $\gamma^i_\varepsilon < \tilde{\gamma}^i_\varepsilon$, we have $u^i_\varepsilon \in M^i_\varepsilon$. Since $\beta(u^i_\varepsilon) \in C^i_\eta$, where C^i_η, $i = 1, 2, \ldots k$ are disjoint, we derive that u^i_ε, $i = 1, 2, \ldots k$ are different. Obviously, u^i_ε is non-negative. By the maximum principle, u^i_ε is positive. □

Now we study the behavior of u^i_ε as $\varepsilon \to 0$.

Lemma 8. *Let $i = 1, 2, \ldots k$. Then there exist $\varepsilon^* \in (0, \varepsilon_0)$, $\{x^i_\varepsilon\} \subset \mathbb{R}^3$, R_0, $\varrho_0 > 0$ satisfying $\int_{B_{R_0}(x^i_\varepsilon)} |u^i_\varepsilon|^2 \mathrm{d}x \geq \varrho_0$ for $\varepsilon \in (0, \varepsilon^*)$.*

Proof of Lemma 8. Otherwise, there exists $\varepsilon_n \downarrow 0$ such that for any $R > 0$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^3} \int_{B_R(x)} |u^i_{\varepsilon_n}|^2 \mathrm{d}x = 0.$$

By the Lions Lemma, $\int_{\mathbb{R}^3} |u^i_{\varepsilon_n}|^t \mathrm{d}x \to 0$ for any $t \in (2, 6)$. Since $I_{\varepsilon_n}(u^i_{\varepsilon_n}) = \gamma^i_{\varepsilon_n} \to \hat{m}$, $I'_{\varepsilon_n}(u^i_{\varepsilon_n}) = 0$, similar to the argument of (19), we get $\hat{m} \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction. □

Lemma 9. *$\varepsilon x^i_\varepsilon \to x^i$ as $\varepsilon \to 0$.*

Proof of Lemma 9. We first prove $|\varepsilon x^i_\varepsilon|$ is bounded. Assume to the contrary that there exists $\varepsilon_n \downarrow 0$ satisfying $|\varepsilon_n x^i_{\varepsilon_n}| \to \infty$. By $I_{\varepsilon_n}(u^i_{\varepsilon_n}) = \gamma^i_{\varepsilon_n} \to \hat{m}$, $I'_{\varepsilon_n}(u^i_{\varepsilon_n}) = 0$, we derive that

$\|u^i_{\varepsilon_n}\|_{\varepsilon_n}$ is bounded. Let $v^i_{\varepsilon_n} = u^i_{\varepsilon_n}(.+x^i_{\varepsilon_n})$. By Lemma 8, we get $v^i_{\varepsilon_n} \rightharpoonup v^i \neq 0$ weakly in H^1. Let

$$\begin{aligned} T_\varepsilon(u) =& \frac{1}{2}\int_{\mathbb{R}^3}\Big(|\nabla u|^2 + V(\varepsilon x + \varepsilon x^i_\varepsilon)|u|^2\Big)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2 \mathrm{d}x \\ & - \int_{\mathbb{R}^3} h(\varepsilon x + \varepsilon x^i_\varepsilon)F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon x^i_\varepsilon)|u|^6\mathrm{d}x,\ u \in H_\varepsilon. \end{aligned}$$

Then $T_{\varepsilon_n}(v^i_{\varepsilon_n}) \to \hat{m}$, $T'_{\varepsilon_n}(v^i_{\varepsilon_n}) = 0$. Furthermore,

$$\begin{aligned} &\int_{\mathbb{R}^3}\Big(\nabla v^i_{\varepsilon_n}\nabla v^i + V(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})v^i_{\varepsilon_n}v^i\Big)\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{v^i_{\varepsilon_n}} v^i_{\varepsilon_n} v^i \mathrm{d}x \\ =& \int_{\mathbb{R}^3} h(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n}) f(v^i_{\varepsilon_n})v^i \mathrm{d}x + \int_{\mathbb{R}^3} g(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})(v^i_{\varepsilon_n})^5 v^i \mathrm{d}x. \end{aligned}$$

By $|\varepsilon_n x^i_{\varepsilon_n}| \to \infty$, we derive that

$$\|v^i\|^2_{V_\infty} + \int_{\mathbb{R}^3}\phi_{v^i}|v^i|^2\mathrm{d}x = \int_{\mathbb{R}^3} h_\infty f(v^i)v^i\mathrm{d}x + \int_{\mathbb{R}^3} g_\infty|v^i|^6\mathrm{d}x,$$

that is, $v^i \in \hat{N}$. By $T_{\varepsilon_n}(v^i_{\varepsilon_n}) \to \hat{m}$, $T'_{\varepsilon_n}(v^i_{\varepsilon_n}) = 0$,

$$\begin{aligned} \hat{m} &= T_{\varepsilon_n}(v^i_{\varepsilon_n}) - \frac{1}{4}\Big(T'_{\varepsilon_n}(v^i_{\varepsilon_n}), v^i_{\varepsilon_n}\Big) + o_n(1) \\ &= \frac{1}{4}\int_{\mathbb{R}^3}\Big(|\nabla v^i_{\varepsilon_n}|^2 + V(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})|v^i_{\varepsilon_n}|^2\Big)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} h(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})\Big(\frac{1}{4}f(v^i_{\varepsilon_n})v^i_{\varepsilon_n} - F(v^i_{\varepsilon_n})\Big)\mathrm{d}x \\ &\quad + \frac{1}{12}\int_{\mathbb{R}^3} g(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})|v^i_{\varepsilon_n}|^6\mathrm{d}x + o_n(1). \end{aligned} \tag{29}$$

Then by Fatou's Lemma and $|\varepsilon_n x^i_{\varepsilon_n}| \to \infty$,

$$\begin{aligned} \hat{m} \geq& \frac{1}{4}\|v^i\|^2_{V_\infty} + \int_{\mathbb{R}^3} h_\infty\Big(\frac{1}{4}f(v^i)v^i - F(v^i)\Big)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} g_\infty|v^i|^6\mathrm{d}x \\ =& \check{I}(v^i) - \frac{1}{4}\Big(\check{I}'(v^i), v^i\Big) = \check{I}(v^i). \end{aligned} \tag{30}$$

Since $v^i \in \hat{N}$, we have $\check{I}(v^i) \geq m_\infty \geq \hat{m}$. Combining (29) and (30), we get $v^i_{\varepsilon_n} \to v^i$ in H^1. Note that

$$\beta(u^i_{\varepsilon_n}) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon_n x)|u^i_{\varepsilon_n}|^6\mathrm{d}x}{\int_{\mathbb{R}^3}|u^i_{\varepsilon_n}|^6\mathrm{d}x} = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon_n x + \varepsilon_n x^i_{\varepsilon_n})|v^i_{\varepsilon_n}|^6\mathrm{d}x}{\int_{\mathbb{R}^3}|v^i_{\varepsilon_n}|^6\mathrm{d}x}.$$

Then by $|\varepsilon_n x^i_{\varepsilon_n}| \to \infty$, we get $\rho = \lim_{n\to\infty}|\beta(u^i_{\varepsilon_n})|$, a contradiction with $\beta(u^i_{\varepsilon_n}) \in C^i_\eta$.

