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Multiple Solutions for Double Phase Problems with Hardy Type Potential

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Abstract: In this paper, we are concerned with the singular elliptic problems driven by the double phase operator and and Dirichlet boundary conditions. In view of the variational approach, we establish the existence of at least one nontrivial solution and two distinct nontrivial solutions under some general assumptions on the nonlinearity f. Here we use Ricceri's variational principle and Bonanno's three critical points theorem in order to overcome the lack of compactness.

Keywords: double phase operator; singular problem; variational methods

MSC: 35J60; 03H10; 35D05

1. Introduction and Main Results

In the present paper, with the aid of variational methods, we establish the existence and multiplicity results for a class of singular elliptic problems, involving a double phase operator, subject to Dirichlet boundary conditions in a smooth bounded domain in $\mathbb{R}^N$. In the recent years, physical models containing a double phase operator have received extensive attention from scientists which is mainly due to applications as a models for describing a feature of strongly anisotropic materials and new examples of Lavrentiev's phenomenon (e.g., see Refs. [1–4]). A number of important results on the existence and multiplicity of nontrivial solutions for double phase problems, have been proved by Papageorgiou-Radulescu-Repovs [5,6], Perera-Squassina [7], Cencelj-Radulescu-Repovs [8], Radulescu [9], Zhang-Radulescu [10], Ge-Chen [11], Ge-Lv-Lu [12], Liu-Dai [13,14] and Colasuonno-Squassina [15]. For related regularity results dealing with minimizers of variational problems with double phase operator, we refer to the works of Baroni-Colombo-Mingione [16,17], De Filippis-Palatucci [18] and Esposito-Leonetti-Mingione [19].

The aim of this paper is to obtain multiple solutions for the following singular double phase problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \mu \dfrac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $\mu \geq 0$, $\lambda > 0$ are two real numbers, Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) containing the origin and with smooth boundary $\partial\Omega$, $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$, $a : \Omega \to [0, +\infty)$ is Lipschitz continuous and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfying certain growth condition.

In particular, when $\mu = 0$ and $\lambda = 1$, problem $(P_{\lambda,\mu})$ is reduced to the following nonlinear problem

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$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u), & \text{in } \Omega. \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$P_{1,0}$}$$

$(P_{1,0})$ has been studied more intensively in the last five years (see [11,13,14] and references therein), where the nonlinear term f being a Carathéodory function provided with suitable growth properties at zero and infinity, respectively. Using the variational method, the authors in [13] proved the existence and multiplicity of weak solutions of problem $(P_{1,0})$ when the nonlinear term has subcritical growth and satisfies the Ambrosetti-Rabinowitz condition. Then, Liu and Dai in [14] also proved the existence of at least three ground state solutions of $(P_{1,0})$ by applying a strong maximum principle for the double phase operator. Recently, based on a direct sum decomposition of a space, Ge and Chen in [11] proved the existence of infinitely many solutions when the nonlinear term has a $q-1$-superlinear growth at infinity and its primitive can be sign-changing. A similar treatment was recently performed by Hou-Ge-Zhang-Wang [20] via the Nehari manifold method. Replacing the strictly monotonicity condition (which was used in Refs. [13,14] to get ground state solution) by a weak version of Nehari type monotonicity condition, they discussed the existence of one ground state sign-changing solution by using the constraint variational method and quantitative deformation lemma. Following this, Zhang, Ge and Hou [21] established the existence of infinitely many positive solutions for the above problem under certain oscillatory conditions on the nonlinearity f at zero.

For the case when $\mu = 0$ and $\lambda > 0$, Ge, Lv and Lu in [12] obtained the existence of infinitely many solutions under the q-superliner condition and quasimonotonicity condition.

When $\mu \neq 0$, the classical variational approach cannot be applied in our treatment due to the presence of the term $\mu\frac{|u|^{p-2}u}{|x|^p}$. This is because the Hardy inequality only ensures that the embedding of the Sobolev space $W_0^{1,H}(\Omega)$ into the weight Lebesgue space $L^p(\Omega, |x|^{-p})$ is continuous, but not compact. However, problems involving p-Laplacian operators have been discussed in several literatures, we refer to [22–24], in which the authors have used different techniques to prove the existence of solutions for problem $(P_{\lambda,\mu})$ in the case $a(x) \equiv 0$. Motivated by the papers mentioned above, in this work we study the existence of solutions for problem $(P_{\lambda,\mu})$ in which the function f is assumed to be subcritical growth condition. Our situation here is different from [11,13,14] in which the authors considered problem $(P_{\lambda,\mu})$ in the case $\mu = 0$ and f is q-superlinear at infinity.

In the remainder of this article, we shall always make the following assumptions:

(h_1) there exist constants $C > 0$ and $r \in (\vartheta, p^*)$, such that

$$|f(x,t)| \leq C(1 + |t|^{r-1}),\ \forall (x,t) \in \Omega \times \mathbb{R},$$

where $p^* = \frac{Np}{N-p}$ is the critical exponent.

(h_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{|t|^{q-2}t} = 0$ uniformly in $x \in \Omega$.

(h_3) $\lim\limits_{t\to \infty} \frac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly in $x \in \Omega$.

(h_4) It holds that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} F(x,t) > 0,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

The main results of this paper are as follow

Theorem 1. *Assume that $f(x,0) \neq 0$ in Ω, and (h_1) with $\vartheta = 1$ hold. Then, for any $\mu \in [0, \frac{pS_H}{q})$ (S_H is given in (3)), there exists a $\lambda_\mu^* > 0$, such that, for any $\lambda \in (0, \lambda_\mu^*)$, problem $(P_{\lambda,\mu})$ has at least one non-trivial weak solution $u_\lambda \in W_0^{1,H}(\Omega)$. Moreover, $\lim\limits_{\lambda\to 0^+} \|u_\lambda\| = 0$ and the function $\lambda \to \varphi_{\lambda,\mu}(u_\lambda)$ is negative and strictly decreasing in $\lambda \in (0, \lambda_\mu^*)$.*

Theorem 2. *Assume that* (h_1) *with* $\vartheta = q$, (h_2)–(h_4) *hold. Then for any* $\mu \in [0, \frac{pS_H}{q})$ *(*S_H *is given in (3)), there is an open interval* $\Lambda_\mu \subset (0, +\infty)$ *and a real number* $\sigma_\mu > 0$, *such that, for every* $\lambda \in \Lambda_\mu$, *problem* $(P_{\lambda,\mu})$ *has at least two nontrivial weak solutions in* $W_0^{1,H}(\Omega)$ *whose* $W_0^{1,H}(\Omega)$*-norms are less than* σ_μ.

