mathematics

Article

From Boolean Valued Analysis to Quantum Set Theory: Mathematical Worldview of Gaisi Takeuti †

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† This paper is an extended version of the article published in *Sūgaku Seminar* **57** (2), 28–33 (Nippon Hyoron Sha, Tokyo, 2018) (in Japanese) by the same author.

Abstract: Gaisi Takeuti introduced Boolean valued analysis around 1974 to provide systematic applications of the Boolean valued models of set theory to analysis. Later, his methods were further developed by his followers, leading to solving several open problems in analysis and algebra. Using the methods of Boolean valued analysis, he further stepped forward to construct set theory that is based on quantum logic, as the first step to construct "quantum mathematics", a mathematics based on quantum logic. While it is known that the distributive law does not apply to quantum logic, and the equality axiom turns out not to hold in quantum set theory, he showed that the real numbers in quantum set theory are in one-to-one correspondence with the self-adjoint operators on a Hilbert space, or equivalently the physical quantities of the corresponding quantum system. As quantum logic is intrinsic and empirical, the results of the quantum set theory can be experimentally verified by quantum mechanics. In this paper, we analyze Takeuti's mathematical world view underlying his program from two perspectives: set theoretical foundations of modern mathematics and extending the notion of sets to multi-valued logic. We outlook the present status of his program, and envisage the further development of the program, by which we would be able to take a huge step forward toward unraveling the mysteries of quantum mechanics that have persisted for many years.

Keywords: Takeuti; Boolean algebras; set theory; Boolean valued models; forcing; continuum hypothesis; cardinal collapsing; Hilbert spaces; von Neumann algebras; AW*-algebras; type I; ortho-complemented lattices; quantum logic; multi-valued logic; quantum set theory; transfer principle; quantum mathematics

Citation: Ozawa, M. From Boolean Valued Analysis to Quantum Set Theory: Mathematical Worldview of Gaisi Takeuti. *Mathematics* **2021**, *9*, 397. https://doi.org/10.3390/math9040397

Academic Editors: Anatoly Georgievich Kusraev and Semën Kutateladze

Received: 31 December 2020
Accepted: 13 February 2021
Published: 17 February 2021

1. Introduction

In 1982, Gaisi Takeuti published a book entitled *Mathematical Worldview: Ideas and Prospects of Modern Mathematics* [1]. In this book, he states that mathematics based on classical logic is an absolute truth, and it has the characteristic that all possible statements always belong to only one of the two, either true or false, and yet in the future, the progress of human culture may lead to the birth of a "new mathematics" that is based on a "new logic". One well-known possibility is the creation of intuitionistic mathematics based on intuitionistic logic. This was advocated by Brouwer as one of the positions of mathematics, but Takeuti has proposed a "new mathematics" that should be called "quantum mathematics" based on quantum logic. In this paper, devoted to the memory of Takeuti, we will try explaining Takeuti's view of the mathematical world, the trajectory of research leading up to this proposal, and the current state of research based on this proposal.

2. Quantum Logic

"Quantum logic" is the logic of quantum mechanics first discovered by Birkhoff and von Neumann [2] in 1936. Among the laws of classical logic, it is known as the logic for which the distributive law does not hold. Birkhoff and von Neumann hypothesized that the modular law holds true in place of the distributive law, but, now, a weaker orthomodular law is hypothesized. On the other hand, in intuitionistic logic, the distributive law holds, but the law of excluded middle, the law of double negation, and De Morgan's law do not hold. In this sense, quantum logic is a logic that contrasts with intuitionistic logic, and Takeuti compared the relationship between classical logic, intuitionistic logic, and quantum logic to "God's logic", "human logic", and "the logic of things".

The fact that the distributive law does not hold in quantum logic is related to the existence of physical quantities that cannot be simultaneously measured in quantum mechanics due to the uncertainty principle. The propositions considered in quantum logic are referred to as observational propositions. The most fundamental of these is the observational proposition that "the value of the physical quantity A is equal to the real number a", which is expressed as $A = a$. In quantum mechanics, a physical quantity is represented by a self-adjoint operator on a Hilbert space, and if it has no continuous spectrum, the possible values are its eigenvalues.