Now we prove $\varepsilon x^i_\varepsilon \to x^i$ as $\varepsilon \to 0$. Since $|\varepsilon x^i_\varepsilon|$ is bounded, we assume $\varepsilon x^i_\varepsilon \to x^i_0$ as $\varepsilon \to 0$. Let $v^i_\varepsilon = u^i_\varepsilon(.+x^i_\varepsilon)$. By Lemma 8, we get $v^i_\varepsilon \rightharpoonup v^i \neq 0$ weakly in H^1. Then by $I'_\varepsilon(u^i_\varepsilon) = 0$, we derive that

$$\begin{aligned} &\int_{\mathbb{R}^3}\Big(|\nabla v^i|^2 + V(x^i_0)|v^i|^2\Big)\mathrm{d}x + \int_{\mathbb{R}^3}\phi_{v^i}|v^i|^2\mathrm{d}x \\ =& \int_{\mathbb{R}^3} h(x^i_0)f(v^i)v^i\mathrm{d}x + \int_{\mathbb{R}^3} g(x^i_0)|v^i|^6\mathrm{d}x. \end{aligned}$$

Let

$$L_0(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x_0^i)|u|^2\right)\mathrm{d}x - \int_{\mathbb{R}^3} h(x_0^i)F(u)\mathrm{d}x$$
$$- \frac{1}{6}\int_{\mathbb{R}^3} g(x_0^i)|u|^6\mathrm{d}x, \;\; u \in H^1.$$

Then $\left(L_0'(v^i), v^i\right) = 0$. By $I_\varepsilon(u_\varepsilon^i) = \gamma_\varepsilon^i \to \hat{m}$, $I_\varepsilon'(u_\varepsilon^i) = 0$,

$$\begin{aligned}\hat{m} =& I_\varepsilon(u_\varepsilon^i) - \frac{1}{4}\left(I_\varepsilon'(u_\varepsilon^i), u_\varepsilon^i\right) + o_\varepsilon(1)\\ =& \frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla v_\varepsilon^i|^2 + V(\varepsilon x + \varepsilon x_\varepsilon^i)|v_\varepsilon^i|^2\right)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} g(\varepsilon x + \varepsilon x_\varepsilon^i)|v_\varepsilon^i|^6\mathrm{d}x\\ &+ \int_{\mathbb{R}^3} h(\varepsilon x + \varepsilon x_\varepsilon^i)\left(\frac{1}{4}f(v_\varepsilon^i)v_\varepsilon^i - F(v_\varepsilon^i)\right)\mathrm{d}x + o_\varepsilon(1).\end{aligned} \tag{31}$$

Then by Fatou's Lemma,

$$\begin{aligned}\hat{m} \geq& \frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla v^i|^2 + V(x_0^i)|v^i|^2\right)\mathrm{d}x + \int_{\mathbb{R}^3} h(x_0^i)\left(\frac{1}{4}f(v^i)v^i - F(v^i)\right)\mathrm{d}x\\ &+ \frac{1}{12}\int_{\mathbb{R}^3} g(x_0^i)|v^i|^6\mathrm{d}x\\ \geq& L_0(v^i) - \frac{1}{4}\left(L_0'(v^i), v^i\right) = L_0(v^i).\end{aligned} \tag{32}$$

Since $\left(L_0'(v^i), v^i\right) = 0$, by (f_1), we have $L_0(v^i) = \sup_{t\geq 0} L_0(tv^i)$. Moreover, there exists a unique $\check{t}^i > 0$ satisfying $\check{t}^i v^i \in \hat{N}$. Then

$$L_0(v^i) = \sup_{t\geq 0} L_0(tv^i) \geq L_0(\check{t}^i v^i) \geq \hat{I}(\check{t}^i v^i) \geq \hat{m}. \tag{33}$$

Combining (31)–(33), we get $h(x_0^i) = h_M$ and $v_\varepsilon^i \to v^i$ in H^1. Then $h(\varepsilon x_\varepsilon^i) \to h_M$. Note that

$$\beta(u_\varepsilon^i) = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x)|u_\varepsilon^i|^6\mathrm{d}x}{\int_{\mathbb{R}^3}|u_\varepsilon^i|^6\mathrm{d}x} = \frac{\int_{\mathbb{R}^3}\chi(\varepsilon x + \varepsilon x_\varepsilon^i)|v_\varepsilon^i|^6\mathrm{d}x}{\int_{\mathbb{R}^3}|v_\varepsilon^i|^6\mathrm{d}x}.$$

If $|x_0^i| \geq \rho$, then $\lim_{\varepsilon\to 0}|\beta(u_\varepsilon^i)| = \rho$, a contradiction with $\beta(u_\varepsilon^i) \in C_\eta^i$. So $|x_0^i| < \rho$, from which we derive that $\lim_{\varepsilon\to 0}\beta(u_\varepsilon^i) = x_0^i \in \overline{C_\eta^i}$. Since $\varepsilon x_\varepsilon^i \to x_0^i$, $h(x_0^i) = h_M$, we get $\varepsilon x_\varepsilon^i \to x^i$ as $\varepsilon \to 0$. □

Proof of Theorem 1. By Lemma 7, problem (11) has at least k different positive solutions u_ε^i, $i = 1, 2, \ldots k$. Let $v_\varepsilon^i = u_\varepsilon^i(. + x_\varepsilon^i)$. Then

$$-\Delta v_\varepsilon^i + V(\varepsilon x + \varepsilon x_\varepsilon^i)v_\varepsilon^i + \phi_{v_\varepsilon^i}v_\varepsilon^i = h(\varepsilon x + \varepsilon x_\varepsilon^i)f(v_\varepsilon^i) + g(\varepsilon x + \varepsilon x_\varepsilon^i)(v_\varepsilon^i)^5. \tag{34}$$

By Lemma 9, $v_\varepsilon^i \to v^i \neq 0$ in H^1, $\varepsilon x_\varepsilon^i \to x^i$. By the argument of Lemmas 3.8 and 3.11 in [15], $\lim_{|x|\to\infty} v_\varepsilon^i(x) = 0$ uniformly for $\varepsilon \in (0, \varepsilon^*)$ and there exists $\acute{C}^i > 0$ independent of $\varepsilon \in (0, \varepsilon^*)$ satisfying $\|v_\varepsilon^i\|_\infty \leq \acute{C}^i$. Furthermore, there exist $\grave{C}^i, c^i > 0$ such that $v_\varepsilon^i(x) \leq \grave{C}^i \exp\left(-c^i|x|\right)$ uniformly for $\varepsilon \in (0, \varepsilon^*)$.

We claim there exists $\gamma_0 > 0$ such that $\|v_\varepsilon^i\|_\infty \geq \gamma_0$ uniformly for $\varepsilon \in (0, \varepsilon^*)$. Otherwise, $\|v_\varepsilon^i\|_\infty \to 0$ as $\varepsilon \to 0$. By (34) and (f_1), we get $\|v_\varepsilon^i\|_{V_0}^2 \leq \frac{V_0}{2}\int_{\mathbb{R}^3}|v_\varepsilon^i|^2\mathrm{d}x + C\int_{\mathbb{R}^3}|v_\varepsilon^i|^6\mathrm{d}x$. Then $\|v_\varepsilon^i\|_{V_0}^2 \leq 2C\|v_\varepsilon^i\|_\infty^4\int_{\mathbb{R}^3}|v_\varepsilon^i|^2\mathrm{d}x \to 0$, a contradiction with $v_\varepsilon^i \to v^i \neq 0$ in H^1. Let z_ε^i be the maximum point of v_ε^i. Then $|v_\varepsilon^i(z_\varepsilon^i)| \geq \gamma_0$. By $\lim_{|x|\to\infty} v_\varepsilon^i(x) = 0$ uniformly for $\varepsilon \in (0, \varepsilon^*)$, we derive that there exists $Z^i > 0$ such that $|z_\varepsilon^i| \leq Z^i$.

