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Review

Mathematical Representation Competency in Relation to Use of Digital Technology and Task Design—A Literature Review

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Abstract: Representations are crucial to mathematical activity, both for learners and skilled mathematicians. Digital technologies (DT) to support mathematical activity offer a plethora of new possibilities, not least in the context of mathematics education. This paper presents a literature review on representations and activation of students' representation competency when using DT in mathematics teaching and learning situations. It does so with a starting point in task designs involving digital tools aiming to activate representation competency, drawing on the notion of Mathematical Digital Boundary Object (MDBO). The 30 studies included in the literature review are analyzed using Duval's registers of semiotic representations and the representation competency from the Danish KOM framework. The results reveal a clear connection between the mathematical topics addressed and the types of representation utilized, and further indicate that certain aspects of the representation competency are outsourced when DT are used. To activate the representation competency in relation to the use of DT, we offer five suggestions for consideration when designing mathematical tasks. Finally, we raise the question of whether DT create new representations or merely new activities.

Keywords: representation competency; dynamic geometry; computer algebra systems; task design

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1. Introduction and Motivation

One of the most obvious affordances of digital technologies (DT) in the teaching and learning of mathematics is their capability to visualize representations of mathematical objects. A quick search on "mathematical representations" and "digital technology" in most relevant databases will reveal this through the mere number of hits. From mathematics education research, we know that mathematical understanding of a given concept—and its associated processes and objects [1,2]—is closely connected to being able to handle different representations of that concept (object). For Sfard [3], for instance, being able to shift between different representations of the same object is part of the condensation of a concept—a step on the way towards reification. For Duval [4] (see later), being able to access and shift between different mathematical representations is the key to all mathematical understanding and activity. Furthermore, mathematical representation is an essential part of mathematical thinking, reasoning and communication [5].

If, however, one adds "task design" to the search mentioned above, the number of hits decreases dramatically (down to perhaps only a handful). Hence, if we are interested in how to design mathematical tasks so that specific features of a given technology can be capitalized upon in relation to students' conceptual mathematical knowledge, it seems we are left with our own ingenuity. Nevertheless, much of the available literature, if scanned more closely, does provide bits and pieces of information on task design in relation to the work with mathematical representations.

In this paper, we make a first attempt at this, using as our focus for identifying work with mathematical representations the components of the so-called mathematical representation competency [5,6], limiting "DT" to Dynamic Geometry Systems (DGS) and Computer Algebra Systems (CAS). More precisely, we address the questions:

- *What does the mathematics education literature have to offer in relation to mathematical representations and representation competency in teaching and learning situations involving DGS or CAS*?
- *How can tasks relying on DGS or CAS be designed in a manner that supports students' activation of mathematical representation competency (according to the literature identified as part of answering question 1)*?

In the following, we first provide the theoretical background of the study. Next, we account for our review methodology. We then present our review results, categorizing and analyzing these according to the representational registers [4] and the different aspects of the representation competency involved [5]. We end the paper with a discussion of these findings using the lens of task design, thus attempting to answer the questions posed above.

2. Theoretical Background

The theoretical background of the present study is a combination of the notion of mathematical competency, drawing on the work of Niss and Højgaard [5,6], in particular the representation competency; Duval's [4,7] work with mathematical representations through the use of semiotic registers; and finally the recent attention brought to task design in digital environments in mathematics education research (e.g., [8])

2.1. The KOM Framework and Its Mathematical Representation Competency

In the Danish mathematical competencies framework [5,6], the so-called KOM-framework, mathematical mastery is characterized as comprising eight mathematical competencies. The framework is integrated in Danish mathematics education curricula and has influenced mathematics education in other parts of the world (e.g., [9,10]).

The KOM-framework defines mathematical competency as an individual's "insightful readiness to act appropriately in response" to a "specific sort of challenge that actually or potentially calls for 'specific kinds of activation' of mathematics in order to answer questions, solve problems, understand phenomena, relationships or mechanisms, or to take a stance or make a decision" [5] (p. 14). Although each competency has its own unique identity, the eight competencies are interwoven, as illustrated in the KOM flower (Figure 1), which also lists the eight competencies.

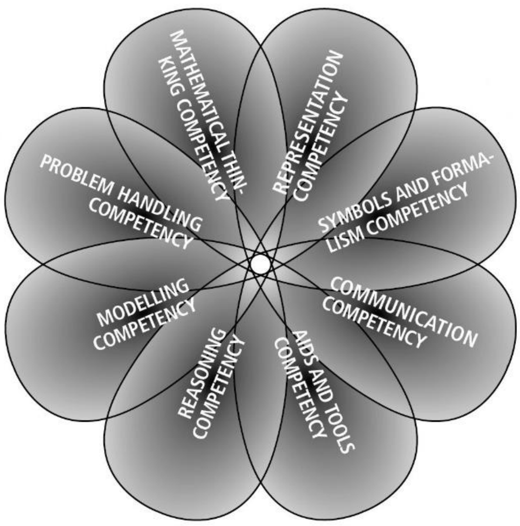

Figure 1. The so-called KOM flower.

Each of the eight competencies has an "investigative" and a "productive" side. The "'productive' side of a competency consists of being able to, by oneself, carry out the processes covered by the competency", while the "'investigative' side comprises an understanding, analysis and critical assessment of the processes already carried out and the products thereof" [6] (pp. 70–71). Since the focus of this study is mathematical representations, we concentrate our further description on the mathematical representation competency and its more closely related competencies.

The representation competency comprises the ability to decode, "interpret as well as translate and move between a wide range of representations (e.g., verbal, material, symbolic, tabular, graphic, diagrammatic or visual) of mathematical objects, phenomena, relationships and processes" [5] (p. 17). The competency further comprises the ability to reflect and decide which representations to utilize when handling mathematical tasks or situations. Moreover, it calls for the ability to reveal the limitations and strengths of these representations, as well as the loss or gain of information when switching between representations (e.g., what is the information loss when shifting between a mathematical expression for a function, a table of values, and a graph). The *symbols and formalism competency* is about decoding symbolic and formal language, translating between mathematical symbolism and natural language, handling and utilizing mathematical symbolism, and transforming symbolic expressions. It also focuses on the nature, role and meaning of symbols (and formal systems) and on the rules for their usage. Both the competency of representation and of symbols and formalism are closely related to the *communication competency*. In short, this consists of the ability to study and interpret others' written, oral or visual mathematical statements, explanations or texts as well as the ability to express oneself mathematically in such ways.

On the use of DT, Niss [11] finds that:

> [It can] help generate student experiences of mathematics-laden processes and phenomena that might be difficult to obtain by other means; create platforms and spaces for exploration in which mathematical entities can be investigated through manipulation and variation; produce static and dynamic images of objects, phenomena, and processes that are otherwise difficult to capture and grasp; create connections between different representations of a given mathematical entity; help solve hard or otherwise inaccessible computational problems; perform rule-based symbolic transformations and manipulations; support the production of mathematical texts... [11] (p. 248).

Adhering to Niss' view on DT, it becomes apparent that representations are key in many of the affordances of DT.

2.2. Mathematical Representations and Semiotic Registers

According to Duval [4], in mathematics we cannot directly access the mathematical objects in the same way as, for instance, in the natural sciences through various measuring instruments etc. We can only access mathematical objects through semiotic representations. For Duval, it is a crucial point that different mathematical representations belong to different semiotic registers, and that mathematical activity—and understanding—essentially is about being able to make shifts between such semiotic registers.

Duval [4] points out that the role played by signs in this regard is not to be placeholders for the mathematical objects but for other signs. In this perspective, signs and transformations between different semiotic representations are the core of mathematical activity—as opposed to the activities of other scientific disciplines. The possibilities of substituting one semiotic representation with another depend on the semiotic system. Every system offers specific possibilities. Hence, the capacity of a given representation does not depend on the individual symbol (or sign), but on the semiotic system of which it is a part. One distinction is for example language (natural and symbolic) versus images (figures, graphs, etc.). However, according to Duval [4], such distinction is too general, and causes us to overlook an important point, namely that while some semiotic systems may only be used to perform mathematical processes, others possess a larger variety of functions.

