Article

Regularized Asymptotics of the Solution of the Singularly Perturbed First Boundary Value Problem on the Semiaxis for a Parabolic Equation with a Rational “Simple” Turning Point [†]

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Abstract: The aim of this study is to develop a regularization method for boundary value problems for a parabolic equation. A singularly perturbed boundary value problem on the semiaxis is considered in the case of a “simple” rational turning point. To prove the asymptotic convergence of the series, the maximum principle is used.

Keywords: singularly perturbed first boundary value problem for a parabolic equation; asymptotic solution; regularization method; rational “simple” turning point

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1. Introduction.

This work consists of seven parts.

The first part is an introduction.

The second part is the nomenclature.

In the third part, the problem is formulated and an asymptotic solution is constructed for a singularly perturbed boundary value problem for a parabolic equation on the semiaxis when the stability conditions for the spectrum of the limit operator are violated—the presence of a “simple” rational turning point.

In the fourth part, an estimate for the remainder of the asymptotic series is given.

In the fifth part (Application), auxiliary lemmas and theorems are proved.

The sixth part is an Conclusion.

The seventh part is an Bibliography.

Among singularly perturbed problems, problems with an unstable spectrum of the limit operator are of particular interest. Various asymptotic methods have been developed to construct solutions to such problems. Such as methods: V.P. Maslov’s school, A.B. Vasilieva’s—V.F. Butuzov’s—N.N. Nefedov’s school and others. One of such methods is the method of regularization S.A. Lomov’s school. This method allows you to build a solution over the entire region of integration. A “simple” pivot point means that the eigenvalues of the limit operator are isolated, and one eigenvalue at separate points of t vanishes. Problems with such features were considered in the works [1–11]. The Cauchy problem for a singularly perturbed parabolic equation with a “simple” rational turning point was considered in [12].

When solving such problems, functions arise that describe the irregular dependence of the solution on ε . The identification and description of these functions is the main problem of the regularization method. Under conditions of a stable spectrum of the limit operator,

i.e., under conditions $\forall t \in [0, T]$, the eigenvalues of the limit operator $\lambda_i(t) \neq \lambda_j(t) \forall i \neq j$, $\lambda_i(t) \neq 0, i = 1, \dots, n, j = 1, \dots, n$, the singularities are:

$$e^{-\varphi_i(t)/\varepsilon}, \quad \varphi_i = \int_0^t \lambda_i(s) ds, \quad i = 1 \div n.$$

In this paper, we construct regularizing functions related to the singularities of the boundary value singularly perturbed problem.

2. Nomenclature.

All quantities in the article are dimensionless:

1. x, t, s, τ -variables;
2. $x \in [0, +\infty)$, $s, t \in [0, T]$ and $0 \leq s \leq t \leq T$, $\tau \in [0, +\infty)$;
3. ε -small parameter varying within $0 < \varepsilon < \varepsilon_0$;
4. $p(x), q(t), b(t), f(x, t), u(x, t), \varphi(x), \psi(t), \sigma_i(t, \varepsilon), a(x, t), Q_s^t(s, t);$
 $v(x, t, \varepsilon), v_k(x, t, \varepsilon), z^i(x, t, \varepsilon), z_k^i(x, t, \varepsilon), W(x, t, \varepsilon);$
 $W(x, t, \varepsilon)_k, H(x, t, \varepsilon), H_1(x, t, \varepsilon), H_2(x, t), C(x, t)$ - functions;
5. G, F, T, L -operators;
6. $M, M_1, M_2, M_3, k, r, \alpha, \beta, \delta, \ell$ - constants;
7. $Q_T = [0, +\infty) \times [0, T], D_\ell = [0, \ell) \times [0, T]$ - problem solution area;
8. m, n -integers.

3. Formulation of the Problem. Construction of an Asymptotic Solution

Let the task be given

$$\begin{cases} \varepsilon \frac{\partial u}{\partial t} - \varepsilon^2 p(x) \frac{\partial^2 u}{\partial x^2} + q(t)u = f(x, t), \\ u(x, 0) = \varphi(x), \quad x \geq 0, \\ u(0, t) = \psi(t) \end{cases} \quad (1)$$

And conditions are met

- (1) $f(x, t) \in C^\infty([0, +\infty) \times [0, T]), \exists M_3 > 0 \forall k |f_x^{(k)}(x, t)| < M_3, \psi(t) \in C^\infty[0, T];$
- (2) $p(x) \in C^\infty[0, +\infty), p(x) \geq p_0 > 0;$
- (3) $q(t) = t^{m/n} b(t), b(t) \in C^\infty[0, T], \operatorname{Re}(b(t)) > 0.$

To prove an estimate for the remainder of the asymptotic series of a solution, additional assumptions are required

- (4) $\exists M_1 > 0 \quad \forall k |f_x^{(k)}(x, t)| < M_1;$
- (5) $\exists M_2 > 0 \quad p(x) < M_2(x^2 + 1);$
- (6) $\exists \mu > 0, r > 0 \forall x \geq 0 \quad u(x, t) > -\mu(x^r + 1).$

Singularly perturbed problems arise in the case when the domain of definition of the initial operator depending on ε , at $\varepsilon \neq 0$, does not coincide with the domain of definition of the limit operator at $\varepsilon = 0$. Under the stability condition for the spectrum of the limit operator, essentially singular singularities are described using exponentials of the form $e^{\frac{\varphi_i(t)}{\varepsilon}}, \varphi_i(t) = \int_0^t \lambda_i(s) ds, i = \overline{1, n}$, where $\lambda_i(t)$ are the limit operator's eigenvalues, $\varphi_i(t)$ are smooth (in general, complex) functions of a real variable t . If the stability conditions for the limit operator are violated for at least one point of the spectrum of the limit operator, then new singularities arise in the solution of the inhomogeneous equation. When studying problems with a "simple" turning point, we are faced with a problem when the range of the original operator does not coincide with the range of the limit operator [12] (p. 1).

To regularize the problem, we introduce an additional variable:

$$\tau(x, \varepsilon) = \frac{1}{\sqrt{\varepsilon}} \int_0^x \frac{ds}{\sqrt{p(s)}}$$

and the extended function $\tilde{u}(x, t, \tau)$, which when narrowed

$$\tilde{u}(x, t, \tau) \Big|_{\tau = \frac{1}{\sqrt{\varepsilon}} \int_0^x \frac{ds}{\sqrt{p(s)}}} \equiv u(x, t)$$

gives an exact solution to the problem (1).