Since $v_\varepsilon^i = u_\varepsilon^i(.+x_\varepsilon^i)$, we know $y_\varepsilon^i := x_\varepsilon^i + z_\varepsilon^i$ is the maximum point of u_ε^i. By $\varepsilon x_\varepsilon^i \to x^i$ as $\varepsilon \to 0$ and $|z_\varepsilon^i| \le Z^i$, we get $\varepsilon(x_\varepsilon^i + z_\varepsilon^i) \to x^i$ as $\varepsilon \to 0$. Moreover, for $\varepsilon \in (0, \varepsilon^*)$,

$$u_\varepsilon^i(x) = v_\varepsilon^i(.-x_\varepsilon^i) \le \grave{C}^i \exp\Big(-c^i|x - x_\varepsilon^i|\Big) \le C^i \exp\Big(-c^i|x - (x_\varepsilon^i + z_\varepsilon^i)|\Big).$$

Let $w_\varepsilon^i = u_\varepsilon^i(\frac{\cdot}{\varepsilon})$, $y_\varepsilon^i = \varepsilon(x_\varepsilon^i + z_\varepsilon^i)$. Then w_ε^i is the positive solution of (1). Furthermore, y_ε^i is the maximum point of w_ε^i, $h(y_\varepsilon^i) \to \sup_{x\in\mathbb{R}^3} h(x)$ and there exist C^i, $c^i > 0$ such that $w_\varepsilon^i(x) \le C^i \exp\Big(-c^i \frac{|x-y_\varepsilon^i|}{\varepsilon}\Big)$ for $\varepsilon \in (0, \varepsilon^*)$. □

4. Proof of Theorems 2 and 3

For simplicity, let $\varepsilon = 1$. Denote $H = \left\{u \in H^1 : \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < +\infty\right\}$ the Hilbert space with the norm $\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$. Let

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \int_{\mathbb{R}^3} h(x)F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x.$$

Let $m = \inf\{I(u) : u \in M\}$, where $M = \{u \in H \setminus \{0\} : (I'(u), u) = 0\}$.

We first prove Theorem 3. Let X be the Banach space. Recall that $\{u_n\} \subset X$ is a $(C)_c$ sequence for the functional I if $I(u_n) \to c$ and $(1 + \|u_n\|_X)\|I'(u_n)\| \to 0$ as $n \to \infty$.

Theorem 4 ([37]). *Let X be the Banach space and $I \in C^1(X, \mathbb{R})$ satisfying*

$$\max\{I(0), I(u_1)\} \le \alpha_2 < \alpha_1 \le \inf_{\|u\|_X = \rho} I(u)$$

for some $\rho > 0$ and $u_1 \in X$ with $\|u_1\|_X > \rho$. Let $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = u_1\}$. Then there exists a $(C)_c$ sequence $\{u_n\}$ for the functional I satisfying $c \ge \alpha_1$.

For any $y \in \mathbb{R}^3$ with $|y| = 1$, by (h_2), there exist σ, $\xi > 0$ such that $h(x) \ge \xi$ for $|x - \frac{3}{2}\rho_0 y| \le \sigma$. Let $r < \min\{\frac{1}{4}\rho_0, \frac{1}{2}\sigma\}$. Define $u_{\delta,\tilde{y}} := u_{\delta,\frac{3}{2}\rho_0 y}(x) = \frac{\psi(x)\delta^{\frac{1}{4}}}{\left(\delta + |x - \frac{3}{2}\rho_0 y|^2\right)^{\frac{1}{2}}}$, where $\psi \in C_0^\infty\left(B_{2r}\left(\frac{3}{2}\rho_0 y\right)\right)$ such that $\psi(x) = 1$ for $|x - \frac{3}{2}\rho_0 y| < r$, $0 \le \psi(x) \le 1$, $|\nabla\psi| \le 2$.

Lemma 10. *Assume that (h_2), (V_1), (g_2') and (f_1') hold. Then there exists $\delta_0 > 0$ independent of $y \in \mathbb{R}^3$ with $|y| = 1$ such that for any $\delta \in (0, \delta_0)$,*

$$\sup_{t\ge 0} I(tu_{\delta,\tilde{y}}) \le \frac{1}{3}S^{\frac{3}{2}} - \delta^{\frac{1}{2}}.$$

Proof of Lemma 10. From Lemma 1 (iv), there exists $C_0 > 0$ such that

$$I(tu_{\delta,\tilde{y}}) \le \frac{t^2}{2}\|u_{\delta,\tilde{y}}\|^2 + \frac{C_0 t^4}{4}\|u_{\delta,\tilde{y}}\|^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} g(x)|u_{\delta,\tilde{y}}|^6 \mathrm{d}x.$$

By Lemma 2 and (g_2'), there exists $\delta_1 > 0$ independent of y such that for $\delta \in (0, \delta_1)$,

$$\|u_{\delta,\tilde{y}}\|^2 \le \frac{3K_1}{2}, \quad \int_{\mathbb{R}^3} g(x)|u_{\delta,\tilde{y}}|^6 \mathrm{d}x = \int_{\mathbb{R}^3} |u_{\delta,\tilde{y}}|^6 \mathrm{d}x \ge \frac{K_2}{2}.$$

Then there exist a small $t_1 > 0$ and a large $t_2 > 0$ independent of $\delta \in (0, \delta_1)$ satisfying

$$\sup_{t\in[0,t_1]\cup[t_2,+\infty)} I(tu_{\delta,\tilde{y}}) < \frac{1}{6}S^{\frac{3}{2}}. \tag{35}$$

By Lemmas 1 and 2, we get

$$\begin{aligned}\sup_{t\in[t_1,t_2]} I(tu_{\delta,\tilde{y}}) \leq& \sup_{t\geq 0}\left(\frac{t^2}{2}\|u_{\delta,\tilde{y}}\|^2 - \frac{t^6}{6}\|u_{\delta,\tilde{y}}\|_6^6\right) + \frac{t_2^4}{4S}\|u_{\delta,\tilde{y}}\|^4_{\frac{12}{5}} \\ &- \xi \inf_{t\in[t_1,t_2]}\int_{\mathbb{R}^3} F(tu_{\delta,\tilde{y}})\mathrm{d}x \\ =&\frac{1}{3}\left(\frac{\|u_{\delta,\tilde{y}}\|^2}{\|u_{\delta,\tilde{y}}\|_6^2}\right)^{\frac{3}{2}} + O(\delta) - \xi \inf_{t\in[t_1,t_2]}\int_{\mathbb{R}^3} F(tu_{\delta,\tilde{y}})\mathrm{d}x \\ \leq&\frac{1}{3}S^{\frac{3}{2}} + \delta^{\frac{1}{2}} - \xi \inf_{t\in[t_1,t_2]}\int_{\mathbb{R}^3} F(tu_{\delta,\tilde{y}})\mathrm{d}x.\end{aligned}$$