Duval [4] lays down four types of semiotic registers that emerge by combining two distinctions. The first distinguishes discursive and non-discursive registers. The discursive registers are languages (both oral and written) that express meaning units of thoughts and thought operations. Hence, these are process-oriented, while the non-discursive registers simply display one or more static, visual objects. The other distinction is between monofunctional and multifunctional semiotic registers: "Some semiotic systems can be used for only one cognitive function: mathematical processing . . . within a monofunctional semiotic system most processes take the form of algorithms" [4] (p. 109). A multifunctional semiotic system "can fulfill a large range of cognitive functions: communication, information processing, awareness, imagination, etc.", and "within a multifunctional semiotic system the processes can never be converted into algorithms" [4] (p. 109).

Duval's [4,7] distinctions are usually illustrated in a two by two diagram, of which Figure 2 is a somewhat reduced version. For the remainder of the text we will refer to each register by its main characteristic type of representation. We do this so the reader can easily recall the registers, however bearing in mind that a register is defined by the two distinctions as described above. From here on, the multifunctional discursive register is referred to as the *linguistic,* the multifunctional non-discursive register as the *figurative*, the monofunctional discursive register as the *symbolic*, and the monofunctional non-discursive register as the *graphic.* These have been added to the diagram in parentheses for clarity. So-called transitional auxiliary representations may serve the purpose of easing the transition between multifunctional and monofunctional discursive registers. As we shall see later, DT may also play the role of such "auxiliaries" between semiotic registers.

	Discursive registers	**Non-discursive registers**
Multifunctional registers (do not take the form of algorithms)	Natural language, spoken or written that creates meaning units (*linguistic*)	Iconic imaging such as drawings, sketches and non-iconic geometric figures (*figurative*)
	Transitional auxiliary representations	
Monofunctional registers (take the form of algorithms)	Symbols, including number systems and formal writing (*symbolic*)	Cartesian diagrams and graphs, including strokes and arrow joining marks or nodes (*graphic*)

Figure 2. Duval's four registers of semiotic representations.

It is, for Duval [4,7], a main point that mathematical activity can take place within one of these four registers or between them (vertically or horizontally). Mathematical work within one register is referred to as a *treatment*, while work between registers—that requires a shift of register—is referred to as a *conversion*. An example of a conversion might be the mathematization of the equation story, "Aya is 3 years older than her brother Ali. Together they are 23 years old. How old are they?" into the algebraic equation $x + (x + 3) = 23$, which takes place between a multifunctional (the linguistic) register and a monofunctional (the symbolic) register. Solving the resulting equation step by step, however, to reveal that $x = 10$, is a treatment, since this takes place within the same register, namely the symbolic system [12]. Therefore, for conversions, source register and target register are different, whereas for treatments they are the same. Duval [4] has found that work within one register, i.e., treatments, typically gives rise to much less (although in no way negligible) difficulty on students' behalf than work requiring a shift of register, i.e., conversions. Duval also notes that there are two different types of conversion. A *congruent conversion* is a straightforward translation—or coding— like the mathematization of the equation story into a symbolic expression above. A *non-congruent conversion* is, however, much more complicated. For example, the opposite translation in our example, i.e., going from the symbolic expression, $2x + 3 = 23$, to an equation story, since there are infinitely many stories to be told based on this equation [12].

When it comes to use of DT in relation to mathematical activities involving representations and shifts between these, Duval [7] is somewhat skeptical:

> [...] the use of a computer for everything that concerns mathematical visualization, both in geometry and in analysis, and geometrical or graphical software opens considerable possibilities of creation and visual exploration. But does software suffice to develop in the students the ability to anticipate the different possible transformations of a given figure into others completely different? Does it make students aware of the one-to-one mapping between graphic visual values and the terms of the equations they represent? [7] (p. xii)

Here, Duval points out that the technology takes over the actual transformation of representation, and consequently takes the knowledge of doing translations away from the students, both in the general overview of possibilities and in the specificities of how the translation is actually obtained.

2.3. Task Design in Relation to Use of Digital Technologies

Nearly two decades ago, Sierpinska [13] called attention to the fact that research reports in mathematics education research seldom include detailed descriptions concerning the design of the tasks being used in the studies. Since then, however, there has been a growing interest in task design as a research area [14], including interest in the specific context of task design in relation to digital tools [15].

According to Leung [16], a generic mathematics task involves asking students to do something of a mathematical nature, which will lead the students to experience mathematics. A tool-based task, then, comprises a design that aims to activate a tool-based environment in which experiences can be produced that may be linked to mathematics [8,16].

Leung and Bolite-Frant [8] describe how a digital tool, such as a DGS or CAS, can be viewed as a Mathematics Digital Boundary Object (MDBO) (The objects of MDBO are not to be confused with usual mathematical objects as referred to by e.g., Duval [4].) that can mediate experiences between mathematical worlds. In this context, mathematical worlds can be viewed as a social community (e.g., a mathematics classroom) or a body of institutionalized/experiential mathematical knowledge (e.g., a mathematics curriculum or mathematical knowledge based on the use of a digital tool). The potentialities of the MDBO can operate as a "bridge" by creating a discourse that allows for communication between mathematical worlds. Considering task design in light of these notions, Leung [16] proposes that

> task design using digital tool to teach and learn mathematics can be thought of as the designing of pedagogical paths (trajectories) to create MDBO-based situated discourses that can reconcile meanings and maintain coherence between mathematical worlds [16] (p. 8).

This understanding of discourse in relation to task design is different from the one represented by Duval [4,7]. With reference to Sfard [17], digital tools act as mediators of discourse. When using discourse in relation to MDBO, two different orientations exist: (1) regarding student's discourse, educators "may find what is missing from the mathematical discourse legitimated by the mathematics community", and (2), "mathematics is a discourse, so learning is participating in discourse" [8] (p. 192). Thus, the first focuses on language and the use of it, while the second looks at participation and changes of participation in relation to learning.

The characteristics and pedagogical features of an MDBO are paramount to its usefulness in mediating and translating between mathematical worlds. Leung and Bolite-Frant [8] suggest five design heuristics that are relevant aspects in tool-based mediation processes of teaching and learning mathematics, which they argue that MDBO should support. One of these heuristics is multiplicity, which can be related to mathematical representations, and hence the representation competency:

> As a boundary object that translates mathematical meaning, if a MDBO supports multi-representations, then meaning can be deepened in a multi-facet way. In a Dynamic and Interactive Mathematics Learning Environments like GeoGebra, multi-windows (e.g., 2-D DGE, 3-D DGE, Spreadsheet) can be presented together on the same screen. The same mathematical idea is represented in each window and elements in these windows can be constructed to co-vary together while a variable is taking different values in certain window. This kind of multiplicity offers mathematical idea/concept diverse communicable expressions/translations that are amiable to different mathematical worlds. [16] (p. 9).(Leung [16] refers to Dynamic Geometry Environments (DGE), which in mathematics education literature is used synonymously with Dynamic Geometry Systems (DGS). Throughout this paper, we use DGS to refer to such dynamic software.)

Integrating this heuristic of multiplicity of DGS and CAS in a task design gives students access to different representations of the same mathematical entity and opens an opportunity for them to deal with different representations and explore their strengths and weaknesses. We will return to this in our discussion (see Section 6.2), with a further elaboration of the idea of using DGS and CAS as MDBOs, specifically for supporting the representation competency, based on the analyses of the literature review.

3. Review Method

We now describe our review strategy, including the criteria for including and excluding publications in our review. We searched for literature in databases. Results were all collected in *Covidence*, a tool to handle literature reviews. The acquired literature was assessed using inclusion and exclusion criteria based on the research questions. The criteria are summarized and presented in Table 1 below.