The rest of this paper is organized as follows. In Section 2, we give some notation. We also include some useful results involving the Musielak-Orlicz-Sobolev space $W_0^{1,H}(\Omega)$ in order to facilitate the reading of the paper. In Section 3, we establish the variational framework associated with problem $(P_{\lambda,\mu})$, and we also establish some lemmas that will be used in the proofs of Theorems 1 and 2. We complete the proofs of Theorems 1 and 2 in Sections 4 and 5, respectively.

2. Preliminaries

In order to study problem $(P_{\lambda,\mu})$, we need some basic concepts on space $W_0^{1,H}(\Omega)$ which are called Musielak-Orlicz-Sobolev space. Based on this reason, we first recall some properties on Musielak-Orlicz spaces. A comprehensive presentation of the theory of such spaces can be found in [15,25–27].

Denote by $N(\Omega)$ the set of all generalized N-function. For $1 < p < q$ and $0 \leq a(\cdot) \in L^1(\Omega)$, we define

$$H(x,t) = t^p + a(x)t^q, \ \forall (x,t) \in \Omega \times [0, +\infty).$$

It is already clear that $H \in N(\Omega)$ is a locally integrable and

$$H(x,2t) \leq 2^q H(x,t), \ \forall (x,t) \in \Omega \times [0, +\infty)$$

which is called condition $(\triangle_2)$ (cf. Definition 2.1 of [15]).

The Musielak-Orlicz space $L^H(\Omega)$ is defined by

$$L^H(\Omega) = \Big\{ u | u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega H(x,|u|)dx < +\infty \Big\},$$

with the Luxemburg norm

$$|u|_H = \inf \Big\{ \lambda > 0 : \int_\Omega H(x, |\frac{u}{\lambda}|)dx \leq 1 \Big\}.$$

In addition, we introduce the Musielak-Orlicz-Sobolev space $W^{1,H}(\Omega)$ is defined by

$$W^{1,H}(\Omega) = \{ u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega) \}$$

which is equipped with the Luxemburg $\|u\|$ given by

$$\|u\| = |u|_H + |\nabla u|_H. \tag{1}$$

The space $W_0^{1,H}(\Omega)$ is defined to be the $\|u\|-$closure of the compactly supported elements of $W^{1,H}(\Omega)$, that is,

$$W^{1,H}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,H}(\Omega)}.$$

With these norms, the spaces $L^H(\Omega)$, $W_0^{1,H}(\Omega)$ and $W^{1,H}(\Omega)$ are separable and reflexive Banach spaces; see [15] for the details.

Proposition 1 ([13], Proposition 2.1). *Set* $\rho_H(u) = \int_\Omega (|u|^p + a(x)|u|^q)dx$. *Let* $u \in L^H(\Omega)$, *then the following facts hold:*

(i) if $u \neq 0$, *then* $|u|_H = \lambda$ *if and only if* $\rho_H(\frac{u}{\lambda}) = 1$;
(ii) $|u|_H < 1 (= 1; > 1)$ *if and only if* $\rho_H(u) < 1 (= 1; > 1)$;
(iii) If $|u|_H \geq 1$, *then* $|u|_H^p \leq \rho_H(u) \leq |u|_H^q$;
(iv) If $|u|_H \leq 1$, *then* $|u|_H^q \leq \rho_H(u) \leq |u|_H^p$.

Proposition 2 ([15], Proposition 2.15, Proposition 2.18).

(1) If $1 \leq s \leq p$, *then the embedding* $W_0^{1,H}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ *is continuous.*
(2) If $1 \leq s < p^* = \frac{Np}{N-p}$, *then the embedding* $W_0^{1,H}(\Omega) \hookrightarrow L^s(\Omega)$ *is compact.*
(3) In $W_0^{1,H}(\Omega)$, *the Poincaré-type inequality holds, this means that there exists constant* $C_0 > 0$ *such that*

$$|u|_H \leq C_0 |\nabla u|_H, \ \forall u \in W_0^{1,H}(\Omega).$$

Due to Proposition 2(3), we deduce that there is a constant $c_s > 0$ such that

$$|u|_s \leq c_s \|u\|, \ \forall u \in W_0^{1,H}(\Omega), \tag{2}$$

where we denote by $|u|_s$ the norm in $L^s(\Omega)$ for all $a \in [1, p^*)$. Hence, from Proposition 2(3), we know that $|\nabla u|_H$ and $\|u\|$ are equivalent norms on $W_0^{1,H}(\Omega)$. So, we will use $|\nabla u|_H$ to replace $\|u\|$ in the following discussion and write $\|u\| = |\nabla u|_H$ for simplicity.

Further, we recall Hardy's inequality, which states that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \frac{1}{S_H} \int_\Omega |\nabla u(x)|^p dx, \ \forall u \in W_0^{1,p}(\Omega),$$

where $S_H := \left(\frac{N-p}{p}\right)^p$ (see [28]). By Proposition 2(1), it follows that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \leq \frac{1}{S_H} \int_\Omega (|\nabla u(x)|^p + a(x)|\nabla u(x)|^q)dx, \ \forall u \in W_0^{1,H}(\Omega). \tag{3}$$

From now on, in the paper we denote by E the space $W_0^{1,H}(\Omega)$. In order to study the problem $(P_{\lambda,\mu})$, we consider the function $J : E \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q)dx.$$

It is standard to check that $J \in C^1(E, \mathbb{R})$ and double phase operator $-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$ is the derivative operator of J in the weak sense. Set $L = J'$, then

$$\langle L(u), v\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v)dx$$

for any $u, v \in E$. Her $\langle \cdot, \cdot \rangle$ is the duality pairing between E and its dual space E^*. Then, we have the following important properties.