By (f_1'), we have $\lim_{u\to+\infty}\frac{F(u)}{u^4} = +\infty$. Then for $L > \frac{2}{\xi t_1^4 \int_{|x|\leq 1}\mathrm{d}x}$, there exists $R_L > 0$ satisfying $F(u) \geq L|u|^4$ for $|u| \geq R_L$. Let $r < \min\{\frac{1}{4}\rho_0, \frac{1}{2}\sigma\}$. Note that $u_{\delta,\tilde{y}}(x) \geq \delta^{-\frac{1}{4}}$ for $|x - \frac{3}{2}\rho_0 y| \leq \delta^{\frac{1}{2}} \leq r$. Let $\delta_0 = \min\{\delta_1, r^2\}$ and $\delta \in (0, \delta_0)$. Then for $t \in [t_1, t_2]$ and $|x - \frac{3}{2}\rho_0 y| \leq \delta^{\frac{1}{2}}$, we have $F(tu_{\delta,\tilde{y}}) \geq Lt_1^4 u_{\delta,\tilde{y}}^4 \geq Lt_1^4\delta^{-1}$. Since $F(tu_{\delta,\tilde{y}}) \geq 0$, we derive that

$$\inf_{t\in[t_1,t_2]}\int_{\mathbb{R}^3} F(tu_{\delta,\tilde{y}})\mathrm{d}x \geq \inf_{t\in[t_1,t_2]}\int_{|x-\frac{3}{2}\rho_0 y|\leq\delta^{\frac{1}{2}}} F(tu_{\delta,\tilde{y}})\mathrm{d}x \geq Lt_1^4\delta^{\frac{1}{2}}\int_{|x|\leq 1}\mathrm{d}x.$$

Thus, for $\delta \in (0, \delta_0)$,

$$\sup_{t\in[t_1,t_2]} I(tu_{\delta,\tilde{y}}) \leq \frac{1}{3}S^{\frac{3}{2}} + \delta^{\frac{1}{2}} - \xi Lt_1^4\delta^{\frac{1}{2}}\int_{|x|\leq 1}\mathrm{d}x \leq \frac{1}{3}S^{\frac{3}{2}} - \delta^{\frac{1}{2}}. \tag{36}$$

Combining (35) and (36), we get Lemma 10. □

Define the functional on H by

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3} g(x)|u|^6\mathrm{d}x.$$

Lemma 11. *Assume that (h_2), (V_1), (Vh), (g_2') and (f_1') hold. If $\{u_n\} \subset H$ such that $u_n \rightharpoonup u$ weakly in H, $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I'(u_n) \to 0$, then $I'(u) = 0$ and $u_n \rightharpoonup u \neq 0$ weakly in H. Moreover, if $I(u) \geq 0$, then $u_n \to u$ in H.*

Proof of Lemma 11. By $u_n \rightharpoonup u$ weakly in H, we have $I'(u) = 0$. Let $v_n = u_n - u$. By Lemma 1.3 in [36], we obtain that $\int_{\mathbb{R}^3} h(x)F(v_n)\mathrm{d}x = \int_{\mathbb{R}^3} h(x)F(u_n)\mathrm{d}x - \int_{\mathbb{R}^3} h(x)F(u)\mathrm{d}x + o_n(1)$. By (h_2) and (f_1'), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $|h(x)F(v_n)| \leq \varepsilon|v_n|^6 + C_\varepsilon h(x)|v_n|^2$. Then for any $R > 0$,

$$\int_{|x|\geq R}|h(x)F(v_n)|\mathrm{d}x \leq \varepsilon\int_{\mathbb{R}^3}|v_n|^6\mathrm{d}x + C_\varepsilon\sup_{|x|\geq R}\frac{h(x)}{V(x)} \times \int_{\mathbb{R}^3} V(x)|v_n|^2\mathrm{d}x.$$

Since $\|v_n\|$ is bounded, by (Vh), for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\sup_{|x|\geq R_\varepsilon}\frac{h(x)}{V(x)} \times \int_{\mathbb{R}^3} V(x)|v_n|^2\mathrm{d}x \leq \varepsilon.$$

By $v_n \rightharpoonup 0$ weakly in H, we have $\int_{|x|\leq R_\varepsilon} h(x)F(v_n)\mathrm{d}x = o_n(1)$. Thus,

$$\int_{\mathbb{R}^3} h(x)F(u_n)\mathrm{d}x - \int_{\mathbb{R}^3} h(x)F(u)\mathrm{d}x = \int_{\mathbb{R}^3} h(x)F(v_n)\mathrm{d}x + o_n(1) = o_n(1). \tag{37}$$

By the Brezis–Lieb Lemma in [34], we get $\|v_n\|^2 = \|u_n\|^2 - \|u\|^2 + o_n(1)$ and $\int_{\mathbb{R}^3} g(x)|v_n|^6 \mathrm{d}x = \int_{\mathbb{R}^3} g(x)|u_n|^6 \mathrm{d}x - \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x + o_n(1)$. Together with Lemma 1 (iii), we have

$$c - I(u) = I(u_n) - I(u) + o_n(1) = J(v_n) + o_n(1). \tag{38}$$

By Lemma 8.9 in [34], we have $\left|\int_{\mathbb{R}^3} g(x)\big(u_n^5 - u^5 - v_n^5\big)\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|$ for any $\varphi \in H$. Similar to Lemma 8.1 in [34], we derive for any $\varphi \in H$, there holds $\left|\int_{\mathbb{R}^3} h(x)[f(u_n) - f(u) - f(v_n)]\varphi \mathrm{d}x\right| = o_n(1)\|\varphi\|$. Together with Lemma 1 (iii), we get $I'(v_n) = o_n(1)$. So $(I'(v_n), v_n) = o_n(1)$. Similar to (37), we get $\int_{\mathbb{R}^3} h(x)f(v_n)v_n \mathrm{d}x = o_n(1)$. Thus, we have $(J'(v_n), v_n) = o_n(1)$.

Assume to the contrary that $u_n \rightharpoonup 0$ weakly in H. Then $v_n = u_n$. Assume $\int_{\mathbb{R}^3} g(x)|v_n|^6 \mathrm{d}x \to l$. Then $\lim_{n\to\infty} \|v_n\|^2 \le l$. If $l > 0$, by $S \le \frac{\|v_n\|^2}{\left(\int_{\mathbb{R}^3} g(x)|v_n|^6 \mathrm{d}x\right)^{\frac{1}{3}}}$, we get $l \ge S^{\frac{3}{2}}$. So $c - I(u) = J(v_n) - \frac{1}{4}(J'(v_n), v_n) + o_n(1) \ge \frac{1}{3}S^{\frac{3}{2}} + o_n(1)$. Since $I(u) = 0$, we get $c \ge \frac{1}{3}S^{\frac{3}{2}}$, a contradiction. So $l = 0$. By $(J'(v_n), v_n) = o_n(1)$, we get $u_n = v_n \to 0$ in H, a contradiction with $I(u_n) \to c > 0$. So $u_n \rightharpoonup u \ne 0$ weakly in H.

If $I(u) \ge 0$, similar to the above argument, we can derive that $u_n \to u$ in H. We omit the proof here. □

Proof of Theorem 3. From (h_2), (f_1'), for $\varepsilon = \frac{V_0}{4}$, there exists $C_\varepsilon = C_{\frac{V_0}{4}} > 0$ such that $|h(x)F(u)| \le \frac{V_0}{4}|u|^2 + C_{\frac{V_0}{4}}|u|^6$. By the Sobolev embedding theorem,

$$I(u) \ge \frac{1}{4}\|u\|^2 - \left(C_{\frac{V_0}{4}} + \frac{1}{6}\right)\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \ge \frac{1}{4}\|u\|^2 - \frac{\left(C_{\frac{V_0}{4}} + \frac{1}{6}\right)\|u\|^6}{S^3}.$$

Then there exist $\rho_0, \gamma_0 > 0$ such that $I(u) \ge \gamma_0$ for $\|u\| = \rho_0$. Furthermore, $I(0) = 0$. By the argument of Lemma 4.1, $\lim_{t\to+\infty} I(tu_{\delta,\bar{y}}) = -\infty$. So by Lemma 10 and Theorem 4, there is $\{u_n\} \subset H$ such that

$$I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right), \ (1 + \|u_n\|)\|I'(u_n)\| \to 0.$$

If $\|u_n\|$ is bounded, by Lemma 11, we have $u_n \rightharpoonup u \ne 0$ weakly in H and $I'(u) = 0$, that is, (1) has a positive solution. So we just need to prove that $\|u_n\|$ is bounded.