First, we searched within MathEduc (https://www.zentralblatt-math.org/matheduc/, accessed on 29 January 2021). We developed two search strings: one on the use of DGS and one on the use of CAS. The string on DGS resulted in 131 hits, whereas the string on CAS resulted in 126 hits:

- any:represent* & cc:U7* & any:("dynamic geometry system" | ICT | "digital technolog*" | DGS | geogebra | dynamic*) & la:english.: la:en & ti:represent* (5 February 2019).
- (any:"CAS" | dynamic* | digital* | cc:U7*) (08 February 2019). The cc:U7* is MathEduc's classification for Technological tool.

Both searches were compiled in *Covidence*, from which we removed 23 duplicates (cf., stage 1, Figure 3). Two of the authors then independently screened publications by abstract and title, clearing out cases on which they did not agree (stage 2, Figure 3). Next, the included publications were analyzed using the following categories: findings; type of article (empirical or theoretical); kind of tool, if the article addresses the interplay between representation and technology; how it is related to the mathematical representation competency; and finally, its relation to task design (stage 3, Figure 3).

Table 1. Summary of inclusion and exclusion criteria for our review.

	Inclusion Criteria	Exclusion Criteria
Representation competency (1)	Studies focusing on representations and students' work with representations and students' representation competency as described in KOM: interpreting and understanding reciprocal relations between representations; knowing the strengths and weaknesses of representations; translating and moving between representations; and reflectively choosing representations when dealing with tasks [5,6].	Studies either not focusing on representations; students' ability to interpret/understand and translate/move between/choose representations; or connecting the focus on representing/representation competency to the use of digital tools (DGS or CAS).
Use of DT (2)	CAS and DGS.	Anything that is not CAS or DGS.
Age group of participants (3)	Primary and secondary (lower + upper secondary).	Kindergarten, adult students, preschool teachers, teachers, university students, college students, engineering students.
Types of students (4)		Students with dyscalculia, deaf students, students with special needs, and talented students.
Types of studies (5)	Empirical or theoretical.	Studies without any documentation.
Content (6)	Geometry, algebra (including functions), and probability and statistics.	Programming, physics, chemistry, biology, engineering.

Stage 1: 257 references imported for screening as 257 studies

23 duplicates removed

Stage 2: 234 studies screened by title and abstract

112 studies excluded

Stage 3: 122 studies assessed for full-text eligibility

92 studies excluded

- 46 excluded studies related to criterion 1
- 15 excluded studies related to criterion 2
- 19 excluded studies related to criterion 3
- 3 excluded studies related to criterion 6
- 9 Not found

Stage 4: 30 studies included

Figure 3. Overview of the identified literature, using *Covidence*. We also excluded studies by criteria 4 and 5, but we did that in stage 2.

As part of the final stage, we conducted a coding of the remaining 30 studies (stage 4, Figure 3). The first coding consisted of categories concerning type of study (theoretical or empirical); type of tool (DGS or CAS and name); age of students involved; mathematical content in play; and theoretical contributions of the study. Then followed two codings of the literature in relation to the first research question. First, a coding was made based on Duval's [4] description of representational transformations (cf., Figure 2) in relation to the use of representations in the literature. This is presented in the following Section 4. Then a final coding was made with more specific categories related to the representation competency. This included considerations about the relationship between mathematical representations in play and the use of DT, explicitly stating how the tool use contributed to or acted together with the students' representation competency. Based on this, four aspects defining representation competency were formulated, drawing closely on Niss and Højgaard's [5,6] description of the competency. These four aspects served as means of structuring the presentation of our review results in Section 5. Any relation to task design was noted along the way. Together with the coding and analyses, these serve as a means of answering our second research question, which is done in Section 6.

4. Results and Analyses Regarding Representational Transformations

To identify patterns in relation to the type of representations addressed in the included studies, we examined the acquired literature from the perspective of Duval's [4] registers of semiotic representations. More specifically, we classified the literature according to which type of register the representations in the studies belong to (see Table 2). Some studies deal with different types of representations and were therefore classified in several registers. For example, three studies [18–20] involve representations that belong to all four semiotic registers. Ofri and Tabach [20] is the only study that includes not only all four semiotic registers, but also transitional auxiliary representations in the form of tables. On the other hand, three other studies [21–23] appertain to one semiotic register only: Özgün-Koca and Edwards [23], whose study revolves around the topic of statistics, includes representations only related to the graphic register, while Laborde and Laborde [22] and Laborde [21], whose topic of focus is geometry, only concern representations related to the figurative register. Yet, both papers (i.e., [21,22]) focus on treatments of iconic and non-iconic visualizations within the figurative register.

Table 2. Overview of the relations between topics and registers in play for the included studies.

Topic	Register / N	Linguistic	Figurative	Symbolic	Graphic	Transitional Auxiliary
Functions (only)	11	4	1	11	11	6
Functions on geometric properties	9	4	9	8	9	2
Geometry (only)	6	1	6	3	0	2
Statistics	1	0	0	0	1	0
Other	3	2	1	3	2	1
Total	30	10	17	24	23	11

Activating only one register might come across as a missed opportunity to draw on the multiplicity of representations in the tools. However, Laborde [21] argues that 3D DGS has potential to assist students in constructing non-iconic visualizations of a 3D object by deconstructing it into units of the same or lower dimension. This idea of deconstructing is in line with Duval [7], who states that

> the first main activity for learning geometry is not to construct figures instrumentally or with a software, but to deconstruct dimensionally all the recognized shapes 2D. This requires a specific operation, which has become reflex for mathematicians and teachers, but it is not all for students at all [7] (p. 61).

The affordance of digital tools is not only to construct a set of examples of geometric figures but, also, and especially, that they can be used to deconstruct and explore the figurative representations as if they were real objects. In this way, DT facilitate a new epistemological function: exploration by simulation [7]. This discussion mainly concerns the figurative representations used in the topic of geometry.

From the analyses of the literature, emerging relationships become evident between the type of representations that are in focus in the studies and the researched topic. To investigate this pattern further, we decided to place the studies in categories that were developed ad hoc. Five categories were established: functions (only); functions on geometric properties; geometry (only); statistics; and a final category named "other". The category "functions (only)" covers algebraic concepts, which relate to different constituent parts of functions such as dependent and independent variables; equations of different order; different kinds of functions; and different elements of differential and integral calculus. The "(only)" refers to the fact that these studies concern functions *only* in the algebraic setting. Lagrange [24] argues that not only a change of representation forms, but also a change of "setting", is an important aspect when considering the use of representations. "A setting is constituted of objects from a branch of mathematics, of relationship between these objects, their various expressions and the mental images associated with these objects" [24] (p. 246). With reference to Duval, Lagrange notes, "in a change of setting, the whole problem is transferred to other objects, while in conversions of the representations the objects do not change" [24] (p. 248). Such change of setting is present for the second category, "functions on geometric properties", where usually a geometric problem is given in a geometric setting but then solved in an algebraic setting (e.g., [24,25]. Despite the geometric problem, the concept in focus is still mainly functions, and geometry is used as a well-known setting to explore the concepts of variables, functions, and calculus. In the category "geometry (only)", the studies concern 2D and 3D geometry (e.g., [22]) and hyperbolic geometry (e.g., [26]). The "statistics" category focuses on data, statistical concepts and diagrams [23]. Lastly, the category "other" contains studies that did not fit into the previous four, such as one concerning convergence and iterations [27] and another concerning complex numbers [28].

The 30 included studies were assigned to only one of the topic categories to distinguish which semiotic registers were in play for which topics as illustrated in Table 2.