For example, if the eigenvalues of such a physical quantity A are a_1 and a_2 and the eigenvalues of a physical quantity B are b_1 and b_2, then $A = a_1 \vee A = a_2$ and $B = b_1 \vee B = b_2$ are true respectively. However, if the physical quantities A and B have no common eigenvectors, for any pair (a_j, b_k) of eigenvalues of A and B, the proposition $A = a_j \wedge B = b_k$ is false. This means that

$$(A = a_1 \vee A = a_2) \wedge (B = b_1 \vee B = b_2)$$

is true, but

$$(A = a_1 \wedge B = b_1) \vee (A = a_1 \wedge B = b_2) \vee (A = a_2 \wedge B = b_1) \vee (A = a_2 \wedge B = b_2)$$

is false; therefore, the distributive law does not hold.

3. Set Theoretical Worldview

Constructing mathematics based on the logic characteristic of these kinds of physical phenomena seems like an absurd undertaking at first. Where did this idea originally come from? According to Takeuti's *Mathematical World View* [1], there are two origins for this idea: first, the idea that all of the objects of mathematics are sets, and the modern mathematics equals ZFC set theory (an axiomatic set theory with Zermelo–Fraenkel's axioms plus the axiom of choice) and, secondly, the idea that the concept of sets, originally considered in strictly two-valued logic, can be generalized to multi-valued logic that allows for intermediate truth values.

The first idea, which began with Frege, went through trials, such as Russell's paradox, Hilbert's formalist program, and Gödel's incompleteness theorem, but came to fruition in Bourbaki's *Éléments de Mathématique* [3]. It can be said that it is the fundamental idea of modern mathematics.

The second idea, which is famous for Zadeh's fuzzy set theory [4,5], is to extend the two-valued logic of the membership relation to a multi-valued logic in defining subsets of a given "classical" set. The method for expanding the universe of sets in the ordinary two-valued logic to multi-valued logic at once was clarified with Cohen's [6,7] independence proof of the continuum hypothesis in the field of foundations of mathematics.

In this independence proof, Cohen [6,7] developed a method, called forcing, of expanding the models of set theory. Scott and Solovay [8] showed that this method can be reformulated in a more accessible way using Boolean valued models of set theory. The Boolean valued models of set theory make all of the sets, as the objects of mathematics, to be "multivalued" in a multi-valued logic with truth values in a given Boolean algebra. This view of multi-valued logicalization of the concept of sets further clarified the relationship between sheaf theory, topos theory, and intuitionistic set theory.

Therefore, it can be said that Takeuti's idea of constructing mathematics based on quantum logic is to first develop this second idea, making all the sets that are the objects of mathematics to be multi-valued at once, and then under the "quantum set theory" obtained in the above, to develop mathematics based on set theory along the first idea. Logic has two aspects, syntax and semantics. Takeuti's program is to quantum-logicalize semantics at once, which leaves the mathematical syntax that is based on the formal system of set theory as it is.

4. Modern Mathematics and ZFC Set Theory

The majority of mathematicians accept that ZFC set theory is the foundation on which modern mathematics is based. Takeuti wrote [9] (p. 171):

> ZFC is a stable and powerful axiomatic system; there is almost no concern about contradictions, and all modern mathematics can be developed within it. At present, it can be said that mathematics is equal to ZFC.

This is an important pillar of Takeuti's view of the mathematical world. However, it is unfortunate that the 20th century, in which these foundations of mathematics were established, became a little distant, and the interest in this idea and the achievements of Bourbaki, which was the basis for it, diminished. Let us add some comments.

Modern mathematics has two characteristics: formal and structural [10]. On the formal side, the whole mathematics can be developed using a formal language, strict deductive rules, and a set of axioms. In other words, the whole mathematics can be formalized by a single first-order predicate theory, known as ZFC set theory, and the only undefined term is the concept of sets. The axioms are exhausted by the axioms of ZFC set theory, and all other mathematical concepts are defined within ZFC set theory. For example, number systems, such as natural numbers and real numbers, geometric spaces, such as Euclidean spaces, and transfinite number systems of ordinal and cardinal numbers are all defined inside ZFC [3].