Otherwise, we have $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $v_n \rightharpoonup v$ weakly in H.

Case 1. $v(x) = 0$ a.e. $x \in \mathbb{R}^3$. Let $\theta \in (4, 6)$ and $\inf_{x\in\mathbb{R}^3} g(x) := g_0$. By (h_2), (f_1') and (g_2'), for $\varepsilon \in \left(0, g_0 \times \left(\frac{1}{\theta} - \frac{1}{6}\right)\right)$, there exists $C_\varepsilon > 0$ such that

$$\left|\frac{1}{\theta}h(x)f(u_n)u_n - h(x)F(u_n)\right| \le \varepsilon|u_n|^6 + C_\varepsilon h(x)|u_n|^2.$$

Then

$$\left(I(u_n) - \frac{1}{\theta}\big(I'(u_n), u_n\big)\right) \ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - C_\varepsilon\int_{\mathbb{R}^3} h(x)|u_n|^2 \mathrm{d}x. \tag{39}$$

From (39), we derive that

$$\frac{1}{\|u_n\|^2}\left(I(u_n) - \frac{1}{\theta}\big(I'(u_n), u_n\big)\right) \ge \left(\frac{1}{2} - \frac{1}{\theta}\right) - C_\varepsilon\int_{\mathbb{R}^3} h(x)|v_n|^2 \mathrm{d}x.$$

Similar to (37), we have $\int_{\mathbb{R}^3} h(x)|v_n|^2 \mathrm{d}x \to 0$. Then by $I(u_n) \to c$, $(I'(u_n), u_n) \to 0$ and $\|u_n\| \to \infty$, we get $0 \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)$, a contradiction.

Case 2. $v(x) \neq 0$. Let $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Then the measure of Ω is positive. For $x \in \Omega$, by $v_n(x) = \frac{u_n(x)}{\|u_n\|} \to v(x)$, we get $\lim_{n\to\infty} |u_n(x)| = +\infty$. Let $q \in (4, 6)$. By (h_2), (f_1') and Fatou's Lemma,

$$\lim_{n\to\infty} \int_\Omega \frac{h(x)F(u_n) + \frac{1}{6}g(x)|u_n|^6}{\|u_n\|^q} \mathrm{d}x \geq \frac{g_0}{6} \lim_{n\to\infty} \int_\Omega |v_n|^q |u_n|^{6-q} \mathrm{d}x = +\infty. \tag{40}$$

Furthermore, for $\varepsilon \in \left(0, \frac{g_0}{6}\right)$, there exists $C_\varepsilon > 0$ such that $|h(x)F(u_n)| \leq \varepsilon|u_n|^6 + C_\varepsilon|u_n|^2$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus\Omega} h(x)F(u_n)\mathrm{d}x + \frac{1}{6}\int_{\mathbb{R}^3\setminus\Omega} g(x)|u_n|^6 \mathrm{d}x \\
&\geq -C_\varepsilon \int_{\mathbb{R}^3\setminus\Omega} |u_n|^2 \mathrm{d}x + \left(\frac{g_0}{6} - \varepsilon\right) \int_{\mathbb{R}^3\setminus\Omega} |u_n|^6 \mathrm{d}x \geq -C_\varepsilon \int_{\mathbb{R}^3\setminus\Omega} |u_n|^2 \mathrm{d}x,
\end{aligned}$$

from which we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3\setminus\Omega} \frac{h(x)F(u_n) + \frac{1}{6}g(x)|u_n|^6}{\|u_n\|^q} \mathrm{d}x \geq -C_\varepsilon \lim_{n\to\infty} \frac{1}{\|u_n\|^{q-2}} \int_{\mathbb{R}^3\setminus\Omega} |v_n|^2 \mathrm{d}x \geq 0. \tag{41}$$

Combining (40) and (41), we derive that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{h(x)F(u_n) + \frac{1}{6}g(x)|u_n|^6}{\|u_n\|^q} \mathrm{d}x = +\infty. \tag{42}$$

On the other hand, by Lemma 1, (iv),

$$I(u_n) + \int_{\mathbb{R}^3} \left(h(x)F(u_n) + \frac{1}{6}g(x)|u_n|^6\right) \mathrm{d}x \leq \frac{1}{2}\|u_n\|^2 + \frac{C_0}{4}\|u_n\|^4.$$

So $\lim\limits_{n\to\infty} \int_{\mathbb{R}^3} \frac{h(x)F(u_n)+\frac{1}{6}g(x)|u_n|^6}{\|u_n\|^q} \mathrm{d}x = 0$, a contradiction with (42). □

Now we prove Theorem 2. By the Lagrange multipliers Theorem, we can derive the following result. Since the proof is standard, we omit it here.

Lemma 12. *Assume that (h_2), (V_1), (g_2) and (f_1) hold. Let $\{u_n\} \subset M$ such that $I(u_n) \to c \in \left(0, \frac{1}{3}S^{\frac{3}{2}}\right)$ and $I|_M'(u_n) \to 0$. Then $I'(u_n) \to 0$. Moreover, $\|u_n\|$ is bounded.*

Lemma 13. *Assume that (V_1), (g_2) and (f_1) hold. Then there exists $\eta_0 > 0$ such that $\int_{\mathbb{R}^3} \frac{x}{|x|}|\nabla u|^2 \mathrm{d}x \neq 0$ for $u \in M_0$ with $J(u) \leq \frac{1}{3}S^{\frac{3}{2}} + \eta_0$, where $M_0 = \{u \in H \setminus \{0\} : (J'(u), u) = 0\}$.*

Proof of Lemma 13. Assume to the contrary that there exists $u_n \in M_0$ such that $J(u_n) \to \frac{1}{3}S^{\frac{3}{2}}$ and $\int_{\mathbb{R}^3} \frac{x}{|x|}|\nabla u_n|^2 \mathrm{d}x = 0$. Then $\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x \leq \int_{\mathbb{R}^3} g(x)|u_n|^6 \mathrm{d}x$. By $S \leq \frac{\int_{\mathbb{R}^3} |\nabla u_n|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^3} g(x)|u_n|^6 \mathrm{d}x\right)^{\frac{1}{3}}}$, we get $\int_{\mathbb{R}^3} g(x)|u_n|^6 \mathrm{d}x \geq S^{\frac{3}{2}}$. So

$$\begin{aligned}\frac{1}{3}S^{\frac{3}{2}}+o_n(1)&=J(u_n)-\frac{1}{6}(J'(u_n),u_n)\\&\geq\frac{1}{3}\int_{\mathbb{R}^3}\left(|\nabla u_n|^2+V(x)|u_n|^2\right)\mathrm{d}x\\&\geq\frac{S}{3}\left(\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\right)^{\frac{1}{3}}\geq\frac{S}{3}\left(\int_{\mathbb{R}^3}g(x)|u_n|^6\mathrm{d}x\right)^{\frac{1}{3}}\geq\frac{1}{3}S^{\frac{3}{2}},\end{aligned}$$

from which we derive that $\lim_{n\to\infty}\int_{\mathbb{R}^3}V(x)|u_n|^2\mathrm{d}x=0$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x=S^{\frac{3}{2}},\quad\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x=\lim_{n\to\infty}\int_{\mathbb{R}^3}g(x)|u_n|^6\mathrm{d}x=S^{\frac{3}{2}}.\tag{43}$$