Proposition 3 ([13], Proposition 3.1). *Let* L *be as above. Then, the following properties hold:*

(1) $L : E \to E^*$ *is a bounded, continuous and strictly monotone operator;*
(2) $L : E \to E^*$ *is a mapping of type* $(S)_+$, *that is, if* $u_n \rightharpoonup u$ *in* E *and* $\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle \leq 0$, *implies* $u_n \to u$ *in* E;
(3) $L : E \to E^*$ *is a homeomorphism.*

3. Variational Setting and Some Preliminary Lemmas

To prove our theorems, we recall the variational setting corresponding to the problem $(P_{\lambda,\mu})$. Now we introduce the Euler Lagrange functional $\varphi_{\lambda,\mu} : E \to \mathbb{R}$ associated with problem $(P_{\lambda,\mu})$ defined by

$$\varphi_{\lambda,\mu}(u) = \Phi_\mu(u) - \lambda\Psi(u), \tag{4}$$

where

$$\Phi_\mu(u) = \int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q)dx - \frac{\mu}{p}\int_\Omega \frac{|u|^p}{|x|^p}dx$$

and $\Psi(u) = \int_\Omega F(x, u(x))dx$. Thus, using Proposition 1 and (3), it is easy to see that, if $\mu \in [0, \frac{pS_H}{q}]$, then

$$\begin{aligned} \frac{pS_H - \mu q}{pqS_H}\|u\|^p \le \Phi_\mu(u) \le \frac{1}{p}\|u\|^q,\ \|u\| \ge 1, \\ \frac{pS_H - \mu q}{pqS_H}\|u\|^q \le \Phi_\mu(u) \le \frac{1}{p}\|u\|^p,\ \|u\| \le 1. \end{aligned} \tag{5}$$

Now, in [13,23], it is shown that $\Phi_\mu(u)$ is a Gâteaux differentiable functional in E, and its Gâteaux derivative is the functional $\Phi_\mu(u) \in E^*$, given by

$$\langle \Phi'_\mu(u), v\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v)dx - \mu\int_\Omega \frac{|u|^{p-2}}{|x|^p}uvdx,$$

for any $v \in E$. Finally, $\Phi_\mu(u)$ is weakly lower semi-continuous and coercive. Moreover, similar to the proof of ([13], proof of Theorem 3.1(ii)), we also can deduce that $\Phi'_\mu : E \to E^*$ is a mapping of type $(S)_+$ for every $\mu \in [0, \frac{pS_H}{q}]$. On the other hand, standard arguments show that Ψ is a well defined and continuously Gâteaux differentiable functional whose Gâteaux derivative

$$\langle \Psi'(u), v\rangle = \int_\Omega f(x, u)vdx,$$

for all $v \in E$.

Definition 1. *Fixing the real parameters μ and λ, we say that $u \in E$ is a weak solution of $(P_{\lambda,\mu})$ if*

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v)dx = \mu\int_\Omega \frac{|u|^{p-2}}{|x|^p}uvdx + \lambda\int_\Omega f(x, u)vdx,$$

for all $v \in E$.

Therefore, the critical points of $\varphi_{\lambda,\mu}$ are exactly the weak solutions of $(P_{\lambda,\mu})$. Next, we give some important lemmas which will play important roles to prove our main results. First of all, let us recall the following the Ricceri's variational principle, which we use in the proof of Theorem 1.

Lemma 1. *Let X be a reflexive real Banach space, and let $G, H : X \to \mathbb{R}$ be two Gâteaux differentiable functionals, such that G is strongly continuous, sequentially weakly lower semi-continuous and coercive. Further, assume that H is sequentially weakly upper semi-continuous. For every $\tau > \inf_{u\in X} G(u)$, put*

$$g(\tau) := \inf_{u\in G^{-1}((-\infty,\tau))} \frac{\sup_{v\in G^{-1}((-\infty,\tau)))} H(v) - H(u)}{\tau - G(u)}.$$

Then, for each $\tau > \inf_{u\in X} G(u)$ *and each* $\lambda \in (0, \frac{1}{g(\tau)})$, *the restriction of* $K_\lambda := G - \lambda H$ *to* $G^{-1}((-\infty, \tau))$ *admits a global minimum, which is a critical point (local minimum) of* K_λ *in* X.

This result is a refinement of the variational principle of Ricceri, see the quoted paper [29]. For the proof of Theorem 2, we need some definitions and results.

Definition 2. *Let* $(X, \|\cdot\|)$ *be a real Banach space,* $\varphi \in C^1(X, \mathbb{R})$. *We say that* $\varphi \in C^1(E, \mathbb{R})$ *satisfies* (PS)*-condition if any sequence* $\{u_n\}_{n\geq 1} \subset X$ *such that*

$$\{\varphi(u_n)\}_{n\geq 1} \text{ is bounded and } \|\varphi'(u_n)\|_{E^*} \to 0 \text{ as } n \to +\infty, \tag{6}$$

has a strongly convergent subsequence.

To prove Theorem 2, we will use the following Bonanno's three critical points theorem.

Lemma 2 ([30], Theorem 2.1)**.** *Let* X *be a separable and reflexive real Banach space, and let* $G, H : X \to \mathbb{R}$ *be two Gâteaux differentiable functionals. Assume that*

(T_1). *There exists* $u_0 \in X$ *such that* $G(u_0) = H(u_0) = 0$ *and* $G(u) \geq 0$ *for every* $u \in X$.

(T_2) *There exists* $u_1 \in X$ *and* $\rho > 0$ *such that*

$$\rho < G(u_1), \quad \sup_{u\in G^{-1}((-\infty,\rho))} H(u) < \rho \frac{H(u_1)}{G(u_1)}.$$

(T_3) *Further, put*

$$\gamma = \frac{\tau\rho}{\rho\frac{H(u_1)}{G(u_1)} - \sup\limits_{G(u)<\rho} H(u)},$$

with $\tau > 1$, *and assume that, for every* $\lambda \in [0, \gamma]$, *the functional* $G - \lambda H$ *is sequentially weakly lower semi-continuous and satisfies the Palais-Smale condition, and*

$$\lim_{\|u\|\to+\infty} (G(u) - \lambda H(u)) = +\infty.$$

Then, there is an open interval $\Lambda \subseteq [0, \gamma]$ *and a number* $\sigma > 0$, *such that for each* $\lambda \in \Lambda$, *the equation* $G'(u) - \lambda H'(u) = 0$ *admits at least three solutions in* X, *having a norm of less than* σ.