On the other hand, on the structural side, what mathematicians actually study is not limited to these kinds of numbers and figures, but, instead, it is generally considered as abstract objects called "mathematical structures". Bourbaki showed that each field of mathematics is reduced to the study of "mathematical structures" that are specific to that field. Those "mathematical structures" are defined within ZFC set theory, and that their research is conducted based on the axioms of ZFC set theory [3].

While it is difficult to explain the concept of "mathematical structure" to the general readership, according to its formal definition, it is a set theoretical object that is given by a set with functions and relations being defined on that set and several number systems (and the sets obtained by repeating direct products and power sets of these). Examples of mathematical structures include algebraic systems (such as groups, rings, and fields), topological spaces, manifolds, and measure spaces, as well as topological algebraic systems (such as Hilbert spaces and C*-algebras).

In this way, the word "axiomatic system" has two different aspects that correspond to the formal and structural aspects of mathematics. In other words, these are the axioms of ZFC set theory, and the axioms that characterizes each mathematical structure and that is included in its definition. The concept of a mathematical structure gradually became clearer through the application of the axiomatic method that was proposed by Hilbert to various fields of mathematics, and it resulted in Bourbaki's *Éléments de mathématique* [3]. Each field of mathematics is relatively consistent from ZFC set theory if there is an example of the "mathematical structure" that the field studies.

Hilbert's formalism program can currently be interpreted as founding all mathematics in ZFC set theory and having a finistic proof for the consistency of ZFC set theory. Hilbert first showed that the consistency of geometry was reduced to the consistency of real number theory, and he questioned the proof of consistency of real number theory in the second problem of his famous 23 problems, and he called for the axiomatizations of physics theories, including probability theory as the sixth problem. As a result, probability theory was axiomatized by Kolmogorov [11], by means of measure theory, and it became the basis for giving mathematical proofs to important hypotheses of statistical mechanics, such as the ergodic hypothesis. Measure theory is a study of the mathematical structure of measure space, and probability theory has been axiomatized as one of the "mathematical structures" in ZFC set theory. In measure theory, the existence of Lebesgue non-measurable sets is a theorem of ZFC set theory, but it is known to be independent of set theory without the axiom of choice. In addition, von Neumann's axiomatization of quantum mechanics shows that the two formulations of quantum mechanics, Heisenberg's matrix mechanics and Schrödinger's wave mechanics, are unified by a common "mathematical structure" of the Hilbert space [12].

Theory of operator algebras on Hilbert spaces plays an important role in the axiomatization of quantum field theory [13]. However, recently, a problem independent of ZFC set theory have been identified within classical problems in operator algebras [14]. In these fields, we can obtain a glimpse of the reality of "modern mathematics equals ZFC". However, Bourbaki's *Éléments de mathématique* [3] does not include "probability theory" or "operator algebras". Nevertheless, it is certain that they are the most Bourbaki-like fields in mathematics.

Returning to Hilbert's program, Gödel's incompleteness theorem showed that Hilbert's program is not feasible as it was. Hence, is Takeuti's trust in ZFC, saying "there is almost no concern about contradictions" unfounded? Putting aside the optimistic viewpoint that as long as contradictions are handled when they occur, existing mathematics will not be lost, Bourbaki states the following regarding the proof of consistency [15] (p. 44).

> The theorem of Gödel does not however shut the door completely on attempts to prove consistency, so long as one abandons (at least partially) the restrictions of Hilbert concerning "finite procedures". It is thus that Gentzen in 1936 [16], succeeds in proving the consistency of formalised arithmetic, by using "intuitively" transfinite induction up to the countable ordinal ε_0. The value of the "certainty" that one can attach to such reasoning is without doubt less convincing than for that which satisfies the initial requirements of Hilbert, and is essentially a matter of personal psychology for each mathematician; it remains no less true that similar "proofs" using "intuitive" transfinite induction up to a given ordinal, would be considered important progress if they could be applied, for example, to the theory of real numbers or to a substantial part of the theory of sets.