Let us calculate the derivatives for the extended function $\tilde{u}(x, t, \tau(x))$:

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= \tilde{u}_t, & \frac{\partial \tilde{u}}{\partial x} &= \tilde{u}_x + \frac{1}{\sqrt{\varepsilon p(x)}} \tilde{u}_\tau, \\ \frac{\partial^2 \tilde{u}}{\partial x^2} &= \tilde{u}_{xx} + \frac{2}{\sqrt{\varepsilon p(x)}} \tilde{u}_{x\tau} + \frac{1}{\varepsilon p(x)} \tilde{u}_{\tau\tau} - \frac{p'(x)}{2\sqrt{\varepsilon} \sqrt{p(x)}^3} \tilde{u}_\tau. \end{aligned}$$

In what follows, $\sim$ will be omitted. Then the problem (1) takes the form

$$\begin{cases} \varepsilon(u_t - u_{\tau\tau}) = -q(t)u + \sqrt{\varepsilon}^3 \left(2\sqrt{p(x)}u_{x\tau} - \frac{p'(x)}{2\sqrt{p(x)}}u_\tau \right) + \varepsilon^2 p(x)u_{xx} + f(x, t), \\ u(x, 0, \tau) \Big|_{\tau(x)} = \varphi(x), \quad u(0, t, 0) = \psi(t). \end{cases} \quad (2)$$

The regularizing functions of the problem (2) have the form

$$\begin{aligned} e^{-\frac{1}{\varepsilon} Q_0^t}, \quad \sigma_i(t, \varepsilon) &= \int_0^t e^{-\frac{1}{\varepsilon} Q_s^t} s^{\frac{i+1}{n}-1} ds, \quad i = \overline{0, p-1}, \quad p = m + n - 1, \\ G(a(x, t)) &= e^{-\frac{1}{\varepsilon} Q_0^t} \frac{2}{\sqrt{\pi}} \int_{\frac{\tau}{2\sqrt{t}}}^\infty a\left(x, t - \frac{\tau^2}{4\xi^2}\right) e^{-\xi^2} d\xi \equiv e^{-\frac{1}{\varepsilon} Q_0^t} F(a(x, t)). \end{aligned}$$

$$\text{Here } Q_s^t = \int_s^t q(s_1) ds_1, \quad F(a(x, t)) = \frac{2}{\sqrt{\pi}} \int_{\frac{\tau}{2\sqrt{t}}}^\infty a\left(x, t - \frac{\tau^2}{4\xi^2}\right) e^{-\xi^2} d\xi.$$

The properties of the operator $F(a(x, t))$ are described in the application. The operator $F(\cdot)$ transfers any smooth function $a(x, t)$ to the solution of the problem:

$$\begin{cases} F_t - F_{\tau\tau} = 0, \\ F(a(x, t)) \Big|_{t=0} = 0, \quad F(a(x, t)) \Big|_{\tau=0} = a(x, t). \end{cases} \quad (3)$$

Note that $\varepsilon = 0$ for $F(a(x, t))$ after restricting to $\tau(x, \varepsilon)$ is essentially special. Following the regularization method, the solution to problem (2) is sought in the form

$$u(x, t, \tau, \varepsilon) = e^{-\frac{1}{\varepsilon} Q_0^t} v(x, t, \varepsilon) + \sum_{i=0}^{p-1} Z^i(x, t, \varepsilon) \sigma_i(t, \varepsilon) + G(a(x, t, \varepsilon)) + W(x, t, \varepsilon). \quad (4)$$

where the functions $v(x, t, \varepsilon), Z^i(x, t, \varepsilon), a(x, t, \varepsilon), W(x, t, \varepsilon)$ —are smooth in x, t and depending on ε .

Substituting (4) in (2), we get a problem with respect to functions $v(x, t, \varepsilon), Z^i(x, t, \varepsilon),$

$a(x, t, \varepsilon), W(x, t, \varepsilon)$:

$$\begin{cases} v_t = \varepsilon p(x) v_{xx}, \\ Z_t^i = \varepsilon p(x) Z_{xx}^i, \quad i = \overline{0, p-1}, \\ F_t(a(x, t, \varepsilon)) - F_{\tau\tau}(a(x, t, \varepsilon)) = \\ \sqrt{\varepsilon} \left(2\sqrt{p(x)} F_{\tau}(a(x, t, \varepsilon)) - \frac{p'(x)}{2\sqrt{p(x)}} F_{\tau}(a(x, t, \varepsilon)) \right) + \varepsilon p(x) F(a_{xx}(x, t, \varepsilon)), \\ q(t)W = f(x, t) - \varepsilon W_t - \varepsilon \sum_{i=0}^{p-1} Z^i(x, t, \varepsilon) t^{\frac{i+1}{n}-1} + \varepsilon^2 p(x) W_{xx}, \\ u(x, 0, \tau, \varepsilon)|_{\tau=\tau(x, \varepsilon)} = \varphi(x), \quad u(0, t, 0, \varepsilon) = \psi(t). \end{cases} \quad (5)$$

Based on the distribution of the parameter $\sqrt{\varepsilon}$ in system (5), it follows that the functions $v(x, t, \varepsilon), Z^i(x, t, \varepsilon), W(x, t, \varepsilon)$ decompose in whole powers ε^k , or $a(x, t, \varepsilon)$ by powers $\sqrt{\varepsilon}^k$:

$$\begin{cases} \dot{v}_k = p(x) v_{k-1}'', \quad \dot{Z}_k^i = p(x) Z_{k-1}^{i''}, \quad i = \overline{0, p-1} \\ q(t)W_k = f(x, t) \delta_k^0 - \dot{W}_{k-1} - \sum_{i=0}^{p-1} Z_{k-1}^i t^{\frac{i+1}{n}-1} + p(x) W_{k-2}'', \\ \dot{F}(a_{\frac{k-1}{2}}(x, t)) - F_{\tau\tau}(a_{\frac{k-1}{2}}(x, t)) = \\ 2\sqrt{p(x)} F_{\tau}(a'_{\frac{k-2}{2}}(x, t)) - \frac{p'(x)}{2\sqrt{p(x)}} F_{\tau}(a_{\frac{k-2}{2}}(x, t)) + p(x) F(a''_{\frac{k-3}{2}}(x, t)), \\ u_k(x, 0, \tau)|_{\tau(x)} = \varphi(x) \delta_k^0, \quad u_k(0, t, 0) = \psi(t) \delta_k^0, \\ k = -1, 0, 1, \dots \end{cases} \quad (6)$$

Note that if $k-1 < -1, \frac{k-i}{2} < -1, i = 2, 3$, then the term with this index is equal to zero.