If we apply a broader view and look at patterns across the papers, we can see that the type of representations most commonly used are representations from the symbolic register (24 papers) and the graphic register (23 papers). The digital tools, in particular DGS, create an environment that can combine all registers and thereby both geometric and algebraic situations for the students [29]. Many of the studies exploit this potential when focusing on the use of a digital tool. Particularly, studies assigned to "functions (only)", N = 11, deal mainly with the symbolic and the graphic registers; only one study (i.e., [30]) also involves representations of the figurative register, although it does not use geometry. The picture is almost the same for the category labeled "functions on geometric properties", which also deals with the symbolic and the graphic registers. Moreover, within this category, all N = 9 studies also involve the figurative register, due to the inclusion of geometric figures and concepts. Of the identified studies, Falcade and colleagues' study [31] is the only one that does not include symbolic representations. Instead, this study focuses on how the students formulate their actions regarding figurative and graphic representations of dependent and independent variables and functions as geometric transformations, such as reflection, as well as how this can lead to definitions that are more formal. Hence, the linguistic representations are better presented in this study than in many of the other studies.

In the category "geometry (only)", N = 6 studies are included. Half of them, N = 3 [21,22,32], mainly involve figurative representations, whereas the other half, N = 3 [26,33,34], connect the figurative register with the symbolic register in forms of algebraic representations.

Considering the use of transitional auxiliary representations, the connections to the given topics are not as evident. The most used type of transitional auxiliary representations is overview tables to look for and translate patterns of geometric relations between length and area, for instance, as a step towards a formal proof [34]. In the studies of Ofri and Tabach [20] and DePeau and Kalder [35], the tables work as a bridge between the geometric problem given in the figurative register and the graphic illustrations of the geometric correlation given in the graphic register. Gray and Thomas [36] discovered that the students in their observations used tables in a similar way to bridge between the graphic and symbolic registers. Whereas most of these studies illustrate how transitional auxiliary representations in the form of tables is used to help conversions, Sack [32] illustrates how students can use tables for a better understanding of 3D DGS representations within the figurative register.

In summary, in Table 2 we see that papers focusing on functions use representations from the symbolic and graphic registers. The same applies to the studies of algebraic concepts in the category "other" whereas, in the studies involving geometry, the representations are primarily in the figurative register. Given that the included literature considers the use of digital tools, this illustrates that representations play a significant role when using DGS and CAS in learning situations. In particular, for functions and similar algebraic concepts, shifts between the two mono-functional registers (symbolic and graphic) are dominant, whereas, for geometry, the multifunctional register (the figurative) plays a bigger role. Among the four registers, the linguistic register receives the least attention. According to Duval [7], interacting with a computer eliminates the use of language, which is also the implication from the overview of relations between topic and registers in play. The linguistic register is the only register being neglected in the included studies. We shall return to this in our discussion in Section 6.

5. Results and Analyses Regarding Representation Competency

The four aspects defining representation competency are listed below. Aspects A and B concern the investigative part of the competency, whereas aspects C and D concern the productive part of the competency.

A. Interpreting and being able to understand the reciprocal relations between different representational forms of the same entity.
B. Knowing about the strengths and weaknesses of representations, including the loss or increase in information.
C. Translating and moving between a wide range of representations.
D. Reflectively choosing representations in dealing with mathematical situations and tasks.

The 30 included studies were divided into four groups in accordance with these four aspects; of course, some studies address more than just one of the aspects. Again, Table 1 served as a way of providing an overview of the involved studies. Findings for each of the 30 studies were scrutinized, potentially to identify tendencies across the studies according to the respective aspects of the representation competency. In the following four subsections, we present the results and analyses.

5.1. Interpreting and Being Able to Understand Reciprocal Relations

In the first aspect, A, interpreting and being able to understand reciprocal relations between different representational forms, N = 20 studies are included. N = 14 concern functions, where six of these combine functions and geometry. Across all 20 studies included in aspect A, DGS, CAS and a Graphical Calculator (GC) are applied, and some of the studies use more than one type of tool (DGS, N = 14; CAS: N = 7, GC, N = 2). Between all 20 studies, Duval's four [4] registers as well as transitional auxiliary representations are present. Using Duval's terminology, this aspect of the representation competency is about being able to understand and follow treatments and conversions, but not necessarily being able to carry them out oneself.

It is well known that dynamic tools such as sliders, dragging and tracing can help students understand and interpret treatments and conversions. With reference to Duval's semiotic registers and transformations, Heid, Thomas and Zbiek [37] argue that "CAS environments are capable assistants in both treatments and conversions. Important conceptual aspects arise from relating, through conversions, corresponding elements of conceptual representations" [37] (p. 606). Thus, studying and understanding connections between different representations is a key element of conceptualization. Many of the included studies stress this point, in particular for the concept of function [20,30,38–41]. Steketee and Scher [30] describe and illustrate activities on functions in five different representations. Composing and manipulating the same function in many different representations emphasizes the connections between the representations and refines the students' concept of function. Davis [38] gives an example of using a dynamic slider for the parameter a in the quadratic function, while keeping the other parameters—b and c—invariant. This makes the exploration of the parameter, a, more comprehensive. Similar set-ups for studying parameter effects on quadratic equations are described by Lin and Hsieh [40] using Geometer's Sketchpad and by Ozgun-Koca and Edwards [41] using the CAS TI-*n*spire. The dynamic features can also be used to investigate invariant properties under variable circumstances (e.g., [22,42]). For instance, investigating "the dynamic parallelogram $ABCD$, constructed on variable points A, B, and C represents two relationships of parallelism between two opposite sides [...] the relationships of parallelism are invariant in the dragging, while points and sides vary." [22] (p. 188)

The above examples illustrate how different representation forms are useful for conceptualization of a mathematical concept. In addition, change of setting is important for students' progress and conceptualization [24]. An example of this could be considering the area of a rectangle as a function of the length of one of the sides. The mathematical object of a rectangle in the geometric setting changes to the mathematical object of a quadratic function in the algebraic setting with help from DGS (e.g., [18,20,24,25,39,43]). Modeling geometric dependencies algebraically, using the reciprocal relations between the representations, helps students foster functional thinking [43] and explore and better understand both concepts represented, i.e., the geometric concept being modeled and the algebraic concept being the modeling tool [25]. The idea of linking geometric and algebraic representations also applies to other mathematical concepts than functions [28,42,44]. For instance, exploring the arithmetic of complex numbers by dragging geometric representations combines the representations of complex numbers as ordered pairs, as points in the complex plane, and as the algebraic form $ai + b$ [28]. If students are only presented with the symbolic representation of algebraic phenomena and objects, they do not learn much about the nature of the object that the symbols represent [44]. Here, a given digital tool with its inter-representational potentials, can act as an "epistemic mediator in order to help students to abstract properties of objects" [44] (pp. 197–198).

Most of the studies included under this aspect point to the multi-representational contribution of the digital tools as a aid in understanding conversions between figurative, symbolic, and graphic registers. However, the studies addressing 3D geometry mainly focus on treatments within the figurative register. Here, the reciprocal relations are between geometric representations of 2D and 3D (e.g., [21,22,32]). For example, going from a representation of two non-intersecting lines in 2D to 3D allows the students to obtain immediate visual evidence that the two lines do not intersect, despite appearing to do so in the 2D diagram [21]. Sack [32] introduces concrete models of 3D figures, which implies that the students need to interpret and understand the relations between the representations within the 2D and 3D environment and concrete representations outside the environment. This is in contrast to the many other examples where the computer helps the students do these interpretations, as the digital environment represents different forms of representation at the same time.