Actually, this achievement hoped for by Bourbaki came to pass afterward, and the results are already in our hands. Takeuti's fundamental conjecture [17] and its solution [18,19] gave a Gentzen-style consistency proof for the theory of real numbers, and Arai [20,21] gave a Gentzen-style consistency proof for a substantial part of ZFC set theory.

5. Set Theory Based on Multi-Valued Logic

According to Takeuti [22], the universe of set theory is composed of the following two principles.

C1. Power set composition principle

C2. Transfinite generation principle

Principle C1 states that for an arbitrary set, the totality of its subsets becomes a set again. Principle C2 states that, when a method for creating a new set is provided, the method can be repeated a transfinite number of times as much as possible. The most typical case of this principle is the generation of ordinal numbers. This starts from the empty set $\emptyset$ and it endlessly repeats the operation of collecting the ordinal numbers constructed so far, and the whole ordinal number is generated, as follows. $0 = \emptyset, 1 = \{0\}, 2 = \{0,1\}, \ldots, \omega = \{0,1,2,\ldots\}, \omega+1 = \{0,1,2,\ldots,\omega\}, \ldots$. Sets, like $\omega = \{0,1,2,3,\ldots\}$, which do not contain the final ordinal number, are called limit ordinal numbers. Subsequently, the order relation for two ordinals α, β is defined as $\alpha < \beta$ if $\alpha \in \beta$. Using these two principles, the universe V of sets can be constructed, as follows.

F1. When ordinal number 0 arises, V_0 is composed as $\emptyset$.

F2. When creating ordinal number $\alpha + 1$ from ordinal number α, $V_{\alpha+1}$ is composed as $\mathcal{P}(V_\alpha)$.

F3. When creating limit ordinal number α, V_α is composed of $\bigcup_{\beta<\alpha} V_\beta$.

F4. As long as the generation of ordinal numbers is continued, the composition of V_α will be continued endlessly. We call the union $\bigcup_\alpha V_\alpha$ of V_α over all the ordinal numbers α, as V.

Consider the following general structure of semantics of a logic to consider a general method for constructing a set theory based on a multi-valued logic. Let $\mathcal{L}$ be a partially ordered set such that every subset S has the supremum $\bigvee S$ and the infimum $\bigwedge S$. For the arbitrary element $x \in \mathcal{L}$, its complement $x^\perp$ is defined with the following properties: $x \wedge x^\perp = 0$, $x \vee x^\perp = 1$, $x^{\perp\perp} = x$, and $x \le y \Leftrightarrow y^\perp \le x^\perp$. Here, we write $x \wedge y = \bigwedge\{x, y\}$, $x \vee y = \bigvee\{x, y\}$. The element $1 = \bigvee \mathcal{L}$ is the maximum element of $\mathcal{L}$ representing "true", the element $0 = \bigwedge \mathcal{L}$ is the minimum element of L representing "false", and the other elements of the set $\mathcal{L}$ represent intermediate truth values. Such a structure $\mathcal{L}$ is called a complete orthocomplemented lattice.

In the process of constructing the universe of sets, the concept of a power set appears in F2 as a sole procedure for creating a new set, so that, to construct the universe of $\mathcal{L}$-valued sets, it suffices to extend this part to the $\mathcal{L}$-valued logic. Incidentally, using the idea of fuzzy sets, for any set X, any function $A : D \to \mathcal{L}$ from any subset D of X to $\mathcal{L}$ can be considered to be an $\mathcal{L}$-valued subset of X. Therefore, the universe $V^{(\mathcal{L})}$ of sets based on $\mathcal{L}$-valued logic is defined, as follows.

(1) $V_0^{(\mathcal{L})} = \emptyset$.

(2) $V_{\alpha+1}^{(\mathcal{L})} = \{u \mid u : \mathrm{dom}(u) \to \mathcal{L},\ \mathrm{dom}(u) \subseteq V_\alpha^{(\mathcal{L})}\}$.

(3) $V_\alpha^{(\mathcal{L})} = \bigcup_{\beta<\alpha} V_\beta^{(\mathcal{L})}$ if α is a limit ordinal.

(4) $V^{(\mathcal{L})} = \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathcal{L})}$.

Here, On stands for the set of ordinals.