Since $f(x, t)$ does not satisfy the theorem on the pointwise solvability of the equation $q(t)W(x, t) = f(x, t)$, then the expansion of the solution begins with ε^{-1} .
 ε^{-1} . System (6) in this case has the form

$$\begin{cases} \dot{v}_{-1} = 0, \quad \dot{Z}_{-1}^i = 0, \quad i = \overline{0, p-1}, \\ q(t)W_{-1} = 0, \\ \dot{F}(a_{-1}(x, t)) - F_{\tau\tau}(a_{-1}(x, t)) = 0. \end{cases}$$

Hence $v_{-1}(x, t) = v_{-1}(x), Z_{-1}^i(x, t) = Z_{-1}^i(x), W_{-1}(x, t) \equiv 0, a_{-1}(x, t)$ is an arbitrary function.

From the initial and boundary conditions, we have

$$\begin{cases} u_{-1}(x, 0) \equiv v_{-1}(x) \equiv 0, \\ u_{-1}(0, t) = \sum_{i=0}^{p-1} Z_{-1}^i(0) \sigma_i(t, \varepsilon) + e^{-\frac{1}{\varepsilon} Q_0^t} a_{-1}(0, t) = 0. \end{cases}$$

From here

$$a_{-1}(0, t) = - \sum_{i=0}^{p-1} Z_{-1}^i(0) \int_0^t e^{\frac{1}{\varepsilon} Q_0^s} s^{\frac{i+1}{n}-1} ds.$$

To determine the unknown functions, consider the following iterative problem.
 $\sqrt{\varepsilon}^{-1}$. System (6) in this case has the form

$$T(F(a_{-1/2}(x, t))) = 2\sqrt{p(x)} F_{\tau}(a'_{-1}(x, t)) - \frac{p'(x)}{2\sqrt{p(x)}} F_{\tau}(a_{-1}(x, t)),$$

here $T = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial \tau^2}$.

As $T(F(a_{-1/2}(x, t))) \equiv 0$, then we get an equation for $a_{-1}(x, t)$ or:

$$\frac{d}{d\tau} F\left(2\sqrt{p(x)}a'_{-1}(x, t) - \frac{p'(x)}{2\sqrt{p(x)}}a_{-1}(x, t)\right) = 0.$$

From here

$$F\left(2\sqrt{p(x)}a'_{-1}(x, t) - \frac{p'(x)}{2\sqrt{p(x)}}a_{-1}(x, t)\right) = C(x, t),$$

where $C(x, t)$ arbitrary function.

From the properties of the operator F we obtain for $\tau \rightarrow \infty$ $F \rightarrow 0$. Then $C(x, t) \equiv 0$. Putting $\tau = 0$, we get

$$2\sqrt{p(x)}a'_{-1}(x, t) - \frac{p'(x)}{2\sqrt{p(x)}}a_{-1}(x, t) = 0. \quad (7)$$

Solution (7) has the form

$$a_{-1}(x, t) = \sqrt[4]{\frac{p(x)}{p(0)}}a_{-1}(0, t).$$

In this way,

$$a_{-1}(x, t) = -\sqrt[4]{\frac{p(x)}{p(0)}} \sum_{i=0}^{p-1} Z_{-1}^i(0) \int_0^t e^{\frac{1}{\varepsilon} Q_0^s} s^{\frac{i+1}{n}-1} ds.$$

The function $a_{-1/2}(x, t)$ at this iteration step is arbitrary. To determine $Z_{-1}^i(0)$, consider the system (6) at step zero.

ε^0 . System (6) looks like:

$$\left\{ \begin{array}{l} \dot{v}_0 = 0, \quad \dot{Z}_0^i = p(x)(Z_{-1}^i)''', \quad i = \overline{0, p-1}, \\ q(t)W_0 = f(x, t) - \sum_{i=0}^{p-1} Z_{-1}^i(x)t^{\frac{i+1}{n}-1}, \\ T(F(a_0(x, t))) = F_\tau\left(2\sqrt{p(x)}a'_{-1/2}(x, t) - \frac{p'(x)}{2\sqrt{p(x)}}a_{-1/2}(x, t)\right) + \\ p(x)F(a_{-1}''(x, t)), \\ u_0(x, 0) = v_0(x) + W_0(x, 0) = \varphi(x), \\ u_0(0, t) = e^{-\frac{1}{\varepsilon} Q_0^t} v_0(0) + \sum_{i=0}^{p-1} Z_0^i(0, t)\sigma_i(t, \varepsilon) + W_0(0, t) + e^{-\frac{1}{\varepsilon} Q_0^t} a_0(0, t) = \psi(t). \end{array} \right. \quad (8)$$

Subordinate the right side of the equation for $W_0(x, t)$ to the conditions of the point solvability theorem. To do this, expand the right-hand side of the equation in the Taylor-Maclaurin formula.

$$q(t)W_0(x, t) = f(x, 0) + \frac{\partial f}{\partial t}(x, 0)t + \dots + \frac{\partial^{[m/n]} f}{\partial t^{[m/n]}}(x, 0)t^{[m/n]} + t^{[m/n]+1}f_0(x, t) - \sum_{i=0}^{p-1} Z_{-1}^i(x)t^{\frac{i+1}{n}-1}.$$

We put

$$\begin{aligned} Z_{-1}^i(x) &= \frac{1}{j!} \frac{\partial^j f}{\partial t^j}(x, 0), & i &= n(j+1) - 1, \quad j = 0, \overline{\left[\frac{m-1}{n}\right]} \\ Z_{-1}^i(x) &\equiv 0, & i &\neq n(j+1) - 1, \quad i = \overline{0, p-1} \end{aligned}$$

Then

$$W_0 = \frac{f(x, t) - \sum_{j=0}^{\left[\frac{m-1}{n}\right]} \frac{1}{j!} \frac{\partial^j f}{\partial t^j}(x, 0) t^j}{q(t)} = t^{1-\left\{\frac{m}{n}\right\}} f_0(x, t).$$

where $[\cdot]$, $\{\cdot\}$ —whole and fractional parts.