Let $v_n=\frac{u_n}{\left(\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\right)^{\frac{1}{6}}}$. We get $\int_{\mathbb{R}^3}V(x)|v_n|^2\mathrm{d}x\to0$, $\int_{\mathbb{R}^3}|\nabla v_n|^2\mathrm{d}x\to S$, $\int_{\mathbb{R}^3}\frac{x}{|x|}|\nabla v_n|^2\mathrm{d}x=0$. By Theorem 1.41 in [34], there exist $y_n\in\mathbb{R}^3$ and $\mu_n\in(0,+\infty)$ such that

$$\left\|v_n-\frac{1}{\mu_n^{\frac{1}{2}}}v_0\left(\frac{x-y_n}{\mu_n}\right)\right\|_{D^{1,2}}\to0.\tag{44}$$

So $\int_{\mathbb{R}^3}|\nabla v_0|^2\mathrm{d}x=S$, $\int_{\mathbb{R}^3}|v_0|^6\mathrm{d}x=1$, that is, S is attained by v_0. By [38], $v_0=\frac{c_0}{(1+|d_0(x-x_0)|^2)^{\frac{1}{2}}}$, where $c_0\neq0$, $d_0>0$, $x_0\in\mathbb{R}^3$. Thus,

$$\left\|v_n-\frac{c_0\mu_n^{\frac{1}{2}}}{\left(\mu_n^2+d_0^2|\cdot-y_n-x_0\mu_n|^2\right)^{\frac{1}{2}}}\right\|_{D^{1,2}}\to0.\tag{45}$$

By $g(0)<1$, there exist $\varrho_0>0$ and $g_m\in(0,1)$ such that $g(x)\leq g_m$ for $|x|\leq\varrho_0$.

Case 1. $\mu_n\to+\infty$ as $n\to\infty$.

Let $x=\mu_nz+y_n$. By $V(x)\geq V_0$, we get $\int_{\mathbb{R}^3}|v_n(x)|^2\mathrm{d}x=\mu_n^2\int_{\mathbb{R}^3}|\mu_n^{\frac{1}{2}}v_n(\mu_nz+y_n)|^2\mathrm{d}z\to0$. Then $\int_{\mathbb{R}^3}|\mu_n^{\frac{1}{2}}v_n(\mu_nz+y_n)|^2\mathrm{d}z\to0$. By (44), we have $\mu_n^{\frac{1}{2}}v_n(\mu_nx+y_n)\to v_0$ a.e. By Fatou's Lemma, we get $\int_{\mathbb{R}^3}|v_0|^2\mathrm{d}z=0$, a contradiction.

Case 2. $\mu_n\to\tilde{\mu}\neq0$ as $n\to\infty$.

Similar to Case 1, we get a contradiction.

Case 3. $\mu_n\to0$ as $n\to\infty$ and $|y_n|\leq\varrho_0$ for large n.

Assume $y_n\to y_0$. Then $|y_0|\leq\varrho_0$. By (43), we have $\int_{\mathbb{R}^3}(1-g(x))|v_n|^6\mathrm{d}x=o_n(1)$. Then by (44),

$$o_n(1)=\int_{\mathbb{R}^3}(1-g(x))\left|\frac{1}{\mu_n^{\frac{1}{2}}}v_0\left(\frac{x-y_n}{\mu_n}\right)\right|^6\mathrm{d}x=\int_{\mathbb{R}^3}(1-g(\mu_nx+y_n))|v_0|^6\mathrm{d}x.$$

By the Lebesgue dominated convergence theorem, we derive that $0=\int_{\mathbb{R}^3}(1-g(y_0))|v_0|^6\mathrm{d}x>0$, a contradiction.

Case 4. $\mu_n\to0$ as $n\to\infty$ and there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ (still denoted by $\{y_n\}_{n=1}^{\infty}$) satisfying $|y_n|>\varrho_0$. By $\int_{\mathbb{R}^3}\frac{x}{|x|}|\nabla v_n|^2\mathrm{d}x=0$ and (45),

$$\begin{aligned}o_n(1)=&\int_{\mathbb{R}^3}\left(\frac{x}{|x|}-\frac{y_n+x_0\mu_n}{|y_n+x_0\mu_n|}\right)\frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2+d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x\\&+\frac{y_n+x_0\mu_n}{|y_n+x_0\mu_n|}\int_{\mathbb{R}^3}\frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2+d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x.\end{aligned}\tag{46}$$

By $\mu_n \to 0$ and $|y_n| > \varrho_0$, we have $|y_n + x_0\mu_n| \geq \frac{|y_n|}{2} \geq \frac{\varrho_0}{2}$ for large n. For any x, $z \in \mathbb{R}^3 \setminus \{0\}$,

$$\left|\frac{x}{|x|} - \frac{z}{|z|}\right| = \frac{|x(|z|-|x|) + |x|(x-z)|}{|x||z|} \leq \frac{2|x-z|}{|z|}.$$

Thus,

$$\begin{aligned}
&\int_{|x-y_n-x_0\mu_n|\leq\mu_n} \left|\frac{x}{|x|} - \frac{y_n + x_0\mu_n}{|y_n + x_0\mu_n|}\right| \frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2 + d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x \\
&\leq \int_{|x-y_n-x_0\mu_n|\leq\mu_n} \frac{2|x-y_n-x_0\mu_n|}{|y_n + x_0\mu_n|} \frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2 + d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x \\
&\leq \frac{4\mu_n}{\varrho_0}\int_{\mathbb{R}^3} \frac{|x|^2}{\left(1 + d_0^2|x|^2\right)^3}\mathrm{d}x \leq C_2\mu_n.
\end{aligned} \tag{47}$$

Furthermore,

$$\begin{aligned}
&\int_{|x-y_n-x_0\mu_n|\geq\mu_n} \left|\frac{x}{|x|} - \frac{y_n + x_0\mu_n}{|y_n + x_0\mu_n|}\right| \frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2 + d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x \\
&\leq \frac{2}{|y_n + x_0\mu_n|}\int_{|x-y_n-x_0\mu_n|\geq\mu_n} \frac{\mu_n|x-y_n-x_0\mu_n|^3}{\left(\mu_n^2 + d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x \\
&\leq \frac{4}{\varrho_0}\int_{|x|\geq 1} \frac{\mu_n|x|^3}{\left(1 + d_0^2|x|^2\right)^3}\mathrm{d}x \leq C_3\mu_n.
\end{aligned} \tag{48}$$

From (46)–(48), we get

$$0 = \lim_{n\to\infty}\left|\int_{\mathbb{R}^3} \frac{x}{|x|} \frac{\mu_n|x-y_n-x_0\mu_n|^2}{\left(\mu_n^2 + d_0^2|x-y_n-x_0\mu_n|^2\right)^3}\mathrm{d}x\right| = \int_{\mathbb{R}^3} \frac{|x|^2}{\left(1 + d_0^2|x|^2\right)^3}\mathrm{d}x,$$

a contradiction. □

Lemma 14. *Assume that (h_2), (V_1), (g_2) and (f_1) hold. Then there exists $h_0 > 0$ such that for $\|h\|_\infty < h_0$ and $u \in M$ with $I(u) < \frac{1}{3}S^{\frac{3}{2}}$, there holds $\int_{\mathbb{R}^3} \frac{x}{|x|}|\nabla u|^2\mathrm{d}x \neq 0$.*