Define the $\mathcal{L}$-valued truth values of any closed formulas of the language of set theory augmented by the names of elements of $V^{(\mathcal{L})}$ as follows [23–25].

(1) $[\![\neg\phi]\!] = [\![\phi]\!]^\perp$.

(2) $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!]$.

(3) $[\![(\forall x \in u)\phi(x)]\!] = \bigwedge_{x \in \mathrm{dom}(u)}(u(x) \to [\![\phi(x)]\!])$.

(4) $[\![(\forall x)\phi(x)]\!] = \bigwedge_{x \in V^{(\mathcal{L})}} [\![\phi(x)]\!]$.

Here, the operation $\to$ on $\mathcal{L}$ is defined by $P \to Q = P^\perp \vee (P \wedge Q)$. We consider the logical symbols $\vee$, $(\exists x \in y)$, and $(\exists x)$ to be defined by

(5) $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$.

(6) $(\exists x \in u)\phi(x) := \neg((\forall x \in u)\neg\phi(x))$.

(7) $(\exists x)\phi(x) := \neg((\forall x)\neg\phi(x))$.

The $\mathcal{L}$-valued truth values of the membership and the equality relations between two elements u, v of $V^{(\mathcal{L})}$ are recursively defined, as follows.

(8) $[\![u \in v]\!] = [\![\exists x \in v(x = u)]\!]$.

(9) $[\![u = v]\!] = [\![\forall x \in u(x \in v) \wedge \forall y \in v(y \in u)]\!]$.

As a result of the above, the $\mathcal{L}$-valued truth value $[\![\phi]\!] \in \mathcal{L}$ is defined for any closed formula ϕ in set theory with constant symbols naming elements of $V^{(\mathcal{L})}$.

Let V be the universe of the standard sets based on the two-valued logic. There exists an $\mathcal{L}$-valued set $\check{a}$ that corresponds to each $a \in V$. In fact, $\check{a}$ is determined as an element of $V^{(\mathcal{L})}$ such that $\mathrm{dom}(\check{a}) = \{\check{x} | x \in a\}$ and that $\check{a}(\check{x}) = 1$ if $x \in a$. Then the relationship between the standard sets a and b is isomorphic to the relationship between $\mathcal{L}$-valued sets $\check{a}$ and $\check{b}$.

When $\mathcal{L}$ satisfies the distributive law,

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R), \quad P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R),$$

$\mathcal{L}$ is called a complete Boolean algebra. If $\mathcal{B}$ is a complete Boolean algebra, $V^{(\mathcal{B})}$ is called a Boolean value model of set theory. In this case, the following theorem holds [23].

Theorem 1 (Scott–Solovay)**.** *If $\phi(x_1, \ldots, x_n)$ is provable in ZFC set theory,*

$$[\![\phi(u_1, \ldots, u_n)]\!] = 1$$

holds for any $u_1, \ldots, n_n \in V^{(\mathcal{B})}$.

Let us show what kind of $\mathcal{B}$ can be used for proving the independence of the continuum hypothesis (CH) (i.e., CH cannot be proved in ZFC set theory); note that the consistency of CH (i.e., the negation of CH cannot be proved in ZFC set theory) was proved by Gödel [26]. Let I be an index set of the cardinality larger than $2^{\aleph_0}$. Let $X = 2^{\aleph_0 \times I}$ be the generalized Cantor space, a product topological space of the direct product of $\aleph_0 \times I$ copies of $2 = \{0, 1\}$. Let $\mathcal{B}(X)$ be the Borel σ-field of subsets of X. Let m be a product measure on $\mathcal{B}(X)$, defined as follows.

$$m(\{p \in X \mid p(j_1) = a_1, \ldots, p(j_n) = a_n\}) = \left(\frac{1}{2}\right)^n,$$

where $a_j \in \{0, 1\}$ $(j = 1, \ldots, n)$. The existence of this kind of measure is based on Kolmogorov's extension theorem. Let $\mathcal{N}$ be the collection of measure zero subsets. Subsequently, the quotient Boolean algebra $\mathcal{B} = \mathcal{B}(X)/\mathcal{N}$ is a complete Boolean algebra, called the measure algebra of m. For this $\mathcal{B}$, the value of $[\![\mathrm{CH}]\!]$ can be calculated, and the value $[\![\mathrm{CH}]\!] = 0$ is obtained [23] (p. 173). If CH can be proved from ZFC, then from Theorem 1, $[\![\mathrm{CH}]\!] = 1$ for any $\mathcal{B}$, so that the independence of the continuum hypothesis is proved in this way.