After $Z_{-1}^i(x)$ is determined, the $u_{-1}(x, t)$ solution is found at the “−1” iteration step:

$$u_{-1}(x, t) = \sum_{j=0}^{\left[\frac{m-1}{n}\right]} \frac{1}{j!} \frac{\partial^j f}{\partial t^j}(x, 0) \sigma_{n(j+1)-1}(t, \varepsilon) + G(a_{-1}(x, t))$$

or

$$u_{-1}(x, t) = \sum_{j=0}^{\left[\frac{m-1}{n}\right]} \frac{1}{j!} \left[\frac{\partial^j f}{\partial t^j}(x, 0) \sigma_{n(j+1)-1}(t, \varepsilon) - \sqrt[4]{\frac{p(x)}{p(0)}} e^{-\frac{1}{\varepsilon} Q_0^t} F \left(\int_0^t e^{\frac{1}{\varepsilon} Q_0^s} s^{n(j+1)-1} ds \right) \frac{\partial^j f}{\partial t^j}(0, 0) \right]. \quad (9)$$

Using the property of the operator $F(\cdot)$, $u_{-1}(x, t)$ can be represented as

$$u_{-1}(x, t) = \sum_{j=0}^{\left[\frac{m-1}{n}\right]} \frac{1}{j!} \left[\frac{\partial^j f}{\partial t^j}(x, 0) - \sqrt[4]{\frac{p(x)}{p(0)}} \frac{\partial^j f}{\partial t^j}(0, 0) \right] \sigma_{n(j+1)-1}(t, \varepsilon) + \underline{O}(\varepsilon^\infty). \quad (10)$$

The solution of the equations of the system (8) have the form

$$\begin{cases} v_0(x, t) = v_0(x), \\ Z_0^i(x, t) = p(x) \frac{1}{j!} \frac{\partial^j f_{xx}}{\partial t^j}(x, 0) t^j + Z_0^j(x, 0), i = n(j+1) - 1, \\ Z_0^i(x, t) = Z_0^i(x), i \neq n(j+1) - 1, \\ j = 0, \left[\frac{m-1}{n}\right]. \end{cases}$$

Let's solve the equation for $a_{-1/2}(x, t)$. Since $T(F(a_0(x, t))) \equiv 0$, then

$$\begin{aligned} \frac{d}{d\tau} \int_{\frac{\tau}{2\sqrt{t}}}^{\infty} \left(2\sqrt{p(x)} a'_{-1/2} - \frac{p'(x)}{2\sqrt{p(x)}} a_{-1/2} \right) \left(x, t - \frac{\tau^2}{4\xi^2} \right) e^{-\xi^2} d\xi = \\ -p(x) \int_{\frac{\tau}{2\sqrt{t}}}^{\infty} a''_{-1} \left(x, t - \frac{\tau^2}{4\xi^2} \right) e^{-\xi^2} d\xi. \end{aligned}$$

From here

$$\begin{aligned} \int_{\frac{\tau}{2\sqrt{t}}}^{\infty} \left(2\sqrt{p(x)} a'_{-1/2} - \frac{p'(x)}{2\sqrt{p(x)}} a_{-1/2} \right) \left(x, t - \frac{\tau^2}{4\xi^2} \right) e^{-\xi^2} d\xi = \\ 2\sqrt{t} p(x) \int_{\frac{\tau}{2\sqrt{t}}}^{\infty} e^{-\xi^2} \int_{\frac{\tau}{2\sqrt{t}}}^{\xi} a''_{-1} \left(x, t - \frac{s^2}{4\xi^2} \right) ds d\xi. \end{aligned}$$

Putting $\tau = 0$, we get

$$2\sqrt{p(x)}a'_{-1/2}(x,t) - \frac{p'(x)}{2\sqrt{p(x)}}a_{-1/2}(x,t) = 2\sqrt{t}p(x) \int_0^\infty e^{-\xi^2} \int_0^\xi a''_{-1}\left(x, t - \frac{s^2}{4\xi^2}\right) ds d\xi. \quad (11)$$

Solution (11) has the form:

As $G(a_{-1/2}(x,t))|_{\tau=0, x=0} = e^{-\frac{1}{\varepsilon}Q_0^t}a_{-1/2}(0,t) = 0$, then

$$a_{-1/2}(x,t) = \sqrt{t} \sqrt[4]{p(x)} \int_0^x \sqrt[4]{p(s_1)} \int_0^\infty e^{-\xi^2} \int_0^\xi a''_{-1}\left(s_1, t - \frac{s^2}{4\xi^2}\right) ds d\xi ds_1.$$

From the initial conditions for $u_0(x,t)$ we have

$$u_0(x,0) = v_0(x) = \varphi(x), \text{ if } \left\{\frac{m}{n}\right\} \neq 0.$$

Thus, at the step “ $-1/2$ ” $u_{-1/2}(x,t)$ is defined. Taking into account the properties of the operator $F(\cdot)$, the solution $u_{-1/2}(x,t)$ can be represented in the form

$$u_{-1/2}(x,t) = e^{-\frac{1}{\varepsilon}Q_0^t}F(a_{-1/2}(x,t)) = e^{-\frac{1}{\varepsilon}Q_0^t}a_{-1/2}(x,t) + O(\varepsilon^\infty).$$

The function $a_0(x,t)$ is arbitrary at this iteration step. To determine $Z_0^i(x,t)$, it is necessary to consider the system at the “1” th iteration step; to determine $a_0(x,t)$, it is necessary to consider the system at the step “ $1/2$ ”. At the zero iteration step, $u_0(x,t)$ has the form

$$u_0(x,t) = e^{-\frac{1}{\varepsilon}Q_0^t}\varphi(x) + \sum_{i=0}^{p-1} Z_0^i(x,t)\sigma_i(t,\varepsilon) + G(a_0(x,t)) + t^{1-\left\{\frac{m}{n}\right\}}f_0(x,t).$$

Submit to the boundary condition:

$$u_0(0,t) = e^{-\frac{1}{\varepsilon}Q_0^t}\varphi(0) + \sum_{i=0}^{p-1} Z_0^i(0,t)\sigma_i(t,\varepsilon) + t^{1-\left\{\frac{m}{n}\right\}}f_0(0,t) + e^{-\frac{1}{\varepsilon}Q_0^t}a_0(0,t) = \psi(t).$$

From here

$$a_0(0,t) = e^{\frac{1}{\varepsilon}Q_0^t}\left(\psi(t) - t^{1-\left\{\frac{m}{n}\right\}}f_0(0,t)\right) - \varphi(0) - \sum_{i=0}^{p-1} Z_0^i(0,t) \int_0^t e^{\frac{1}{\varepsilon}Q_0^s} s^{\frac{i+1}{n}-1} ds.$$