4. The Proof of Theorem 1

In this section, we will prove Theorem 1. Firstly, we show that $\varphi_{\lambda,\mu}$ possesses a nontrivial global minimum point in E.

Lemma 3. *For every* $\mu \in [0, \frac{pS_H}{q})$, *the functional* Φ_μ *is coercive on* E.

Proof. Using inequalities (3), we obtain that for any $u \in E$,

$$\begin{aligned}\Phi_\mu(u) &= \int_\Omega (\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q)dx - \frac{\mu}{p}\int_\Omega \frac{|u|^p}{|x|^p}dx \\ &\geq \Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx \\ &\geq \Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\min\{\|u\|^p, \|u\|^q\}.\end{aligned} \tag{7}$$

From this and $0 \leq \mu < \frac{pS_H}{q}$, we conclude that

$$\Phi_\mu(u) \to +\infty, \text{ as } \|u\| \to +\infty,$$

This means that $\Phi_\mu(u)$ is coercive and this ends the proof. □

Lemma 4. *For every $\mu \in [0, \frac{pS_H}{q})$, the functional Φ_μ is sequentially weakly lower semi-continuous on E.*

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence that converges weakly to u in E. In view of (3), we can deduce that $\{\frac{|u_n(x)|}{|x|}\}_{n\geq 1}$ is bounded in $L^p(\Omega)$, so that the sequence has a weak limit, and, since $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^p(\Omega)$, and $u_n(x) \to u(x)$ a.e. in Ω, it holds that

$$\frac{|u_n(x)|}{|x|} \rightharpoonup \frac{|u(x)|}{|x|}, \text{ in } L^p(\Omega).$$

Assume that 0 is the weak limit of the sequence, because if we denote by $v_n = u_n - u$, Brezis-Lieb lemma [31] yields for all $\xi \in C_c(\Omega)$

$$\int_\Omega \frac{|\xi(x)|^p |u_n(x)|^p}{|x|^p} dx - \int_\Omega \frac{|\xi(x)|^p |v_n(x)|^p}{|x|^p} dx \to \int_\Omega \frac{|\xi(x)|^p |u(x)|^p}{|x|^p} dx,$$

where $C_c(\Omega)$ is the space of those functions on $\mathbb{R}$ that are indefinitely differentiable and have compact support contained in Ω.

Note that the sequence $\{u_n\}_{n\geq 1}$ is bounded in E, while the sequence $\{\frac{|u_n(x)|}{|x|}\}_{n\geq 1}$ is bounded in $L^p(\Omega)$, so that the weak* limits of the sequences in the measure space exist. Due to P.L. Lions (see [32,33]), we have

$$\frac{|u_n(x)|^p}{|x|^p} dx \to \nu \text{ and } \left(|\nabla u_n|^p + a(x)|\nabla u_n|^q\right) dx \to \chi, \tag{8}$$

in the $*$-weak convergence of measures.

Given any $\xi \in C_c(\Omega)$, using the functions ξu_n in the Hardy inequality, we have

$$\int_\Omega |\xi(x)|^p \frac{|u_n(x)|^p}{|x|^p} dx \leq \frac{1}{S_H} \int_\Omega |\nabla(\xi u_n)|^p dx. \tag{9}$$

The left-hand side member of (9) goes to $\int_\Omega |\xi(x)|^p d\nu$ as n goes to $+\infty$. On the other hand, the right-hand side member is estimated as follows

$$\left| \left(\int_\Omega |\nabla(\xi u_n)|^p dx \right)^{\frac{1}{p}} - \left(\int_\Omega |\xi \nabla(u_n)|^p dx \right)^{\frac{1}{p}} \right| \leq \left(\int_\Omega |u_n \nabla(\xi)|^p dx \right)^{\frac{1}{p}}.$$

Using the fact that ξ has compact support, and the Rellich theorem, we see that this bound goes to 0 as n goes to $+\infty$. Hence, passing to the limit in (9), we find for all $\xi \in C_c(\Omega)$

$$\int_\Omega |\xi(x)|^p d\nu \leq \frac{1}{S_H} \int_\Omega |\xi(x)|^p d\chi. \tag{10}$$

Choosing ξ, such that $0 \notin \text{supp}(\xi)$, we have that

$$\begin{aligned} \int_\Omega \frac{|\xi(x)|^p |u_n(x)|^p}{|x|^p} dx &= \int_{\text{supp}(\xi)} \frac{|\xi(x)|^p |u_n(x)|^p}{|x|^p} dx \\ &\leq C \int_{\text{supp}(\xi)} |u_n(x)|^p dx \to 0, \end{aligned}$$

since the function $\{\frac{|\xi(x)|}{|x|}\}$ belongs to $L^p(\mathrm{supp}(\xi))$, $u_n \rightharpoonup 0$ in E and the embedding $E \hookrightarrow L^p(\Omega)$ is compact. The above information implies that ν is a measure concentrated in 0 and is absolutely continuous with respect to a Dirac mass (since χ contains Dirac masses). Hence, it holds that

$$\begin{aligned} \nu &= \frac{|u_n(x)|^p}{p|x|^p}dx + \nu_0\delta_0, \\ \chi &\geq \Big(\frac{|\nabla u_n|^p}{p} + a(x)\frac{|\nabla u_n|^q}{q}\Big)dx + \chi_0\delta_0, \end{aligned} \tag{11}$$

Fixing a function $\xi \in C_c(\Omega)$, such that

$$\xi \simeq I_{B\varepsilon(0)} := \begin{cases} 1, & x \in B\varepsilon(0), \\ 0, & x \notin B\varepsilon(0). \end{cases}$$