6. Boolean Valued Analysis

Let ϕ be a logical formula in ZFC set theory representing a mathematical theorem. The proposition "$[\![\phi]\!] = 1$" is a new, different theorem, according to Theorem 1. Because the proposition $[\![\phi]\!] = 1$ is constructed by recursive rules, we can analyze its meaning, so that we will be able to restate $[\![\phi]\!] = 1$ using familiar concepts. If ψ is a mathematical proposition written with our familiar concepts, and if $[\![\phi]\!] = 1$ and ψ can be proved equivalent, ψ will also become a new mathematical theorem. In other words, if we prove that $[\![\phi]\!] = 1$ and ψ are equivalent, then we can prove ψ only by proving ϕ without proving ψ directly. In this case, ϕ is a far simpler proposition than ψ, and proving the equivalence of $[\![\phi]\!] = 1$ and ψ is similar to language translation in many ways. As a result, this approach is known to often lead to very promising prospects.

To apply this method to analysis, in *Two Applications of Logic to Mathematics* [27], Takeuti studied the structure of the real numbers in $V^{(\mathcal{B})}$ for a complete Boolean algebra $\mathcal{B}$ of projections on a Hilbert space $\mathcal{H}$ and showed that the real numbers in $V^{(\mathcal{B})}$ are in one-to-one correspondence with the self-adjoint operators on $\mathcal{H}$ such that their spectral projections belong to $\mathcal{B}$. Here, the real numbers in $V^{(\mathcal{B})}$ are defined as the set of elements u of $V^{(\mathcal{B})}$ satisfying $[\![\mathrm{R}(u)]\!] = 1$, where $\mathrm{R}(x)$ is a logical formula in ZFC meaning "x is a real number". This shows that, from theorems ϕ related to real numbers, the theorem $[\![\phi]\!] = 1$ for these kinds of self-adjoint operators can be systematically obtained. For example, from the theorem stating "Every upper bounded set of real numbers has its supremum", we obtain the theorem stating "Every upper bounded set of mutually commuting self-adjoint operators has its supremum".

This is a very powerful method, which Takeuti referred to as Boolean valued analysis. In his paper *Von Neumann algebras and Boolean valued analysis* [28], Takeuti took this a step further, showing that the class of all Hilbert spaces in $V^{(\mathcal{B})}$ are in one-to-one correspondence with the class of all normal *-representations of the commutative von Neumann algebra $\mathcal{A}$ generated by $\mathcal{B}$; see also [29]. In addition, he further showed that the class of all von Neumann factors (von Neumann algebras with trivial centers) in $V^{(\mathcal{B})}$ are in one-to-one correspondence with the class of von Neumann algebras such that their centers are isomorphic to $\mathcal{A}$.

Based on this finding, theorems for general von Neumann algebras are systematically derived from theorems on von Neumann factors. In operator algebras, there has been a well-known method, called the reduction theory, for deriving theorems on general von Neumann algebras from theorems on von Neumann factors by making use of the direct integral decompositions of von Neumann algebras into von Neumann factors, but there are restrictions, such as separability. The Boolean valued analysis method is a powerful approach that does not have these restrictions.

In order to show that Boolean valued analysis is a truly powerful method, the author explored the von Neumann factors in $V^{(\mathcal{B})}$ for the completely general complete Boolean algebra $\mathcal{B}$, and showed that the type I von Neumann factors in $V^{(\mathcal{B})}$ are in one-to-one correspondence with the type I AW*-algebras whose central projections are isomorphic to $\mathcal{B}$ [30,31]; see also [29,32]. There is a theorem that "every type I von Neumann factor is isomorphic to the algebra of all bounded operators on a Hilbert space, and the cardinal number representing the dimension of that Hilbert space is a complete invariant". While considering $[\![\phi]\!] = 1$ for ϕ to be the above theorem, we obtain the theorem stating "the isomorphic invariants of type I AW*-algebras corresponds to the cardinal numbers in $V^{(\mathcal{B})}$".