Proof of Lemma 14. By (f_1), for any $u \in M$, there exists a unique $t_u > 0$ such that $t_u u \in M_0$. Then

$$\begin{aligned}
\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\mathrm{d}x &= \int_{\mathbb{R}^3} h(x)f(u)u\mathrm{d}x + \int_{\mathbb{R}^3} g(x)|u|^6\mathrm{d}x, \\
t_u^2\|u\|^2 + t_u^4\int_{\mathbb{R}^3} \phi_u u^2\mathrm{d}x &= t_u^6\int_{\mathbb{R}^3} g(x)|u|^6\mathrm{d}x.
\end{aligned}$$

If $t_u < 1$, then

$$\begin{aligned}
t_u^4\left(\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\mathrm{d}x\right) &\leq t_u^6\left(\int_{\mathbb{R}^3} h(x)f(u)u\mathrm{d}x + \int_{\mathbb{R}^3} g(x)|u|^6\mathrm{d}x\right) \\
&= t_u^6\left(\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\mathrm{d}x\right),
\end{aligned}$$

that is, $t_u \geq 1$, a contradiction. So $t_u \geq 1$. By (49),

$$t_u^2 \leq \frac{\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2\mathrm{d}x}{\int_{\mathbb{R}^3} g(x)|u|^6\mathrm{d}x}. \tag{49}$$

Since $u \in M$, by (h_2), (f_1), for $\varepsilon = V_0$, there exists $C_\varepsilon = C_{V_0} > 0$ such that $\|u\|^2 \leq \|h\|_\infty \int_{\mathbb{R}^3} V_0|u|^2 \mathrm{d}x + C_{V_0}\|h\|_\infty \int_{\mathbb{R}^3} |u|^6 \mathrm{d}x + \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x$. Then for $\|h\|_\infty < 1$,

$$S\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}} \leq C_{V_0}\|h\|_\infty \int_{\mathbb{R}^3} |u|^6 \mathrm{d}x + \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x. \tag{50}$$

Since $g(x) \leq 1$, we obtain that there exists $\eta_0 > 0$ such that $\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \geq \eta_0$. By $I(u) < \frac{1}{3}S^{\frac{3}{2}}$ with $u \in M$, we get $\|u\|^2 \leq \frac{4}{3}S^{\frac{3}{2}}$. Then $\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \leq \frac{\|u\|^6}{S^3} \leq \frac{64}{27}S^{\frac{3}{2}}$. So $\eta_0 \leq \int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \leq \frac{64}{27}S^{\frac{3}{2}}$. By (50), there exists $h_0' \in (0,1)$ such that for $\|h\|_\infty < h_0'$,

$$\frac{S}{2}\left(\int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x\right)^{\frac{1}{3}} \leq \frac{S}{2}\left(\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x\right)^{\frac{1}{3}} \leq \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x.$$

Then $\int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x \geq (\frac{S}{2})^{\frac{3}{2}}$. Together with (49) and $\|u\|^2 \leq \frac{4}{3}S^{\frac{3}{2}}$, we derive that there exists $T_0 > 0$ such that $t_u \leq T_0$. By (f_1), we obtain that for $u \in M$ with $I(u) < \frac{1}{3}S^{\frac{3}{2}}$,

$$\frac{1}{3}S^{\frac{3}{2}} > I(u) = \sup_{t \geq 0} I(tu) \geq I(t_u u) \geq J(t_u u) - \|h\|_\infty \int_{\mathbb{R}^3} F(t_u u)\mathrm{d}x.$$

By $t_u \leq T_0$ and $\|u\|^2 \leq \frac{4}{3}S^{\frac{3}{2}}$, there exists $h_0 \in (0, h_0')$ such that $J(t_u u) \leq \frac{1}{3}S^{\frac{3}{2}} + \eta_0$ for $\|h\|_\infty < h_0$. Since $t_u u \in M_0$, by Lemma 13, we get $\int_{\mathbb{R}^3} \frac{x}{|x|}|\nabla(t_u u)|^2 \mathrm{d}x \neq 0$, that is, $\int_{\mathbb{R}^3} \frac{x}{|x|}|\nabla u|^2 \mathrm{d}x \neq 0$. $\square$

We introduce the Lusternik–Schnirelman category.

Definition 1. *For a topological space X, a nonempty,closed subset $A \subset X$ is contractible to a point y in X if and only if there exists a continuous mapping $\eta : [0,1] \times A \to X$ such that $\eta(0,x) = x$ for $x \in A$ and $\eta(1,x) = y$ for $x \in A$.*

Definition 2. *Define*

$$\begin{aligned} \mathrm{cat}(X) = \min\{k \in \mathbb{N} : &\text{ there exist closed subsets } A_1, \ldots, A_k \subset X \text{ such that} \\ & A_i \text{ is contractible to a point in } X \text{ for all } i \text{ and} \\ & \cup_{i=1}^k A_i = X\}. \end{aligned}$$

In particular, if there does not exist finitely many closed subsets $A_1, \ldots, A_k \subset X$ such that $\cup_{i=1}^k A_i = X$ and A_i is contractible to a point in X for all i, denote $\mathrm{cat}(X) = \infty$.

The following two lemmas are introduced to prove Theorem 2.

Lemma 15 (Lemma 2.5 in [39])**.** *Let X be a topological space. Assume there exist two continuous mapping*

$$P : \mathbb{S}^2 = \{y \in \mathbb{R}^3 : |y| = 1\} \to X, \quad Q : X \to \mathbb{S}^2$$

such that $Q \circ P$ is homotopic to identity, that is, there is a continuous mapping $\sigma : [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$ such that $\sigma(0,x) = (Q \circ P)(x)$ for $x \in \mathbb{S}^2$ and $\sigma(1,x) = x$ for $x \in \mathbb{S}^2$. Then $\mathrm{cat}(X) \geq 2$.

Lemma 16 (Proposition 2.4 in [39])**.** *Let M be a Hilbert manifold and $I \in C^1(M, \mathbb{R})$. If there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $I(u)$ satisfies the Palais–Smale condition for $c \leq c_0$ and* $\mathrm{cat}(\{u \in M : I(u) \leq c_0\}) \geq k$*, then $I(u)$ admits at least k critical points in $\{u \in M : I(u) \leq c_0\}$.*

Proof of Theorem 2. We note that $I(0) = 0$. By the proof of Theorem 3, there exist ρ_0, $\gamma_0 > 0$ such that

$$I(u) \geq \gamma_0, \ \forall \, \|u\| = \rho_0.$$

By the argument of Lemma 10, $\lim_{t\to+\infty} I(tu_{\delta,\tilde{y}}) = -\infty$. Then $I(tu_{\delta,\tilde{y}})$ attained its maximum at a $t_y > 0$. So $\frac{d}{dt} I(tu_{\delta,\tilde{y}})|_{t=t_y} = 0$. We note that

$$\frac{d}{dt} I(tu) = t^3 \left[\frac{1}{t^2}\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2 \mathrm{d}x - \frac{1}{t^3} \int_{\mathbb{R}^3} h(x) f(tu) u \mathrm{d}x - t^2 \int_{\mathbb{R}^3} g(x)|u|^6 \mathrm{d}x \right].$$

By (f_1), we get t_y is unique. Moreover, $t_y u_{\delta,\tilde{y}} \in M$. By Lemma 10, for $\delta \in (0, \delta_0)$,

$$\sup_{t\geq 0} I(tu_{\delta,\tilde{y}}) = I(t_y u_{\delta,\tilde{y}}) \leq \frac{1}{3} S^{\frac{3}{2}} - \delta^{\frac{1}{2}}. \tag{51}$$