At the $\sqrt{\varepsilon}$ step, we have:

as $T(F(a_{1/2}(x,t))) = 0$, then

$$\begin{aligned} & \frac{d}{d\tau} \int_{\frac{\tau}{2\sqrt{t}}}^\infty \left(2\sqrt{p(x)}a'_0 - \frac{p'(x)}{2\sqrt{p(x)}}a_0 \right) \left(x, t - \frac{\tau^2}{4\xi^2} \right) e^{-\xi^2} d\xi + \\ & p(x) \int_{\frac{\tau}{2\sqrt{t}}}^\infty a''_{-1/2} \left(x, t - \frac{\tau^2}{4\xi^2} \right) e^{-\xi^2} d\xi = 0. \end{aligned} \quad (12)$$

Solution (12) has the form

$$\begin{aligned} a_0(x,t) = & \sqrt{t} \sqrt[4]{p(x)} \int_0^x \sqrt[4]{p(s_1)} \int_0^\infty e^{-\xi^2} \int_0^\xi a''_{-1/2} \left(s_1, t - \frac{s^2}{4\xi^2} \right) ds d\xi ds_1 + \\ & \sqrt[4]{\frac{p(x)}{p(0)}} a_0(0,t). \end{aligned} \quad (13)$$

To determine $Z_0^i(x, t)$, consider the equation at step ε for $W_1(x, t)$

$$q(t)W_1(x, t) = -t^{-\{\frac{m}{n}\}}\tilde{f}_0(x, t) - \sum_{i=0}^{p-1} Z_0^i(x, t)t^{\frac{i+1}{n}-1}. \quad (14)$$

where $\tilde{f}_0(x, t)$ is determined from the relation $\frac{\partial(t^{1-\{\frac{m}{n}\}}f_0(x, t))}{\partial t} = t^{-\{\frac{m}{n}\}}f_0(x, t) + t^{1-\{\frac{m}{n}\}}\frac{\partial f_0(x, t)}{\partial t} = t^{-\{\frac{m}{n}\}}\tilde{f}_0(x, t)$

Subordinate the right side of the Equation (14) to the conditions of solvability. For this we expand $\tilde{f}_0(x, t)$ by the Taylor-Maclaurin formula.

$$\begin{aligned} q(t)W_1(x, t) = & -\tilde{f}_0(x, 0)t^{-\{\frac{m}{n}\}} - \frac{\partial \tilde{f}_0}{\partial t}(x, 0)t^{1-\{\frac{m}{n}\}} - \dots \\ & - \frac{1}{([\frac{m}{n}] + 1)!} \frac{\partial^{[\frac{m}{n}] + 1} \tilde{f}_0}{\partial t^{[\frac{m}{n}] + 1}}(x, 0)t^{[\frac{m}{n}] + 1 - \{\frac{m}{n}\}} + f_1(x, t)t^{[\frac{m}{n}] + 2 - \{\frac{m}{n}\}} - \sum_{i=0}^{p-1} Z_0^i(x, t)t^{\frac{i+1}{n}-1}. \end{aligned}$$

We put

$$\begin{aligned} Z_0^i(x, 0) = & -\frac{1}{j!} \frac{\partial^j \tilde{f}_0}{\partial t^j}(x, 0), \quad i = n(j+1) - 1 - n\{\frac{m}{n}\}, j = 0, \overline{[\frac{m-1}{n} + \{\frac{m}{n}\}]} \\ Z_0^i(x, 0) = & 0, \quad i \neq n(j+1) - 1 - n\{\frac{m}{n}\}, i = 0, p-1. \end{aligned}$$

Then

$$W_1(x, t) = -\frac{t^{-\{\frac{m}{n}\}}\tilde{f}_0(x, t) + \sum_{i=0}^{p-1} Z_0^i(x, t)t^{\frac{i+1}{n}-1}}{q(t)} = t^{2(1-\{\frac{m}{n}\})}f_1(x, t).$$

Thus the terms for the solution at the zero iteration step have been finally found:

$$\begin{aligned} Z_0^i(x, t) = & p(x)\frac{1}{j!} \frac{\partial^j f_{xx}}{\partial t^j}(x, 0)t, \quad i = (j+1)n - 1, j = 0, \overline{[\frac{m-1}{n}]}, \\ Z_0^i(x) = & -\frac{1}{j!} \frac{\partial^j \tilde{f}_0}{\partial t^j}(x, 0), \quad i = n(j+1) - 1 - n\{\frac{m}{n}\}, j = 0, \overline{[\frac{m-1}{n} + \{\frac{m}{n}\}]} \\ Z_0^i(x) = & 0 \text{ in other cases.} \end{aligned}$$

Thus, at this iteration step, after restricting the solution to $\tau(x, \varepsilon)$, we obtain the main term of the regularized asymptotics of the solution to the boundary value problem on the semiaxis for the parabolic equation:

$$\begin{aligned} u_{\text{main}} = & \frac{1}{\varepsilon} \sum_{j=0}^{[\frac{m-1}{n}]} \left[\frac{1}{j!} \left(\frac{\partial^j f_{xx}}{\partial t^j}(x, 0)\sigma_{n(j+1)-1}(t, \varepsilon) - \sqrt{\frac{p(x)}{p(0)}} e^{-\frac{1}{\varepsilon}Q_0^t} F\left(\int_0^t e^{\frac{1}{\varepsilon}Q_0^s} s^{n(j+1)-1} ds\right) \frac{\partial^j f_{xx}}{\partial t^j}(0, 0) \right) \right] + \\ & \frac{1}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon}Q_0^t} F(a_{-1/2}(x, t)) + e^{-\frac{1}{\varepsilon}Q_0^t} \varphi(x) + \sum_{j=0}^{[\frac{m-1}{n}]} \frac{1}{j!} \left(p(x) \frac{\partial^j f_{xx}}{\partial t^j}(x, 0)t^j \right) \sigma_{n(j+1)-1}(t, \varepsilon) \\ & - \sum_{j=0}^{[\frac{m-1}{n} + \{\frac{m}{n}\}]} \frac{1}{j!} \frac{\partial^j \tilde{f}_0}{\partial t^j}(x, 0)\sigma_{n(j+1)-1-n\{\frac{m}{n}\}}(t, \varepsilon) + e^{-\frac{1}{\varepsilon}Q_0^t} F(a_0(x, t)). \end{aligned} \quad (15)$$