By the way, when $\mathcal{A}$ is a type I von Neumann algebra, $\mathcal{B}$ satisfies the countable chain condition locally, so that the cardinal numbers in $V^{(\mathcal{B})}$ can be represented by the step functions of the standard cardinal numbers; this fact is known as the absoluteness of the cardinality in $V^{(\mathcal{B})}$ for complete Boolean algebras $\mathcal{B}$ satisfying the countable chain condition [23] (p. 162). In this case, it was already known in the theory of operator algebras that "the step functions of the standard cardinal numbers forms an isomorphic invariant of the type I von Neumann algebras". However, it had been an open problem whether the isomorphic invariants of the type I AW*-algebras can be represented by the step functions of the standard cardinal numbers, since Kaplansky [33] made a negative conjecture in 1952. This conjecture was eventually settled in 1983 by the method of Boolean valued analysis [30].

In fact, the case where the cardinals in $V^{(\mathcal{B})}$ cannot be represented by the step functions of the standard cardinal numbers is known by the forcing method as "cardinal collapsing" [34] (Ch. 5): for any two infinite cardinals α and β in V, we can construct a complete Boolean algebra $\mathcal{B}$ such that $\check{\alpha}$ and $\check{\beta}$ have the same cardinality in $V^{(\mathcal{B})}$. Therefore, when the central projections forms such a complete Boolean algebra, it is derived that the isomorphic invariants of those type I AW*-algebras cannot be represented by the step functions of the standard cardinal numbers. Thus, Kaplansky's conjecture is settled [30,31]. For example, for arbitrary infinite cardinal numbers α and β, we can construct a commutative AW*-algebra $\mathcal{Z}$ such that the two type I AW*-algebras, the $\alpha \times \alpha$ matrix algebra over $\mathcal{Z}$ and $\beta \times \beta$ matrix algebra over $\mathcal{Z}$, are isomorphic [32].

In this way, Takeuti's Boolean valued analysis makes it possible to systematically apply the forcing method to analysis, and it plays a role of a bridge between the two fields by applying the results in the foundations of mathematics to analysis. We refer the reader to [24,35–38] for further developments of Boolean valued analysis in operator algebras.

7. Quantum Set Theory

If $\mathcal{L}$ is a lattice consisting of the projections on a Hilbert space $\mathcal{H}$, then $\mathcal{L}$ is called the standard quantum logic. In general, $\mathcal{L}$ is called orthomodular if $P \leq Q$ implies that there exists a Boolean subalgebra of $\mathcal{L}$ including P and Q. The standard quantum logic is a complete orthomodular lattice, and a complete orthomodular lattice is considered to be a general model of quantum logic.

Takeuti started his research of quantum set theory, while he introduced the universe $V^{(\mathcal{Q})}$ of sets based on the standard quantum logic $\mathcal{Q}$ on a Hilbert space $\mathcal{H}$ in his seminal paper *Quantum Set Theory* [24] published in 1981.

A remarkable fact pointed out by Takeuti regarding quantum set theory is that the reals defined in $V^{(\mathcal{Q})}$ corresponds bijectively to the self-adjoint operators on $\mathcal{H}$, as a straightforward consequence of Boolean valued analysis based on complete Boolean algebras of projections, developed by Takeuti in *Two Applications of Logic to Mathematics* [27]. We refer to the above correspondence between the self-adjoit operators and reals in $V^{(\mathcal{Q})}$ as the Takeuti correspondence.

What formulas hold in $V^{(\mathcal{Q})}$? In order to answer this question, Takeuti introduced the commutator $\perp\!\!\!\perp(S)$ for any subset S of $\mathcal{Q}$ in [24], and used it to define the commutator $\underline{\vee}(u_1, \ldots, u_n)$ of any elements $u_1, \ldots, u_n$ in $V^{(\mathcal{Q})}$. In addition, he showed that, roughly speaking, if one rewrites an axiom of ZFC by replacing $\forall x\phi(x)$ by $\forall x(\underline{\vee}(x) \rightarrow \phi(x))$ and replacing $\exists x$ by $\exists x(\underline{\vee}(x) \wedge \phi(x))$, the modified axiom holds in $V^{(\mathcal{Q})}$. Then how theorems of ZFC hold true in $V^{(\mathcal{Q})}$? Using Takeuti's technique, the present author obtained the following theorem [25,39,40].