One of the potentials of DT is their ability to show different representational windows at the same time [23,24,39]. Furthermore, DT can provide fully linked representations. Hence, they can carry out the heavy translations between the representations so that the students can focus on how changes of one representation affects other representations [41]. With the cognitive load removed, the students can concentrate on the purpose of the task and the relevant concepts. However, it is important to be aware of the syntax level of CAS, as the students do not always know the demands of the tool, nor how to decode the outcome. In addition, the tool's representations might not always be consistent with the conventional mathematical representations, which is an important aspect to be aware of when including digital tools in the classroom [45]. If the students do not know how to interpret the feedback from the digital tool, the interaction with the representations does not necessarily develop their *representational fluency* (*Representational fluency* "includes the ability to interact with these representations, using them as conceptual tools and to demonstrate the flexibility of being able to move from one representation to another, recognizing invariant properties, etc." [35] (p. 7)). In a study with grade 10 students, Gray and Thomas [36] found that the use of GC did not support the students in connecting graphic and algebraic representations of the quadratic function. In their conclusion, they stress the need for finding suitable pedagogical formats that use GC to support representational fluency and conception. Lagrange and Psycharis [43] also emphasize the central role of tool design for successful work with graphic and algebraic representations.

For this aspect of the representation competency, the results indicate that DT have the potential to play a central role. Mathematics educators and task designers can draw on the multiplicity of the tools for concept formation and development, as the tools can help carry out the actual representational transformations. In this way, the students can focus on the interpretation and understanding of the inter-relational connections in the transformations. With dynamic features such as sliders and dragging, the students interact with the representations of the concept. With the possibility of accessing multiple representational windows at the same time, students can interpret the direct changes of one representation compared to the other and using the reciprocal relations between the representations can support students' functional thinking. Furthermore, the results illustrate how this aspect of the competency is a key element for conception, in particular of functions and algebraic structures and phenomena. To profit from these potentials of using DT, well-produced task designs are needed, and mathematics educators should be aware of supporting students' interpretations of feedback from the tool. The interpretations should be coherent with the mathematical concepts, and the students should learn to relate the use of symbols within the tool to the use of symbols in conventional mathematics.

5.2. Knowing about the Strengths and Weaknesses of Representations

The second aspect, B, concerning knowing about the strengths and weaknesses of different representations, including loss or increase in information, includes N = 4 studies. N = 2 concern geometry, N = 1 focuses on a combination of algebra and geometry, and N = 1 on statistics. Both CAS and DGS are present (CAS, N = 1; DGS, N = 2; both CAS and DGS, N = 1). N = 2 studies do not present any empirical data. N = 2 present empirical data from students in grade 6 and 10, respectively.

In relation to the strengths and weaknesses of different representations, being critical is important. The four studies emphasize how DT can work as a tool to compare and search for discrepancies between representations [19,21,23,34]. Hershkowitz and Kieran [19] identify two methods in which students join tool-based representations: "the mechanistic-algorithmic" and "the meaningful". The former does not include a critical aspect, whereas the latter includes consideration of which crucial properties of the functions are relevant. Laborde [21] states that it is important to be aware of the differences from 2D to 3D representations in geometry. For instance, as also mentioned in the above section, two non-intersecting lines in 3D may appear to intersect in 2D. The differences between these two types of representation can be accessed by breaking down the 3D constructions into non-iconic figures, if a DGS provides 2D and 3D representations. As for representations of geometric and algebraic entities, representations of statistical data sets also highlight different characteristics while hiding others. With the potential to split screens in CAS, different representations of the data set can be shown simultaneously. "Studying multiple representations will help students not only see connections among representations but also understand why one representation might be more appropriate than another" [23] (p. 511). This aspect seems so have received less attention in the literature; however, it is crucial for students to develop this. The few studies indicate that it is possible to draw attentions to information and information loss between and within registers by the use of DT. However, both task designers and mathematics educators must be aware of the need to include this aspect.

5.3. Translating and Moving between a Wide Range of Representations

For aspect C, i.e., translating and moving between representations, N = 12 studies are included. Most of these studies concern mathematical functions (N = 7), and both DGS and CAS are applied (CAS, N = 6; DGS, N = 7). Across all studies included in aspect C, all four of Duval's [4] registers are present: using a digital tool, e.g., Casyopee, students are to translate within and across all semiotic registers—both when solving tasks in algebra and geometry [29].

Translations between symbolic and figurative should preferably take place in both directions in order to obtain the best conditions for visualization. In particular, DGS supports visualization of iconic drawing, since algorithms control DGS and therefore students' manipulations become rule-governed [33]. The use of DT makes it possible to move between different representations, thus supporting students' linking and understanding of the relations between representations (e.g., [27,29,35]).

Still, when students translate and move between representations using DT, they systematically outsource the transformations of different representations—both treatments of, e.g., numbers and conversions between registers—to the tool [19,29,34,37,40,46]. This opens a possibility to make connections between different representations of an object [46]. As an example, when students work with vectors, their treatments cause direct changes of associated graphs and equations [40]. The use of DT is expected to provide security for the students when calculating and moving between representations [29]. Some studies illustrate that students can easily create and represent graphs using DT, but it is the interpretation of these that is difficult [27]. Hershkowitz and Kieran [19] describe two different ways of using a given tool: a mechanical–algorithmic way and a more meaning-oriented way. With a mechanical–algorithmic use of DT, students rely on the calculations and transformations of representation done by the tool. According to the researchers, with such outsourcing follows no critical thinking, and in some cases, students end up with "false representations" of the mathematical objects at play.

Similar to aspect A, several studies focus on students' use of dragging, tracing and using sliders, and illustrate how these actions transform the meaning of the involved mathematical objects. When dragging, the relationship between spatio–graphical and theoretical aspects of a figure change, since DGS mediates the theoretical aspects of the figure during dragging in the form of invariants—which is not relevant in a static figure [22]. Furthermore, dragging within the coordinate system makes it possible for the students to translate between mathematical representations [40]. Students can use the trace function to find a trajectory for a point. This is useful for identifying dependent and independent variables in a function and for inventing and constructing a function for the trajectory [31].

The use of DT results in outsourcing/assistance of systematic translations of mathematical representations, thus making it possible for students to easily move and translate between representations (although in an assisted manner). When dragging points, using a slider or tracing tool, students have the opportunity to realize and move between representations, which may result in better connecting and understanding different representations of objects. Despite the potentials of the systematic translations that ease the students' use and give them access to a wide range of representations, mathematics educators should be aware of the extent of outsourcing that takes place and orchestrate students' use of the tool to be meaning-oriented.

5.4. Reflectively Choosing Representations

For aspect D, reflectively choosing representations in dealing with mathematical situations and tasks, N = 4 studies are included. N = 2 of these utilize CAS, N = 1 utilizes DGS, and N = 1 includes both CAS and DGS. The use of different tools is also evident from the mathematical content in play. One focuses on functions, one on geometry, and two studies are a combination of geometry and algebra/calculus/functions. All four studies include the symbolic register, and one study (i.e., [19]) includes all four registers.

Hershkowitz and Kieran [19] point out that even though students are aware of the algebraic expression being key to solving growth problems, the shape of the algebraic model of functional growth might fall in the background when they apply strong graphic tools. This is because graphic tools allow the students to produce a graph and its algebraic expression by applying regression analysis rather than modeling the algebraic expression based on the problem situation. Hence, students' choice of representation is influenced by the potential solution strategies available in the tool as well as the strategies accessible to the students.

Despite visual representations being the main form of representation to access geometric problems, Kadunz and Sträßer [33] argue that geometric problem solving cannot only be done through the visual representations of the figurative in a DGS. In order to link conceptual knowledge of geometry to visual iconic representations, continual shifts between the figurative and symbolic registers have to go both ways. Based on similar ideas, drawing on both algebraic and geometric argumentation, Santos-Trigo and colleagues [26] provide an example of how the figurative geometric representations in a DGS can lead to conjectures and discoveries of relationships, which in turn must be argued clearly. They obtain this argumentation by choosing CAS to support the algebraic proof of the conjectures and discoveries.

Weigand [47] concentrates on students' difficulties with using DT in terms of choosing adequate representations by considering the relation between students' written solutions and their solutions obtained in a CAS environment in test situations. This relation demands that students are able to detect how to translate the use of digital representations to well-documented solutions, which points out that, in general, the work with digital tools poses a new aspect of the representation competency. This aspect entails the documentation of solutions obtained by digital tools, as the students have to consider which representations and processes should be documented, and which (traditional) representations should be added to present a coherent solution.