Then, by (10), one easily deduces

$$\nu(B\varepsilon(0)) \leq \frac{1}{S_H}\chi(B\varepsilon(0)).$$

Since ε is arbitrary, we have $0 \leq \nu_0 \leq \frac{\chi_0}{S_H}$. Then from (11), we deduce that

$$\begin{aligned} \liminf_{n\to+\infty} \Phi_\mu(u_n) =& \liminf_{n\to+\infty} \Big[\frac{1}{q}\int_\Omega \big(|\nabla u_n|^p + a(x)|\nabla u_n|^q\big)dx \\ &+ (\frac{1}{p} - \frac{1}{q})\int_\Omega |\nabla u_n|^p dx - \frac{\mu}{p}\int_\Omega \frac{|u_n|^p}{|x|^p}dx\Big] \\ \geq& \liminf_{n\to+\infty} \Big[\frac{1}{q}\int_\Omega \big(|\nabla u_n|^p + a(x)|\nabla u_n|^q\big)dx \\ &- \frac{\mu}{p}\int_\Omega \frac{|u_n|^p}{|x|^p}dx\Big] + (\frac{1}{p} - \frac{1}{q})\int_\Omega |\nabla u|^p dx \\ =& \frac{1}{q}\Big(\int_\Omega \big(|\nabla u|^p + a(x)|\nabla u|^q\big)dx + \chi_0 + \text{nonnegative terms}\Big) \\ &- \frac{\mu}{p}\Big(\int_\Omega \frac{|u|^p}{|x|^p}dx + \nu_0\Big) + (\frac{1}{p} - \frac{1}{q})\int_\Omega |\nabla u|^p dx \\ \geq& \Phi_\mu(u_n) + \frac{\chi_0}{q} - \frac{\mu}{p}\nu_0 \\ \geq& \Phi_\mu(u_n) + \chi_0(\frac{1}{q} - \frac{\mu}{pS_H}), \end{aligned}$$

and this ends the proof. □

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. Let $X = E$, $G = \Phi_\mu$ and $H = \Psi$. Fix $\mu \in [0, \frac{pS_H}{q})$. Clearly, $\inf_{u\in E} \Phi_\mu(u) = 0$, and $\Phi_\mu, \Psi : E \to \mathbb{R}$ are continuously Gâteaux differentiable. In view of Lemmas 3 and 4, we can deduce that the functional Φ_μ is coercive and weakly lower semi-continuous on E.

Taking into account of (h_1), we obtain

$$F(x,t) \leq C(|t| + |t|^r), \ \forall (x,t) \in \Omega \times \mathbb{R}. \tag{12}$$

Let

$$\lambda_\mu^* := \frac{1}{C}\sup_{\tau>0}\Big(\frac{\tau^{\frac{q-1}{q}}}{c_1\big[\Theta^{\frac{1}{p}-\frac{1}{q}}+\Theta^{\frac{1}{q}}\big]+2^r c_r^r\big[\Theta^{\frac{r}{p}}\tau^{\frac{r}{p}-\frac{1}{q}}+\Theta^{\frac{r}{q}}\tau^{\frac{r-1}{q}}\big]}\Big),$$

where $\Theta := \frac{pqS_H}{pS_H-\mu q}$ and c_1, c_r are defined in (2). Then for each $\lambda \in (0, \lambda_\mu^*)$, there exists $\tau_0 > 0$ such that

$$\lambda < \lambda_\mu^*(\tau_0) := \frac{\tau_0^{\frac{q-1}{q}}}{Cc_1\big[\Theta^{\frac{1}{p}-\frac{1}{q}}+\Theta^{\frac{1}{q}}\big]+C2^r c_r^r\big[\Theta^{\frac{r}{p}}\tau_0^{\frac{r}{p}-\frac{1}{q}}+\Theta^{\frac{r}{q}}\tau_0^{\frac{r-1}{q}}\big]}. \tag{13}$$

Now, let us introduce the functional $h : (0, +\infty) \to \mathbb{R}$, defined by

$$h(\tau) := \frac{\sup\limits_{u\in\Phi_\mu^{-1}((-\infty,\tau)))}\Psi(u)}{\tau}.$$

Combining (2) and (12), we conclude that

$$\begin{aligned}\Psi(u) = \int_\Omega F(x,u)dx \le C(|u|_1 + |u|_r^r)\\ \le C(c_1\|u\| + c_r^r\|u\|^r).\end{aligned} \tag{14}$$

On the other hand, we deduce from (5) that

$$\|u\| \le \begin{cases}\Theta^{\frac{1}{p}}\tau^{\frac{1}{p}}, & \|u\| \ge 1,\\ \Theta^{\frac{1}{q}}\tau^{\frac{1}{q}}, & \|u\| \le 1,\end{cases}$$

for every $u \in E$ and $\Phi_\mu(u) < \tau$. Thus, for all $u \in E$ with $\Phi_\mu(u) < \tau$, one has

$$\|u\| \le \Theta^{\frac{1}{p}}\tau^{\frac{1}{p}} + \Theta^{\frac{1}{q}}\tau^{\frac{1}{q}}. \tag{15}$$

Combining the above inequality with (14) we get

$$\sup_{u\in\Phi_\mu^{-1}((-\infty,\tau))}\Psi(u) \le Cc_1(\Theta^{\frac{1}{p}}\tau^{\frac{1}{p}} + \Theta^{\frac{1}{q}}\tau^{\frac{1}{q}}) + 2^r Cc_r^r(\Theta^{\frac{r}{p}}\tau^{\frac{r}{p}} + \Theta^{\frac{r}{q}}\tau^{\frac{r}{q}}),$$

which implies that

$$h(\tau) \le Cc_1(\Theta^{\frac{1}{p}}\tau^{\frac{1}{p}-1} + \Theta^{\frac{1}{q}}\tau^{\frac{1}{q}}) + 2^r Cc_r^r(\Theta^{\frac{r}{p}}\tau^{\frac{r}{p}-1} + \Theta^{\frac{r}{q}}\tau^{\frac{r}{q}-1}),$$

for every $\tau \in (0, +\infty)$. Then, we have that

$$h(\tau_0) \le Cc_1(\Theta^{\frac{1}{p}}\tau_0^{\frac{1}{p}-1} + \Theta^{\frac{1}{q}}\tau_0^{\frac{1}{q}}) + 2^r Cc_r^r(\Theta^{\frac{r}{p}}\tau_0^{\frac{r}{p}-1} + \Theta^{\frac{r}{q}}\tau_0^{\frac{r}{q}-1}), \tag{16}$$