Define $P : \mathbb{S}^2 \to M$ by $P(y) = t_y u_{\delta,\tilde{y}}$. Then P is continuous and $P(\mathbb{S}^2) \subset \left\{ u \in M : I(u) \leq \frac{1}{3} S^{\frac{3}{2}} - \delta^{\frac{1}{2}} \right\}$. Define

$$Q : \left\{ u \in M : I(u) \leq \frac{1}{3} S^{\frac{3}{2}} - \delta^{\frac{1}{2}} \right\} \to \mathbb{S}^2$$

by $Q(u) = \frac{\int_{\mathbb{R}^3} \frac{x}{|x|} |\nabla u|^2 \mathrm{d}x}{\left| \int_{\mathbb{R}^3} \frac{x}{|x|} |\nabla u|^2 \mathrm{d}x \right|}$. By Lemma 14, we know Q is continuous. Define $\sigma(\theta, y) : [0,1] \times \mathbb{S}^2 \to \mathbb{S}^2$ such that $\sigma(\theta, y) = Q\big((1-2\theta)P(y) + 2\theta u_{\delta,\tilde{y}}\big)$ for $\theta \in [0, \frac{1}{2})$, $\sigma(\theta, y) = Q\Big(u_{2(1-\theta)\delta,\tilde{y}}\Big)$ for $\theta \in [\frac{1}{2}, 1)$ and $\sigma(\theta, y) = y$ for $\theta = 1$. By the argument of Case 4 in Lemma 13, we have

$$\left| \frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 y}{|\frac{3}{2}\rho_0 y|} \right| \leq \frac{2|x - \frac{3}{2}\rho_0 y|}{|\frac{3}{2}\rho_0 y|}.$$

By a direct calculation, for $|x - \frac{3}{2}\rho_0 y| \leq r$,

$$|\nabla u_{2(1-\theta)\delta,\tilde{y}}|^2 = \frac{(2(1-\theta)\delta)^{\frac{1}{2}} |x - \frac{3}{2}\rho_0 y|^2}{\big(2(1-\theta)\delta + |x - \frac{3}{2}\rho_0 y|^2\big)^3}.$$

Then

$$\begin{aligned}
&\int_{|x-\frac{3}{2}\rho_0 y|\leq r} \left| \frac{x}{|x|} - \frac{\frac{3}{2}\rho_0 y}{|\frac{3}{2}\rho_0 y|} \right| |\nabla u_{2(1-\theta)\delta,\tilde{y}}|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}^3} \frac{4|x - \frac{3}{2}\rho_0 y|}{3\rho_0} \frac{(2(1-\theta)\delta)^{\frac{1}{2}} |x - \frac{3}{2}\rho_0 y|^2}{\big(2(1-\theta)\delta + |x - \frac{3}{2}\rho_0 y|^2\big)^3} \mathrm{d}x \\
&= \frac{4(2(1-\theta)\delta)^{\frac{1}{2}}}{3\rho_0} \int_{\mathbb{R}^3} \frac{|x|^3}{\big(1 + |x|^2\big)^3} \mathrm{d}x.
\end{aligned}$$

By a direct calculation, for $r \leq |x - \frac{3}{2}\rho_0 y| \leq 2r$,

$$\begin{aligned}
|\nabla u_{2(1-\theta)\delta,\tilde{y}}|^2 &\leq \frac{2|\nabla\psi|^2 (2(1-\theta)\delta)^{\frac{1}{2}}}{2(1-\theta)\delta + |x - \frac{3}{2}\rho_0 y|^2} + \frac{2\psi^2 (2(1-\theta)\delta)^{\frac{1}{2}} |x - \frac{3}{2}\rho_0 y|^2}{\big(2(1-\theta)\delta + |x - \frac{3}{2}\rho_0 y|^2\big)^3} \\
&\leq \frac{10(2(1-\theta)\delta)^{\frac{1}{2}}}{2(1-\theta)\delta + |x - \frac{3}{2}\rho_0 y|^2}.
\end{aligned}$$

Then

$$
\begin{aligned}
&\int_{r\leq|x-\frac{3}{2}\rho_0 y|\leq 2r}\left|\frac{x}{|x|}-\frac{\frac{3}{2}\rho_0 y}{|\frac{3}{2}\rho_0 y|}\right||\nabla u_{2(1-\theta)\delta,\bar{y}}|^2\mathrm{d}x\\
\leq &\int_{r\leq|x-\frac{3}{2}\rho_0 y|\leq 2r}\frac{40|x-\frac{3}{2}\rho_0 y|}{3\rho_0}\frac{(2(1-\theta)\delta)^{\frac{1}{2}}}{2(1-\theta)\delta+|x-\frac{3}{2}\rho_0 y|^2}\mathrm{d}x\\
\leq &\frac{40(2(1-\theta)\delta)^{\frac{1}{2}}}{3\rho_0 r}\int_{r\leq|x|\leq 2r}\mathrm{d}x.
\end{aligned}
$$

Thus, we have $\lim_{\theta\to 1-}\int_{\mathbb{R}^3}\left(\frac{x}{|x|}-\frac{y}{|y|}\right)|\nabla u_{2(1-\theta)\delta,\bar{y}}|^2\mathrm{d}x=0$. By Lemma 2,

$$
\lim_{\theta\to 1-}\int_{\mathbb{R}^3}\frac{x}{|x|}|\nabla u_{2(1-\theta)\delta,\bar{y}}|^2\mathrm{d}x=\lim_{\theta\to 1-}\frac{y}{|y|}\int_{\mathbb{R}^3}|\nabla u_{2(1-\theta)\delta,\bar{y}}|^2\mathrm{d}x=K_1 y.
$$

Then by the continuity of Q, we obtain that $\sigma(\theta,y)\in C([0,1]\times\mathbb{S}^2,\mathbb{S}^2)$, $\sigma(0,y)=Q\circ P(y)$ for $y\in\mathbb{S}^2$ and $\sigma(1,y)=y$ for $y\in\mathbb{S}^2$. By Lemma 15, we have

$$
\mathrm{cat}\left(\left\{u\in M: I(u)\leq\frac{1}{3}S^{\frac{3}{2}}-\delta^{\frac{1}{2}}\right\}\right)\geq 2.
$$

By Lemmas 11 and 12, we know I satisfies the (PS) condition for $c\in\left(0,\frac{1}{3}S^{\frac{3}{2}}\right)$. Then by Lemma 16, we obtain that I has two nonnegative critical points $u_{i,h}$, $i=1,2$. By the maximum principle, $u_{i,h}$ is positive. □

5. Conclusions

We first study multiplicity of solutions of the singularly perturbed Schrödinger–Poisson equation with critical growth. When the perturbed coefficient is small, we establish the relationship between the number of solutions and the profiles of the coefficients, which is different from the existing results. We pointed out that, when we seek multiplicity of solutions, it is crucial to prove the compactness of the Palais–Smale sequence. Many authors solved the problem by imposing the Rabinowitz type assumption, which is restrictive. In this paper, we remove the technical assumption. Furthermore, we study multiplicity of solutions without any restriction on the perturbed coefficient. By using the Lusternik–Schnirelman category and developing some techniques, we obtain a multiplicity result. Besides, we study the existence of solutions of non-autonomous Schrödinger–Poisson equations without the classical (AR) condition or the monotony condition. We introduce a new argument to solve the problem.

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