It is noteworthy that the studies included under this aspect do not directly address students' ability to choose representations reflectively, but rather emphasize the importance of this ability for solving mathematical problems. The studies mostly point to the fact that DT can be useful for aspect B, knowing the strengths and weaknesses of representations, which is a key element to this aspect of the representation competency. As for aspect B, aspect D is less investigated in the literature. Both task designers and mathematics educators can however be conscious of the implications of the strong graphic tools on students' problem-solving solutions.

6. Discussion

With our two analyses using Duval's [4] registers of semiotic representations and the representation competency of the KOM framework [5,6], we illustrate the role of representations in relation to the use of DGS and CAS. In the first analysis, looking at representational transformations, we found that the registers in use are strongly connected to the mathematical topic in focus, and that representations within the linguistic register are underrepresented in all topics. In the second analysis, regarding the representation competency, we found that the multi-representational possibilities and dynamic features of DGS and CAS hold potentials and pitfalls depending on the four different aspects of the representation competency. Notably, only a few included studies directly investigate aspects of the representation competency: Weigand [47] studies aspect D (i.e., choosing appropriate representations), and Gray and Thomas [36] examine representational fluency.

Looking deeper into the potentials and pitfalls for the use of DT, Niss [11] distinguishes between how DT can enhance students' mathematical capacities or replace them. Applying this point of view to our study, we can consider whether the potentialities offered by DT replace or enhance students' capacities in relation to the representation competency. A potential of DT is to produce static and dynamic images of objects and create connections between different representations of a mathematical entity. Many included studies take advantage of these potentials, for instance using split screen (e.g., [24]) or exploiting the potential that representations are interconnected in the software and, therefore, manipulating one representation leads to an instantaneous and continuous change of any linked representation (e.g., [41]). The potentialities of DT to produce static and dynamic images and create connections between different representations may therefore provide students with tools to interpret and understand how the different representations of the same mathematical entity are connected, which is closely related to aspect A, interpreting and understanding reciprocal relations between representations (N = 20). If these potentialities of DT are exploited in a designed teaching/learning sequence, it is possible that DT can contribute to enhancing students' capacities in relation to representation competency. The aforementioned potentialities are also utilized in the studies included in aspect C—translating and moving between a wide range of representations (N = 12). However, using DGS and CAS without designing appropriate tasks and teaching sequences may have the effect of replacing students' capacities in relation to aspect C rather than enhancing them, since the use of DT often results in outsourcing the transformations between representations (cf., Section 5.3). Despite the intention of using DT to support translations, DT rather support interpreting and understanding the connections (aspect A). Furthermore, the outsourcing of transformations does not automatically support critical thinking regarding the different representations, which relates to aspect B, knowing about the strengths and weaknesses of representations (N = 4), and aspect D, reflectively choosing representations in dealing with mathematical situations and tasks (N = 4) (cf., Sections 5.2 and 5.4). However, this does not imply that DT cannot be used to activate these aspects of the representation competency, but the outcome of using DT depends on the properties of the DT at issue, the teacher's orchestration—which is a discussion we will not go further into here—and the task designs involving DT [11,24]. The first subsection of the discussion addresses task design involving use of DT, including five suggestions/considerations when implementing DT. The second subsection concerns whether or not new registers or representations arise due to the implementation of DT.

6.1. Task Design including Representation Competency and Use of Digital Tools

The use of DT influences students' development differently with respect to the four aspects of representation competency (cf., Section 5). In this section, we discuss how DT, such as CAS or DGS, may be used in mathematical task designs in order to activate the students' representation competency. We rely on the literature included for research question 1, analyses and results from Section 4 and 5, and the theoretical perspectives defined in Section 2.

As described earlier, mathematical tasks involving DT serve the purpose of creating experiences for the students in which the use of DT makes it possible to experience mathematics [8,16]. Duval [4,7] points out that the most essential part of mathematical activity is the ability to conduct treatments and conversions within and between different representations. However, as mentioned in the introduction of the discussion, aspects of the representation competency are often outsourced to DT (i.e., aspect C, translating and moving between representations). This is because the multiplicity of an MDBO (Mathematical Digital Boundary Object) readily produces transformations of representation done by DT, which is noted by several authors (e.g., [19,29,46]). As transformations are fundamental parts of mathematical activity, the implementation of DT changes the mathematical activities at play. As presented in Section 5, some included studies consider the multi-representational opportunities of DGS and CAS as a gateway to conceptualization [20,30,37–39,41]. With regard to task design and conceiving of DGS and CAS as MDBOs, dragging, sliders, and tracing are features within DGS and CAS, and thus can be part of the MDBOs. An essential part of MDBOs is multiplicity, meaning that tools offer different representations simultaneously [8,16]. Yet, multiplicity does not necessarily lead to learning, nor to students automatically interpreting and connecting representations, even though it is a common idea in educational research and teaching practice. Multiplicity may lead to confusion if the user is not already familiar with the ways of thinking and working in mathematics. Multiplicity allows for the design of mathematical tasks involving DT, which may lead the students to connect and relate the multiple representations embedded in the MDBOs. According to Duval [7], the primary key to any mathematical activity is, however, conversion between any two registers; hence, essential aspects of mathematical activities change with the implementation of DT.

Another important perspective on using MDBOs is that one register is often being neglected, i.e., the linguistic, which is stressed by Duval [7] and shown in the results presented in Section 4. DT do not represent the linguistic register, but this register is still an essential part of mathematical nature (e.g., reasoning and communicating) and semiotic representations. Hence, task designs involving DT must also include the linguistic register if the aim is to construct a bridge between mathematical representations and activating the mathematical representation competency.

In the following, we discuss the activation of the representation competency while using DGS and CAS in relation to task design. We group its forms using the four aspects of the competency.

For aspect A, interpreting and being able to understand reciprocal relations, multiplicity being part of MDBOs is presented as highly important for this aspect. Particularly, manipulation of different representations holds great potential in understanding mathematical objects and the reciprocal relations of representations (e.g., [18,20,23,30,34,39]). For aspect A, sliders, dragging, and tracing are marked as fundamental parts of the MDBOs helping the students to relate and understand the reciprocal relations between the different representations. The multiplicity, however, does not necessarily imply that students interact with and relate the representations involved [7]. Therefore, it is an important part of task design involving tracing, dragging and sliders to request the students to reflect upon their actions with these features and to investigate and relate the reciprocal relationships of the representations. In addition, Lagrange [24] suggests that new windows and tools in DT should be introduced gradually to avoid confusion among students when studying functions on geometric relationships. In relation to geometry, Laborde [21] states that, when using DT, students should break down a representation of a given object into different units in order to identify the nature of the different units leading to the reconstruction of the object. The notions of pragmatic and epistemic value of instrumented techniques [48] are relevant to consider when discussing the use of tools for concept development. Epistemic use mediates insight about a given problem, whereas pragmatic use solely mediates a solution to a given problem. Artigue (2002) argues that for education use, epistemic techniques should precede pragmatic techniques. In a recent study, Iversen, Misfeldt and Jankvist [49] argues for the need to also take into account students' identity as a third component, next to epistemic and pragmatic mediations, in relation to CAS-related work and being mathematically proficient.

A feature of MDBOs presented in the literature with positive results is the possibility of exploiting feedback. When using DT, students are provided with constant feedback, for instance, when calculating, solving or drawing graphs that introduce them to a mathematical world. This leads to students being able to explore failures and mistakes, test their conjectures and so forth (e.g., [18,34,43]).

For aspect B, knowing about the strengths and weaknesses of representations, being critical is important, e.g., by comparing representations and investigating their strengths and weaknesses [19,21,23,26]. Multiplicity can again be used as a part of MDBO for this aspect. Yet, compared to aspect A, aspect B focuses more on the differences and strengths/weaknesses of each representation [5,6]. An example that offers different perspectives is splitting 3D representations into 2D representations using a tool such as GeoGebra. In this case, the tool opens a mathematical world building on Euclidean geometry, which the students can explore.