Using the properties of the operator $F(\cdot)$ (Lemma 3) can be represented as

$$\begin{aligned} u_{\text{main}} = & \frac{1}{\varepsilon} \sum_{j=0}^{\lfloor \frac{m-1}{n} \rfloor} \left[\frac{1}{j!} \left(\frac{\partial^j f_{xx}}{\partial t^j}(x, 0) - \sqrt[4]{\frac{p(x)}{p(0)}} \frac{\partial^j f_{xx}}{\partial t^j}(0, 0) \right) \sigma_{n(j+1)-1}(t, \varepsilon) \right] + \frac{1}{\sqrt{\varepsilon}} e^{-\frac{1}{\varepsilon} Q_0^t} a_{-1/2}(x, t) + \\ & e^{-\frac{1}{\varepsilon} Q_0^t} \varphi(x) + \sum_{j=0}^{\lfloor \frac{m-1}{n} \rfloor} \frac{1}{j!} \left(p(x) \frac{\partial^j f_{xx}}{\partial t^j}(x, 0) t^j \right) \sigma_{n(j+1)-1}(t, \varepsilon) - \\ & \sum_{j=0}^{\lfloor \frac{m-1}{n} \rfloor + \lfloor \frac{m}{n} \rfloor} \frac{1}{j!} \frac{\partial^j \tilde{f}_0}{\partial t^j}(x, 0) \sigma_{n(j+1)-1-n\lfloor \frac{m}{n} \rfloor}(t, \varepsilon) + e^{-\frac{1}{\varepsilon} Q_0^t} a_0(x, t) + \mathcal{O}(\varepsilon^\infty). \end{aligned}$$

4. Remainder Estimate

Let

$$\begin{aligned} u(x, t, \varepsilon) = & e^{-\frac{1}{\varepsilon} Q_0^t} \sum_{k=0}^n \varepsilon^k v_k(x, t) + \sum_{i=0}^{p-1} \sigma_i(t, \varepsilon) \sum_{k=-1}^n \varepsilon^k Z_k^i(x, t) + \\ & \sum_{k=-2}^{2n} \sqrt{\varepsilon}^k G(a_{k/2}(x, t)) + \sum_{k=0}^n \varepsilon^k W_k(x, t) + \varepsilon^{n+1} R_n(x, t, \varepsilon). \end{aligned} \quad (16)$$

Substituting (16) into (1) and taking into account the solutions of iterative problems, we get

$$\begin{cases} \dot{R}_n + \frac{1}{\varepsilon} q(t) R_n = e^{-\frac{1}{\varepsilon} Q_0^t} p(x) v_n'' + p(x) \sum_{i=0}^{p-1} Z_n^i \sigma_i''(t, \varepsilon) + p(x) W_n'' + p(x) G(a_n'') - \\ \quad - \frac{1}{\varepsilon} \left[\dot{W}_n - p(x) W_{n-1}'' + \sum_{i=0}^{p-1} t^{\frac{i+1}{n}-1} Z_n^i \right] + \varepsilon p(x) R_n'', \\ R_n(x, 0, \varepsilon) = 0, \quad R_n(0, t, \varepsilon) = 0. \end{cases}$$

Let's introduce the notation:

$$H_1(x, t, \varepsilon) = e^{-\frac{1}{\varepsilon} Q_0^t} p(x) v_n'' + p(x) \sum_{i=0}^{p-1} Z_n^i \sigma_i''(t, \varepsilon) + p(x) W_n'' + p(x) G(a_n''),$$

$$H_2(x, t, \varepsilon) = \dot{W}_n - p(x) W_{n-1}'' + \sum_{i=0}^{p-1} t^{\frac{i+1}{n}-1} Z_n^i.$$

Then

$$\begin{cases} \varepsilon \dot{R}_n - \varepsilon^2 p(x) R_n'' + q(t) R_n = H(x, t, \varepsilon), \\ R_n(x, 0, \varepsilon) = 0, \quad R_n(0, t, \varepsilon) = 0, \quad \text{where } H(x, t, \varepsilon) = -H_2(x, t) + \varepsilon H_1(x, t, \varepsilon) \end{cases} \quad (17)$$

By the estimates of singular integrals and the conditions of the problem (1), the right-hand side has the estimate

$$|H(x, t, \varepsilon)| = | -H_2(x, t) + \varepsilon H_1(x, t, \varepsilon) | < M_1.$$

The remainder estimate is based on the maximum principle for parabolic problems [13]. This principle is used in the form of generality that we need to estimate the remainder. The classical solution to the problem (17) is a function $R_n(x, t, \varepsilon)$, continuous in $Q_T = [0, \infty) \times [0, T] \times (0, \varepsilon_0]$, having continuous $\frac{\partial R_n}{\partial t}$, $\frac{\partial R_n}{\partial x}$, $\frac{\partial^2 R_n}{\partial x^2}$ in Q_T and satisfies the Equation (17) and the initial conditions at $t = 0$ at all points of Q_T . [12] (p. 6).

Theorem 1. Let the problem (17) be given and the following conditions are satisfied:

- (1) $R_n(x, 0, \varepsilon) = 0$;
- (2) $\exists M_1 > 0$ $|H| < M_1$;

- (3) conditions (1) $\div$ (3) of the problem (1);
 (4) $\exists M_2 > 0$ $p(x) < M_2(x^2 + 1)$;
 (5) $\exists \mu > 0, r > 0 \forall x \geq R_n(x, t, \varepsilon) > -\mu(x^r + 1)$.

Then $\exists M > 0$ $|R_n| \leq M$.

Proof. In the proof of this theorem, to estimate the solution, ideas from [13] (In the proof of this theorem, ideas from [13] are used to estimate the solution). We carry out the proof in two stages.

Step I.