Theorem 2 (Quantum Transfer Principle)**.** *If a theorem $\phi(x_1, ..., x_n)$ of ZFC contains only bounded quantifiers, then the relation*

$$[\![\phi(u_1, \ldots, u_n)]\!] \geq [\![\underline{\vee}(u_1, \ldots, u_n)]\!]$$

holds for any $u_1, \ldots, u_n \in V^{(\mathcal{Q})}$.

This theorem implies that any commuting family of quantum sets $u_1, \ldots, u_n$ satisfies ZFC theorems, since in this case, $[\![\underline{\vee}(u_1, \ldots, u_n)]\!] = 1$. The above theorem more generally states that any family of quantum physical quantities $u_1, \ldots, u_n$ satisfies ZFC theorems at least the truth value $[\![\underline{\vee}(u_1, \ldots, u_n)]\!]$.

In quantum mechanics, to every quantum system **S**, there corresponds a Hilbert space $\mathcal{H}$, and the physical quantities of **S** are represented by the self-adjoint operators on $\mathcal{H}$. Accordingly, Q represents the logic of quantum system **S**, and the real numbers in $V^{(\mathcal{Q})}$ represent the physical quantities of **S**. Based on this, all of the observational propositions on a quantum system **S** can be translated into logical formulas in ZFC referring to real numbers in $V^{(\mathcal{Q})}$ by using the Takeuti correspondence [41]. Therefore, the mathematics based on quantum logic is nothing but a mathematics in which the reals are quantum physical quantities.

A method for calculating the probability of an observational proposition ϕ in quantum mechanics has been known as the Born rule. However, the probability of A and B having the same value was not defined in quantum mechanics until recently because the equality relation $A = B$ for two physical quantities A and B is not included in the observational propositions under the known quantum rule. In order to overcome this difficulty, we have shown that by translating an observational proposition ϕ into the corresponding ZFC logical formula $\hat{\phi}$ using the Takeuti correspondence between physical quantities A and reals $\hat{A}$ in $V^{(\mathcal{Q})}$, the probability calculation can be expressed as $\Pr\{\phi \| \psi\} = \|[\![\hat{\phi}]\!]\psi\|^2$ [41]. To any physical quantities A and B, the truth value $[\![\hat{A} = \hat{B}]\!]$ for the corresponding real numbers $\hat{A}$ and $\hat{B}$ in $V^{(\mathcal{Q})}$ is well defined in $V^{(\mathcal{Q})}$, so that by $\Pr\{A = B\|\psi\} = \|[\![\hat{A} = \hat{B}]\!]\psi\|^2$, the probability of two physical quantities A and B having the same value has been newly defined [41]. This theory was used to introduce the most basic condition in quantum measurement theory requiring that "the measured quantity A and the meter quantity B in the measuring instrument match". It has contributed a great deal to reforming the uncertainty principle and other progress in this field [41]. In this way, quantum set theory is a mathematical theory that has the power to extend conventional quantum mechanics, allowing it to be applied to new phenomena.

We have finally reached the starting point for constructing mathematics based on quantum logic; we were able to clarify the equality relation between real numbers, and the study of the order relation between real numbers has just begun [42,43]. Because these relations show observable relationships that hold between the physical quantities of the quantum system, the results of quantum set theory can be immediately experimentally verified. By inheriting and developing such a wonderful heritage of Takeuti's mathematical achievements, it must be possible to greatly advance the elucidation of the interpretational problem of quantum mechanics, which has been regarded as a mystery for many years.

Funding: This research was funded by JSPS KAKENHI, No. 17K19970. The author acknowledges the support of the IRI-NU collaboration.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data sets were created or analyzed in this study.

Conflicts of Interest: The author declares no conflict of interest.

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