If we aim to avoid that multiplicity is meaningless, as often indicated by students using DT in a mechanical–algorithmic way [19], the mathematical tasks must contain a critical dimension. Thus, it is important to encourage students to use DT in a meaningful way by comparing representations and possibly having multiple representations presented simultaneously within a given tool.

Related to aspect B, Santos-Trigo and colleges [26] do not only focus on the representations themselves, but also on the strengths and weaknesses of different digital tools. Different tools make use of different representations and mediate different perspectives of the conventional mathematical world to the students. Thus, for students to know about the strengths and weaknesses of different representations, the ability to understand the different tools and the strengths and weaknesses of these is important as well. Such an argument is in line with the aids and tool competency of the KOM [5,6], and the statement indicates a strong relationship between the representation and the aids and tool competencies, which are constantly engaged in task designs involving DT and activating the representation competency. Using different types of DT as MDBOs in mathematical tasks can evoke the previously mentioned critical aspect, because technological environments may bring students to use non-appropriate representations if they do not know the crucial properties of the tools in play [19]. New strides in mathematical digital tools expand the possibilities of students' access to feedback, as automated feedback, and additionally assessment tools are developed to support students' work with DT. This opens up possibilities for students to receive feedback on their appropriation of representations in solutions (e.g., [50,51]).

For aspect C, translating and moving between a range of representations, parts of the representation competency are outsourced to the tools doing constant treatments and conversions [19,29,34,37,40,46]. Due to the outsourcing of treatments and conversions, which are easily accessed due to the multiplicity offered by MDBOs, it is easy to move between different representations. A feature, previously mentioned, in the MDBOs is feedback, which provides security for the students concerning their transformations of representations used in a given task [22,34]. A way of exploiting feedback, when focusing on aspect C, is to make the students write down their expectations and predictions of changes in representations and thereby keep the students active [41].

Again, dragging, tracing and using sliders are mentioned as ways of manipulating mathematical objects (e.g., [22,31,37,40]). In this way, the MDBO introduces new mathematical activities [16] in which students can manipulate mathematical objects as if they were real [7] by using DGS and/or CAS to translate and move between representations [22]. When using DT, the ability to translate representations becomes a manner of learning to use the tool, for example, by pressing a button, opening a window, drawing a line, and so forth. However, an exception occurs with regard to the linguistic register, which is not usually part of MDBO [7]. If the linguistic register should be embedded into the task design, students would still activate aspect C of the representation competency. For instance, by having a description of a function in natural language (i.e., a linguistic representation), students would have to transform it into an equation (i.e., a symbolic representation). Alternatively, at the end of a task, students could be asked to give a description of a graph (i.e., of the graphic register) in natural written language (i.e., in the linguistic register).

For aspect D, reflectively choosing representations, the included studies rather concern the importance of such ability than directly answer how students reflectively choose representation for a given task. In relation to MDBO, Hershkowitz and Kieran [19] state that it is the possible solution strategies offered by a certain technology that influence which representations students choose to use in an answer. This is also in line with Santos-Trigo and colleges [26], who identify that students must be aware of what different tools offer in relation to representations. Therefore, educators and task designers also have to be aware of the affordances of the specific digital tool, and of what to expect of students' solutions and representations. This emphasizes the mediation of the MDBO regarding students' mathematical experiences [16]. Another aspect of this is presented by Weigand [47], who argues that students must be able to translate and understand the representations provided by DT, which may be influenced by the use of DT. For instance, bringing the answers of mathematical problems to a form, where CAS' solve command can take care of the rest and presenting this as the solution to a task is not part of conventional mathematical discourse. Novices of mathematical activities might not see this difference, since CAS has influenced their view of how symbolic expressions and equations can be represented [52]. An MDBO presents a new mathematical activity for the students that mediates the mathematical nature. When looking at mathematical tasks, for instance, students are asked to do treatments within the symbolic register, from CAS expressions of the DT to expressions accepted in the mathematical discourse they participate in, such as written exams. Furthermore, they are asked to discuss and critically relate to which representations are best suited to answer a certain question. Related to aspect B, knowing strength and weaknesses, the students must be aware of the differences between the representations of a formal mathematical nature and representations generated by the use of DT.

Regarding the representation competency as a whole, we propose the following when designing tasks aiming at activating the representation competency to ensure that all aspects are covered. The suggestions take the mathematical worlds presented by the MDBO into account.

1. Include the discursive multifunctional register (linguistic), both at the beginning and at the end of each task, to make sure that the register is not neglected. For instance, by posing questions in natural language (i.e., in the linguistic register) to be converted to another register, or asking for answers expressed using natural language (i.e., in the linguistic register).
2. Exploit the feedback opportunities provided by the DT as a fundamental part of the MDBO. As an example, ask the students to predict changes for representations before using DT to ensure the ability to translate representations and understand their reciprocal relationships.
3. Break objects/representations into smaller units or gradually introduce new windows and features within a given MDBO.
4. Use sliders, dragging and tracing as they hold potentials for students' ability to move and translate (using the tools) as well as interpreting and understanding the representations and their reciprocal relations. To exploit the multiplicity of an MDBO, ask students to relate and explore such relations, as they do not necessarily understand the relationship just because many representations are presented simultaneously.
5. Focus on activating students' critical thinking by encouraging them to investigate strengths and weaknesses of representations, focusing on representations both inside and outside an MDBO, but also between MDBOs. Furthermore, this emphasizes the use of different tools in teaching mathematics, but it also includes discussions among students concerning which representations are most appropriate in the given task.

The review results indicated that transformations within and between registers were outsourced to DT, and that the linguistic register was often left out. Our task design includes five suggestions for designing mathematical tasks using DT and aiming to activate students' representation competency. Educators must be aware of what specific DT offer and exploit the affordances of using an MDBO. This means that the educator must know features such as sliders, tracing, and dragging, as they may help students to interpret and connect representation (i.e., aspects A and B), as well as moving between representations (i.e., aspect C). Educators should also be aware that the multiplicity in itself does not automatically bring students to understand and connect representations. Educators should be able to support students in making these connections, since the multiplicity in itself does not support such abilities in a mathematical task design. The MDBO does not automatically stimulate critical thinking, but tasks using MDBO, designed to activate the representation competency, can encourage students to do so. The five suggestions for task design using DT take into account the pitfalls and potentials of DT listed in the included literature, when activating representation competency in primary and secondary school.

6.2. New Questions Emerging from the Literature Review and Analyses

In the two analyses and our above discussion on task design in relation to representation competency, we have found that dynamic representations play a crucial role when involving DGS and CAS. Laborde and Laborde [22] consider the dynamic representations of dragging as a new kind of representation in mathematics and mathematics education. Duval [7] strongly disagrees, arguing that the computer displays the same representations as produced with pen and paper. Duval considers the computer to have an

> unlimited treatment power compared with the possibilities of the graphic-visual mode. We obtain immediately, much more than anything we could get by writing calculations or constructing geometrical figures and graphs, for several hours or days. [7] (p. 100).

Thus, a consequence of the computer's treatment power is the dynamic aspect. With this, Duval [7] questions the degree of cognitive activity needed for a novice in mathematics to process such a load of information. In their study, Gray and Thomas [36] found that using a graphic calculator with multi-representations for the study of quadratic equations did not support students in building representational fluency. To do so, it is important to be able to interpret the feedback of the DT. In line with this, Weigand [47] argues that an important aspect of using DT is the requirement of the user to know the relations between the digital representations in the tool and the representations used for documentation, for example in a written exam. This leaves us with the question of whether or not representations of DT are a new kind of representation, a new semiotic register, or merely new types of mathematical activities.