Consider a homogeneous equation

$$L(R_n) = \varepsilon \dot{R}_n - \varepsilon^2 p(x) R_n'' + q(t) R_n = 0.$$

Build a function $u = R_n + e^{\frac{\alpha t}{\varepsilon}} \frac{2\mu}{\ell^{2\beta-r}} (x^2 + kt)^\beta$, $2\beta > r$:

$$\text{Then } L(u) = L(R_n) + \frac{2\mu}{\ell^{2\beta-r}} L(e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^\beta) = \frac{2\mu}{\ell^{2\beta-r}} L(e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^\beta).$$

Let's calculate and estimate $L(e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^\beta)$.

$$\begin{aligned} \text{We have a chain of inequalities } L(e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^\beta) &= \\ e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-2} [(\alpha + q(t))(x^2 + kt)^2 + \varepsilon \beta k (x^2 + kt) - \varepsilon^2 p(x)(2\beta(x^2 + kt) + \\ 4x^2 \beta(\beta - 1))] &\geq e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-2} \\ [(\alpha + q(t))(x^2 + kt)^2 + \varepsilon \beta k (x^2 + kt) - \varepsilon^2 M_2(x^2 + 1)(2\beta(x^2 + kt) + 4x^2 \beta(\beta - 1))] &\geq \\ e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-1} [\alpha(x^2 + kt) + \varepsilon \beta k - \varepsilon^2 M_2(x^2 + 1)2\beta(2\beta - 1)] \end{aligned}$$

(a) $x > 1$.

$$\begin{aligned} L(u) &\geq \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-1} [\alpha(x^2 + kt) + \varepsilon \beta k - \varepsilon^2 M_2(x^2 + 1)2\beta(2\beta - 1)] \geq \\ \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^\beta [\alpha - \varepsilon^2 8M_2\beta^2] &\geq 0, \\ \text{if } \alpha &> \varepsilon_0^2 8M_2\beta^2. \end{aligned}$$

(b) $0 < x < 1$.

$$\begin{aligned} L(u) &\geq \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-1} [\alpha(x^2 + kt) + \varepsilon \beta k - \varepsilon^2 M_2(x^2 + 1)2\beta(2\beta - 1)] \geq \\ \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (x^2 + kt)^{\beta-1} (\varepsilon \beta k - 8\varepsilon^2 M_2\beta^2) &\geq 0, \\ \text{if } k &> \varepsilon_0 8M_2\beta. \end{aligned}$$

We estimate $u(x, t, \varepsilon)$ on the border of the region D_ℓ :

$$\begin{aligned} u(x, 0, \varepsilon) &= 2\mu \frac{x^{2\beta}}{\ell^{2\beta-r}} \geq 0, \\ u(0, t, \varepsilon) &= \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (kt)^\beta \geq 0, \\ u(\ell, t, \varepsilon) &= R_n + \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (\ell^2 + kt)^\beta \geq -\mu(\ell^r + 1) + \frac{2\mu}{\ell^{2\beta-r}} e^{\frac{\alpha t}{\varepsilon}} (\ell^2 + kt)^\beta \geq, \\ -\mu(\ell^r + 1) + 2\mu\ell^r &= \mu(\ell^r - 1) \geq 0. \end{aligned}$$

Hence, $L(u) \geq 0 \implies u \geq 0$ in D_ℓ : In other words

$$R_n + e^{\frac{\alpha t}{\varepsilon}} \frac{2\mu}{\ell^{2\beta-r}} (x^2 + kt)^\beta \geq 0.$$

$\forall x \in (0, +\infty) \quad \exists \ell \geq x$. Then directing $\ell \rightarrow \infty$, we get $\forall x \in (0, +\infty) R_n \geq 0$.

Step II.

Consider the function $u = \pm R_n + \frac{M_1 t}{\varepsilon}$. Then

$$L(u) = \pm H + M_1 + \frac{q(t)M_1 t}{\varepsilon} \geq 0.$$

From here

$$u = \pm R_n + \frac{M_1 t}{\varepsilon} \geq 0 \implies |R_n| \leq \frac{M_1 t}{\varepsilon}.$$

In this way

$$|R_n| \leq \frac{M_1 t}{\varepsilon}.$$

Let's write $R_n = u_{n+1} + \varepsilon R_{n+1}$. Then $|R_n| \leq |u_{n+1}| + \varepsilon \frac{M_1 t}{\varepsilon} \leq M$. Hence

$$\left| u - \sum_{k=-1}^n u_k \right| \leq \varepsilon^{n+1} M.$$

□

5. Application

Lemma 1. $\forall z > 0 \quad \operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\xi^2/2} d\xi < \sqrt{2} e^{-z^2/2}.$

Proof.

$$\begin{aligned} \operatorname{erfc}(z) &= \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\xi^2/2 - \xi^2/2} d\xi \leq \frac{2}{\sqrt{\pi}} e^{-z^2/2} \int_z^\infty e^{-\xi^2/2} d\xi \leq e^{-z^2/2} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = \sqrt{2} e^{-z^2/2}. \end{aligned}$$

□

Lemma 2. $\forall \psi(t) \in \mathbb{C}[0, T]$ the estimate is correct

$$|F(\psi(t))| \leq \frac{2}{\sqrt{\pi}} \int_{x/(2\varepsilon\sqrt{t})}^\infty \left| \psi\left(t - \frac{x^2}{4\varepsilon^2\xi^2}\right) \right| e^{-\xi^2} d\xi \leq M.$$

Proof.

if $\frac{x}{2\varepsilon\sqrt{t}} \leq \xi < +\infty$, then $t - \frac{x^2}{4\varepsilon^2\xi^2} \in [0, t] \subseteq [0, T]$

Since the function is continuous by condition, the estimate

$$|F(\psi(t))| \leq \frac{2}{\sqrt{\pi}} \int_{x/(2\varepsilon\sqrt{t})}^\infty \left| \psi\left(t - \frac{x^2}{4\varepsilon^2\xi^2}\right) \right| e^{-\xi^2} d\xi \leq M \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-\xi^2} d\xi = M.$$

□

Lemma 3. $\forall \psi(t) \in \mathbb{C}[0, T]$ the estimate is correct $F(\psi(t)) = \psi(t) + \underline{O}(\varepsilon^\infty)$ at $\varepsilon \rightarrow 0$, $\forall x \in [\delta, +\infty)$ where $\delta > 0$.