Recalling that $\Phi_\mu(0) = \Psi(0) = 0$ and $0 \in \Phi_\mu^{-1}((-\infty, \gamma_0))$, it holds that

$$\begin{aligned} g(\tau_0) :=& \inf_{u \in \Phi_\mu^{-1}((-\infty,\tau_0))} \frac{\sup_{v \in \Phi_\mu^{-1}((-\infty,\tau_0)))} \Psi(v) - \Psi(u)}{\tau - \Phi_\mu(u)} \\ \leq& \frac{\sup_{v \in \Phi_\mu^{-1}((-\infty,\tau_0)))} \Psi(v)}{\tau} \\ =& h(\tau_0). \end{aligned} \tag{17}$$

Taking into account (13), (16) and (17), we deduce that

$$\begin{aligned} g(\tau_0) \leq& h(\tau_0) \leq Cc_1(\Theta^{\frac{1}{p}} \tau_0^{\frac{1}{p}-1} + \Theta^{\frac{1}{q}} \tau_0^{\frac{1}{q}}) + 2^r Cc_r^r(\Theta^{\frac{r}{p}} \tau_0^{\frac{r}{p}-1} + \Theta^{\frac{r}{q}} \tau_0^{\frac{r}{q}-1}) \\ =& \frac{Cc_1\left[\Theta^{\frac{1}{p}-\frac{1}{q}} + \Theta^{\frac{1}{q}}\right] + C2^r c_r^r\left[\Theta^{\frac{r}{p}} \tau_0^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}} \tau_0^{\frac{r-1}{q}}\right]}{\tau_0^{\frac{q-1}{q}}} \\ <& \frac{1}{\lambda}. \end{aligned}$$

From this we conclude that

$$\lambda \in (0, \lambda_\mu^*(\tau_0)) \subseteq (0, \frac{1}{g(\tau_0)}).$$

These above facts enable us to apply Lemma 1 in order to find that there exists a $u_\lambda \in \Phi_\mu^{-1}((-\infty, \tau_0))$, a global minimizer of the restriction of $\varphi_{\lambda,\mu}$ to $\Phi_\mu^{-1}((-\infty, \tau_0))$, such that

$$\varphi'_{\lambda,\mu}(u_\lambda) = \Phi'_\mu(u_\lambda) - \lambda \Psi'_\mu(u_\lambda) = 0.$$

Moreover, by $f(x, 0) \neq 0$ in Ω, it follows that u_λ is not trivial, that is, $u_\lambda \neq 0$. Therefore, for any $\mu \in [0, \frac{pS_H}{q})$, there exists a $\lambda_\mu^* > 0$ such that for any $\lambda \in (0, \lambda_\mu^*)$ problem $(P_{\mu,\lambda})$ admits at least one non-trivial weak solution $u_\lambda \in E$.

Our next goal is to prove that,

(A_1) $\|u_\lambda\| \to 0$ as $\lambda \to 0^+$;

(A_2) the function $\lambda \to \varphi_{\lambda,\mu}(u_\lambda)$ is negative and strictly decreasing in $(0, \lambda_H^*)$.

Let us first examine the fact (A_1). To this end, fix $\mu \in [0, \frac{pS_H}{q})$ and let us consider $\overline{\lambda} \in (0, \lambda_\mu^*)$. Arguing as before, let $\overline{\tau} > 0$, such that

$$\overline{\lambda} < \lambda_\mu^*(\overline{\tau}) := \frac{\overline{\tau}^{\frac{q-1}{q}}}{Cc_1\left[\Theta^{\frac{1}{p}-\frac{1}{q}} + \Theta^{\frac{1}{q}}\right] + C2^r c_r^r\left[\Theta^{\frac{r}{p}} \overline{\tau}^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}} \overline{\tau}^{\frac{r-1}{q}}\right]}.$$

Then, for every $\lambda \in (0, \lambda_\mu^*(\overline{\tau}))$, we deduce that the functional $\varphi_{\lambda,\mu}$ has a nontrivial critical point $u_\lambda \in \Phi_\mu^{-1}((-\infty, \overline{\tau}))$. Note that the functional Φ_μ is coercive and $u_\lambda \in \Phi_\mu^{-1}((-\infty, \overline{\tau}))$ for every $\lambda \in (0, \lambda_\mu^*(\overline{\tau}))$. Hence, we obtain that there exists a $C_0 > 0$, such that

$$\|u_\lambda\| \leq C_0, \ \forall \lambda \in (0, \lambda_\mu^*(\overline{\tau})).$$

Recall that operator Ψ' is compact. So, there exists constant $M > 0$, such that

$$|\langle \Psi'(u_\lambda), u_\lambda \rangle| \leq \|\Psi'(u_\lambda)\|_{E^*} \|u_\lambda\| < MC_0^2, \ \forall \lambda \in (0, \lambda_\mu^*(\overline{\tau})). \tag{18}$$

Moreover, since $\varphi'_{\lambda,\mu}(u_\lambda) = 0$ for every $\lambda \in (0, \lambda^*_\mu(\overline{\tau}))$, and so $\langle \Psi'(u_\lambda), u_\lambda \rangle = 0$, this means that

$$\langle \Phi'_\mu(u_\lambda), u_\lambda \rangle = \lambda \int_\Omega f(x, u_\lambda) u_\lambda dx, \ \forall \lambda \in (0, \lambda^*_\mu(\overline{\tau})). \tag{19}$$

Hence, from (18) and (20), it follows that

$$\lim_{\lambda \to 0^+} \langle \Phi'_\mu(u_\lambda), u_\lambda \rangle = 0. \tag{20}$$