The dynamic feature of dragging figurative representations is, using Duval's [7] terminology, regarded as unlimited treatment within the figurative register. Similarly, a calculation done by CAS, such as a "solve" command applied to various forms of equations, can be viewed as a treatment within the symbolic register. These activities imply that students need to be able to carry out new forms of treatment such as dragging or CAS commands, but still within the figurative and symbolic registers. It is not as straightforward when we consider the multi-representational possibilities of dynamic environments. For instance, using a digital tool to study the connection between the parabola and the quadratic function with a slider for a coefficient of the function, see Figure 4, the computer does conversions between the parabola in the graphic register and the algebraic expression in the symbolic register. However, where does this leave the slider, which in this case is a representation of a given parameter in the quadratic equation? An answer could be that it is simply a transitional auxiliary representation, helping the user to navigate the conversions between the representations of the two registers.

In this way, we can place the different elements of dynamic representations within the four registers of semiotic representations and the category of transitional auxiliary representations, just as Duval [7] argues. However, because of the importance of these dynamic representations and activities, we suggest that they need their own name in order to distinguish between how we work and deal with static and with dynamic representations. Such terminology should emphasize the action of the dynamic representation. Considering the example of dragging in the figurative register, such dynamic representations could be termed "continuously interactive treatment", since the digital tool performs the treatment, and the treatments are shown continuously by the interaction of the user. In contrast, having CAS doing symbol manipulation and carrying out calculations is not as interactive. In this case, the user only types in a command once, executes the command,

and obtains a single result. Moreover, solving geometric problems in DGS does not necessarily require the continuous interaction of the user. For instance, using a measuring tool or constructing a geometric figure also results in one output only—which then can be controlled interactively afterwards depending on the task. These actions also require some interaction from the user, but not in the same continuous way. Rather such actions are solitary interactive treatments. These two suggestions address dynamic representations within the same semiotic register, i.e., for treatments.

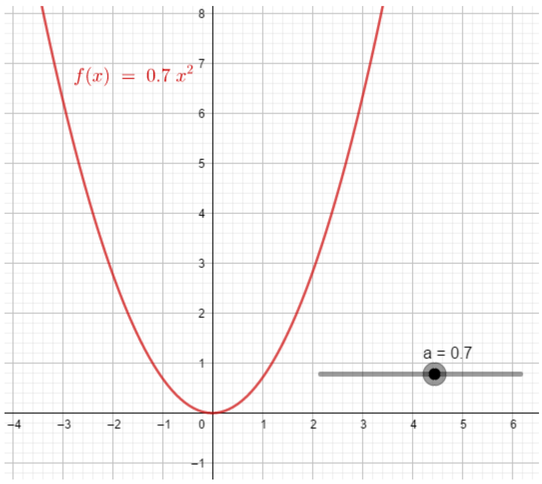

Figure 4. An example of a parabola and the quadratic function connected with a slider for parameter *a*, represented in GeoGebra (our own construction).

Considering the example of the parabola and the quadratic equation connected with the slider, the digital tool in use illustrates a conversion and, as with dragging, the student is continuously interactive. Hence, this would be called a continuously interactive conversion. This further indicates that the tool illustrates the conversion between representations of different registers, and that the activities of the student are expected to be different from static conversion activities. Additionally, this notion entails all elements of the conversion and unites the slider of the parameter with the graphic representation of the parabola as well as with the algebraic representation of the quadratic function. The change of the slider instantaneously affects the parabola and the algebraic expression via the user's activity. Similar to the category of treatments, we also have solitary interactive conversions, such as typing in an algebraic formula in the symbolic register, whereupon the given digital tool converts it into a graph in the graphic register.

All the examples given above and the cases identified in the included studies are of this interactive character. However, DT can also be used to illustrate different treatments and conversions such as animations, where the user is passive rather than interacting with the tool. Whether a representation is passive or interactive depends not only on the setup of the MDBO, but also on how it is used. For instance, if the educator uses a continuously interactive conversion to illustrate the conversion itself or the properties of the included mathematical concepts, the representation will be passive for the students, whereas when they work with it themselves it will be interactive.

7. Conclusions

The results of the included literature illustrate that the multi-representational affordances of DGS and CAS have great potential in teaching and learning mathematics and mathematical conceptualization. Only a few of the included studies address the potential of DT in relation to dealing particularly with representations. With the use of Duval's [4,7] semiotic registers and the representations competency of the KOM framework [5,6], the analyses focus on the elements particularly relevant to dealing with representations.

In relation to the four registers of semiotic representations [4,7], there is a clear connection between the topic at issue and which of the registers are involved. Studies of functions mainly involve representations of the symbolic and graphic registers, whereas studies of geometry make use of figurative representations. In contrast, the linguistic register does not relate specifically to any topic, and in general does not receive much attention, which is also one of Duval's [7] points of critique on introducing DT.

The perspective of the representation competency is divided into four aspects (A, B, C and D, cf., Section 5), as none of the included studies covers the whole competency. From the analysis, we found that 20 out of the 30 studies have elements pointing toward aspect A, interpreting and being able to understand the reciprocal relations between different representational forms of the same entity. Using multi-representations with DGS and CAS in particular support this aspect, since the multiplicity can constitute a beneficial environment for investigating the connection between representations of the same mathematical entity. Considering aspect C, translating and moving between a wide range of representations, the analysis and the discussion hereof point to this aspect not being as easy to promote with the use of DT, as the translations between representations are often outsourced to the digital tool in use. Therefore, using DT often replaces this aspect rather than enhancing it.

To ensure that all aspects of the representation competency are covered, we give five suggestions for consideration when implementing DT in task design with the particular purpose of activating the representation competency. For using DGS or CAS as an MDBO to mediate between two mathematical worlds, such as a mathematics curricula and a mathematics class of students, we suggest being aware of the following in task design: (1) to include the D/Multi register; (2) to exploit the feedback opportunities provided by the DT as a fundamental part of the MDBO; (3) to break down representations into smaller units and gradually introduce new windows and features of the DT in use; (4) to exploit the multiplicity of the MDBO by asking students to relate and explore relations between representations by using dynamic features such as sliders, dragging and tracing; and finally (5) to focus on activating students' critical thinking by encouraging them to investigate the strengths and weaknesses of representations.

Since even well-produced task designs do not ensure constructive use of DT on its own, mathematics educators play a crucial role when using DT in the teaching and learning of mathematics. From our analysis using the four aspects of the representation competency, we suggest educators to be aware of that using the multiplicity of DT does not automatically make students connect and understand different representation. Educators should support the students in interpreting the automatic feedback from the tool in coherence with mathematical concepts and conventional mathematical discourse. Furthermore, educators should be aware of the affordances of different digital technologies, such that as aspect B, knowing about the strengths and weaknesses of representations, and aspect D, reflectively choosing representations, also can be included. Lastly, the educators should be aware of the extent of outsourcing that takes place when introducing DT, and orchestrate students' use of tools to be meaning-oriented and epistemic rather than only pragmatic.

Our analyses and discussions gave rise to a question of whether DT introduce new representations or even new registers, or just new activities. We found that even if representations in digital tools are not new in relation to the semiotic registers, they have become so important for mathematics teaching and learning that framing them in a new terminology can emphasize the change in activities expected of the involved students. With Duval's [4,7] distinction between treatments within and conversions between registers, we suggest furthermore to distinguish between whether a digital treatment or conversion in DGS or CAS is continuously or solitarily interactive. For instance, using sliders, dragging or tracing, the students are often continuously interacting with the tool, exploring the changes of using the dynamic features, whereas typing in an input and obtaining an output is a solitary interaction. In addition, such representations are either interactive, activating the user, or passive, illustrated as an animation or by someone else being active. This means that representations in DGS and CAS can illustrate either a treatment or a conversion. Furthermore, the activity can be continuous or solitary and interactive or passive, leaving us 2^3 new classifications of representations given by DT.

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