Proof. Taking into account the results of Lemmas 1 and 2, we obtain

Let's write

$$F(\psi(t)) = \psi(t) + \frac{2}{\sqrt{\pi}} \int_{x/(2\varepsilon\sqrt{t})}^\infty \left(\psi\left(t - \frac{x^2}{4\varepsilon^2\xi^2}\right) - \psi(t) \right) e^{-\xi^2} d\xi,$$

Let us estimate the second term modulo

$$\frac{2}{\sqrt{\pi}} \left| \int_{x/(2\varepsilon\sqrt{t})}^\infty \left(\psi\left(t - \frac{x^2}{4\varepsilon^2\xi^2}\right) - \psi(t) \right) e^{-\xi^2} d\xi \right| \leq 2M \frac{2}{\sqrt{\pi}} \int_{x/(2\varepsilon\sqrt{t})}^\infty e^{-\xi^2} d\xi \leq$$

$2\sqrt{2}M e^{-\frac{x^2}{8\varepsilon^2 t}}.$ Hence follows the estimate

$$F(\psi(t)) = \psi(t) + \underline{O}(\varepsilon^\infty) \text{ at } \varepsilon \rightarrow 0 \quad \forall x \in [\delta, +\infty) \quad \delta > 0.$$

□

Lemma 4. $\forall \psi(t)$ real-analytic on $[0, T]$ the point $\varepsilon = 0$ is an essential singular point for

$$F(\psi(t)) = \frac{2}{\sqrt{\pi}} \int_{x/(2\varepsilon\sqrt{t})}^{\infty} \psi\left(t - \frac{x^2}{4\varepsilon^2\zeta^2}\right) e^{-\zeta^2} d\zeta.$$

Proof. When $\frac{x}{2\varepsilon\sqrt{t}} \leq \zeta < \infty$ expression $\frac{x^2}{4\varepsilon^2\zeta^2}$ varies within $0 \leq \frac{x^2}{4\varepsilon^2\zeta^2} \leq t \leq T$. Expanding $\psi\left(t - \frac{x^2}{4\varepsilon^2\zeta^2}\right)$ in a series, we get

$$\psi\left(t - \frac{x^2}{4\varepsilon^2\zeta^2}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\psi^{(k)}(t)}{k!} \left(\frac{x}{4\varepsilon^2}\right)^k \frac{1}{\zeta^{2k}}.$$

Substituting $\psi\left(t - \frac{x^2}{4\varepsilon^2\zeta^2}\right)$ into the integral $\int_{x/(2\varepsilon\sqrt{t})}^{\infty} \psi\left(t - \frac{x^2}{4\varepsilon^2\zeta^2}\right) e^{-\zeta^2} d\zeta$ and after simple calculations we get

$$F(\psi(t)) = \operatorname{erfc}(z) \sum_{k=0}^{\infty} \frac{t^k 4^k \psi^{(k)}(t)}{(2k)!} z^{2k} + \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{m=0}^{\infty} z^{2m+1} \sum_{k=m+1}^{\infty} \frac{(-1)^{k+m} t^k 2^{k+m} (2k-2m-3)!!}{(2k)!} \psi^{(k)}(t). \quad (18)$$

Here $z = \frac{x}{2\varepsilon\sqrt{t}}$. Row (18) consists of two rows. If $|\psi^{(k)}| < C, k = \overline{0, \infty}$, then:

a. The first series converge $0 \leq z < +\infty$ by the d'Alembert's criterion. It is obvious;

b. Let us prove that the second series also converges for any z . The series

$$\sum_{k=m+1}^{\infty} \frac{(-1)^k t^k 2^k}{(2k)!} (2k-2m-3)!! \psi^{(k)}(t) \text{ converges according to the Leibniz criterion,}$$

therefore, admits the estimate

$$\left| \sum_{k=m+1}^{\infty} \frac{(-1)^k t^k 2^k}{(2k)!} (2k-2m-3)!! \psi^{(k)}(t) \right| < C \frac{t^{m+1} 2^{m+1}}{(2m+2)!}.$$

As a consequence of this estimate, the series $\sum_{m=0}^{\infty} (-1)^m \frac{t^{m+1} 2^{m+1}}{(2m+2)!} z^{2m+1}$ converges absolutely for $\forall z \geq 0$.

Thus, $\varepsilon = 0$ —is an essentially singular point.

□

Theorem 2 (on the solvability of iterative problems). *Let the equation be given*

$$t^{m/n} Z(x, t) = F(x, t) \quad (19)$$

and conditions are met

$$F(x, t) \in C^\infty(R \times [0, T]).$$

Then the Equation (19) is solvable in the class of smooth functions if and only if

$$\frac{\partial^k F}{\partial t^k}(x, 0) = 0, \quad k = \overline{0, \left[\frac{m}{n}\right]}.$$

Proof. We expand $F(x, t)$ by Maclaurin's formula in t :

$$F(x, t) = F(x, 0) + \frac{\partial F}{\partial t}(x, 0)t + \dots + \frac{\partial^{\left[\frac{m}{n}\right]} F}{\partial t^{\left[\frac{m}{n}\right]}}(x, 0)t^{\left[\frac{m}{n}\right]} + t^{\left[\frac{m}{n}\right]+1} F_0(x, t).$$

Then the Equation (19) has the form

$$t^{m/n}Z(x, t) = F(x, 0) + \frac{\partial F}{\partial t}(x, 0)t + \dots + \frac{\partial^{[\frac{m}{n}]} F}{\partial t^{[\frac{m}{n}]}(x, 0)}t^{[m/n]} + t^{[m/n]+1}F_0(x, t).$$

Since by condition all derivatives up to the order $[\frac{m}{n}]$ are equal to zero, the Equation (19) takes the form $t^{m/n}Z(x, t) = t^{[m/n]+1}F_0(x, t)$. Hence it is solvable. The solution to the equation is written in the form $Z(x, t) = t^{1-\{\frac{m}{n}\}}F_0(x, t)$.

Necessity. Let the Equation (19) have a solution. For $t = 0$ we have $0 = F(x, 0)$:

$$t^{m/n}Z(x, t) = F(x, t) - F(x, 0).$$

Divide the equation by t if $[\frac{m}{n}] > 1$:

$$t^{\frac{m}{n}-1}Z(x, t) = \frac{F(x, t) - F(x, 0)}{t}.$$

For $t = 0$ we get $\frac{\partial F}{\partial t}(x, 0) = 0$.

Continuing this process to step $[\frac{m}{n}]$, we get $\frac{\partial^k F}{\partial t^k}(x, 0) = 0 \forall k = 0, [\frac{m}{n}]$.

□

6. Conclusions

In this paper, the regularization method is developed for a singularly perturbed parabolic boundary value problem on the semiaxis. It is found that, in addition to the functions describing the irregular dependence of the solution on the parameter ε using the spectrum of the limit operator, a singular operator is also needed to describe the parabolic boundary layer. The introduction of these regularizing functions and the operator made it possible to construct a uniform asymptotics for the solution of the parabolic problem. To estimate the remainder, the maximum principle was used, modified for a singularly perturbed parabolic equation. The regularization method developed in this paper can be used both in the study of applied problems containing a “simple” turning point, and in the construction of numerical algorithms for solving problems with spectral features.

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