Moreover, it is easy to compute directly that

$$\begin{aligned} \langle \Phi'_\mu(u_\lambda), u_\lambda \rangle &= \int_\Omega (|\nabla u_\lambda|^p + a(x)|\nabla u_\lambda|^q) dx - \mu \int_\Omega \frac{|u_\lambda(x)|^p}{|x|^p} dx \\ &\geq (1 - \frac{\mu}{S_H}) \int_\Omega (|\nabla u_\lambda|^p + a(x)|\nabla u_\lambda|^q) dx \\ &> \frac{q-p}{q} \int_\Omega (|\nabla u_\lambda|^p + a(x)|\nabla u_\lambda|^q) dx \\ &\geq \begin{cases} \frac{q-p}{q} \|u_\lambda\|^p, & \|u_\lambda\| \geq 1, \\ \frac{q-p}{q} \|u_\lambda\|^q, & \|u_\lambda\| \leq 1, \end{cases} \end{aligned} \tag{21}$$

for all $\lambda \in (0, \lambda^*_\mu(\overline{\tau}))$. Consequently, (20) and (21) yield

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0.$$

Now, it remains to prove (A_2). Recall that $\varphi_{\lambda,\mu}(0) = 0$ and the restriction of the functional $\varphi_{\lambda,\mu}$ to $\Phi_\mu^{-1}((-\infty, \overline{\tau}))$ admits a global minimum, which is a critical point (local minimum) of $\varphi_{\lambda,\mu}$ in E. The above information implies that the map $\lambda \to \varphi_{\lambda,\mu}(u_\lambda)$ is negative in $\lambda \in (0, \lambda^*_\mu(\overline{\tau}))$.

Finally, we prove that $\varphi_{\lambda,\mu}$ is strictly decreasing in $(0, \lambda^*_\mu(\overline{\tau}))$. We observe that

$$\varphi_{\lambda,\mu}(u) = \lambda \Big(\frac{\Phi_\mu(u)}{\lambda} - \Psi(u) \Big),$$

for each $u \in E$. Fix $0 < \lambda_1 < \lambda_2 < \lambda^*_\mu(\overline{\tau})$ and consider

$$m_{\lambda_i} := \inf_{u \in \Phi_\mu^{-1}((-\infty, \overline{\tau}))} \Big(\frac{\Phi_\mu(u)}{\lambda_i} - \Psi(u) \Big), \ i = 1, 2.$$

Arguing as above, we see that there exist $u_{\lambda_1}, u_{\lambda_2} \in E$ such that

$$m_{\lambda_1} = \frac{\Phi_\mu(u_{\lambda_1})}{\lambda_1} - \Psi(u_{\lambda_1}) \text{ and } m_{\lambda_2} = \frac{\Phi_\mu(u_{\lambda_2})}{\lambda_2} - \Psi(u_{\lambda_2}).$$

It is obvious that $m_{\lambda_2} \leq m_{\lambda_1} < 0$, so it follows that

$$\varphi_{\lambda_2,\mu}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = \varphi_{\lambda_1,\mu}(u_{\lambda_1}).$$

From this, we conclude that the map $\lambda \to \varphi_{\lambda,\mu}(u_\lambda)$ is strictly decreasing in $(0, \lambda^*_\mu(\overline{\tau}))$. By the arbitrariness of $\overline{\lambda} \in (0, \lambda^*_\mu)$, it follows that the above conclusions are still true in $(0, \lambda^*_\mu)$. This completes the proof of the theorem. □

Remark 1. *It is important to point out that accurate estimation of parameters λ_μ^* are very important in Theorem 1. In order to obtain such an estimation, let us fix $\mu \in [0, \frac{pS_H}{q})$. It is easy to compute directly that*

$$\lambda_\mu^* = \begin{cases} +\infty, & 1 < r < p, \\ \dfrac{1}{2^p c_p^p \Theta C}, & r = p, \\ \dfrac{\tau_1^{\frac{q-1}{q}}}{Cc_1\left[\Theta^{\frac{1}{p}-\frac{1}{q}} + \Theta^{\frac{1}{q}}\right] + C2^r c_r^r\left[\Theta^{\frac{r}{p}}\tau_1^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}}\tau_1^{\frac{r-1}{q}}\right]}, & p < r < q, \\ \dfrac{\tau_2^{\frac{q-1}{q}}}{Cc_1\left[\Theta^{\frac{1}{p}-\frac{1}{q}} + \Theta^{\frac{1}{q}}\right] + C2^r c_r^r\left[\Theta^{\frac{r}{p}}\tau_2^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}}\tau_2^{\frac{r-1}{q}}\right]}, & r = q, \\ \dfrac{\tau_3^{\frac{q-1}{q}}}{Cc_1\left[\Theta^{\frac{1}{p}-\frac{1}{q}} + \Theta^{\frac{1}{q}}\right] + C2^r c_r^r\left[\Theta^{\frac{r}{p}}\tau_3^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}}\tau_3^{\frac{r-1}{q}}\right]}, & q < r < p^*, \end{cases}$$

where τ_1, τ_2, τ_3 are positive constants and satisfy the following equations

$$\Theta^{\frac{r}{q}-\frac{r}{p}}(1-\frac{r}{q})\tau_1^{\frac{r-1}{q}} + \frac{c_1(q-1)(\Theta^{\frac{1}{p}-\frac{1}{q}}+\Theta^{\frac{1}{q}})}{q2^r c_r^r}\Theta^{\frac{r}{p}} = (\frac{r}{p}-1)\tau_1^{\frac{r}{p}-\frac{1}{q}}, \text{ if } p < r < q;$$

$$\tau_2 = \Big[\frac{c_1 p(q-1)(\Theta^{\frac{1}{p}-\frac{1}{q}}+\Theta^{\frac{1}{q}})}{q(q-p)2^q c_q^q \Theta^{\frac{q}{p}}}\Big];$$

$$(\frac{r}{p}-1)\tau_3^{\frac{r}{p}-\frac{1}{q}} + \Theta^{\frac{r}{q}-\frac{r}{p}}(\frac{r}{q}-1)\tau_3^{\frac{r-1}{q}} = \frac{c_1(q-1)(\Theta^{\frac{1}{p}-\frac{1}{q}}+\Theta^{\frac{1}{q}})}{q2^r c_r^r}\Theta^{\frac{r}{p}}, \text{ if } q < r < p^*.$$

Remark 2. *We observe that, if f is a $(r-1)$-sublinear growth at infinity, with $r \in (1, p)$, then for all $\lambda > 0$, Theorem 1 shows that problem $(P_{\lambda,\mu})$ admits at least one nonzero solution. We explicitly observe that, in this case, the existence of at least one nontrivial solution can be obtained by classical direct methods.*

5. The Proof of Theorem 2

In the section, we use Lemma 2 to prove Theorem 2, so we will first prove these lemmas.

Lemma 5. *Suppose that the assumption (h_1) is satisfied. Then, for any $\mu \in [0, \frac{pS_H}{q})$ and $\lambda \in \mathbb{R}$, $\varphi_{\lambda,\mu}$ is sequentially weakly lower semi-continuous on E.*

Proof. Recall that the embedding $W_0^{1,H}(\Omega) \hookrightarrow L^r(\Omega)$ is compact. So, from (h_1) it follows that Ψ has sequentially weak continuity. As a result of Lemma 4, we obtain that Φ_μ is sequentially weakly lower semi-continuous for every $\mu \in [0, \frac{pS_H}{q})$. Hence, $\varphi_{\lambda,\mu}$ is sequentially weakly lower semi-continuous on E. This completes the proof. □

Lemma 6. *Suppose that the assumption (h_3) is satisfied. Then, for every $\mu \in [0, \frac{pS_H}{q})$ and $\lambda \in \mathbb{R}$, the functional $\varphi_{\lambda,\mu}$ is coercive and satisfies the (PS) condition.*

Proof. Let us fix $\mu \in [0, \frac{pS_H}{q})$ and $\lambda \in \mathbb{R}$ arbitrarily. In view of assumption (h_3), there exists $\delta = \delta(\mu, \lambda) > 0$ such that

$$|f(x,t)| < \frac{1}{c_p^p}\Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\frac{1}{1+|\lambda|}|t|^{p-1}, \tag{22}$$

for every $x \in \Omega$ and $|t| > \delta$.

Using assumption (h_1) and (22), we deduce that

$$\begin{aligned} |F(x,t)| \leq & \frac{1}{c_p^p}\Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\frac{1}{1+|\lambda|}|t|^p + \max_{|s|\leq\delta}|f(x,s)||t| \\ \leq & \frac{1}{c_p^p}\Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\frac{1}{1+|\lambda|}|t|^{p-1} + C(1+\delta^{r-1})|t|, \end{aligned} \tag{23}$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Thus, by (2), (3) and (22), it follows that for every $u \in E$ with $\|u\| > 1$, we have

$$\begin{aligned} \varphi_{\lambda,\mu}(u) = & \int_\Omega (\frac{1}{p}|\nabla u(x)|^p + \frac{a(x)}{q}|\nabla u(x)|^q)dx - \frac{\mu}{p}\int_\Omega \frac{|u(x)|^p}{|x|^p}dx \\ & - \lambda \int_\Omega F(x,u(x))dx \\ \geq & \Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\|u\|^p - |\lambda| \int_\Omega |F(x,u(x))|dx \\ \geq & \Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\|u\|^p - \frac{1}{c_p^p}\Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\frac{|\lambda|}{1+|\lambda|}\int_\Omega |u(x)|^p dx \\ & - C|\lambda|(1+\delta^{r-1})\int_\Omega |u(x)|dx \\ \geq & \Big(\frac{1}{q} - \frac{\mu}{pS_H}\Big)\frac{1}{1+|\lambda|}\|u\|^p - C|\lambda|(1+\delta^{r-1})c_1\|u\|. \end{aligned} \tag{24}$$

From this we conclude that

$$\varphi_{\lambda,\mu}(u) \to +\infty \text{ as } \|u\| \to +\infty.$$

Therefore, $\varphi_{\lambda,\mu}$ is coercive.

We now turn to proving that $\varphi_{\lambda,\mu}$ satisfies the (PS) condition. Let $\{u_n\}_{n\geq 1} \subseteq E$ be a sequence such that

$$\{\varphi_{\lambda,\mu}(u_n)\}_{n\geq 1} \text{ is bounded and } \varphi'_{\lambda,\mu}(u_n) \to 0 \text{ as } n \to +\infty. \tag{25}$$

Recalling that $\varphi_{\lambda,\mu}$ is coercive, we deduce that the sequence $\{u_n\}_{n\geq 1}$ is bounded. Then, by the reflexivity of E, there exists $u_0 \in E$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in E, noticing that

$$\begin{aligned} |\langle \varphi'_{\lambda,\mu}(u_n), u_n - u_0\rangle| \leq & |\langle \varphi'_{\lambda,\mu}(u_n), u_n\rangle| + |\langle \varphi'_{\lambda,\mu}(u_n), u_0\rangle| \\ \leq & \|\varphi'_{\lambda,\mu}(u_n)\|\|u_n\| + \|\varphi'_{\lambda,\mu}(u_n)\|\|u_0\|. \end{aligned} \tag{26}$$

So, from (25), (26) and the fact that $\{u_n\}$ is bounded in E, we deduce that

$$\lim_{n\to+\infty}\langle \varphi'_{\lambda,\mu}(u_n) - \varphi'_{\lambda,\mu}(u_0), u_n - u_0\rangle = 0. \tag{27}$$

Next, we shall prove that

$$\lim_{n\to+\infty}\langle\Phi'_\mu(u_n)-\Phi'_\mu(u_0),u_n-u_0\rangle=0. \tag{28}$$

Indeed, combining assumption (h_1) and Proposition 2(2), we calculate

$$\begin{aligned}&\int_\Omega|f(x,u_n)||u_n-u_0|dx\\ \leq& C\int_\Omega(1+|u_n|^{r-1})|u_n-u_0|dx\\ =&C|u_n-u_0|_1+C|u_n|_r^{r-1}|u_n-u_0|_r\\ \to&0,\ \text{as } n\to\infty,\end{aligned} \tag{29}$$

and

$$\begin{aligned}&\int_\Omega|f(x,u_0)||u_n-u_0|dx\\ \leq& C\int_\Omega(1+|u_0|^{r-1})|u_n-u_0|dx\\ =&C|u_n-u_0|_1+C|u_0|_r^{r-1}|u_n-u_0|_r\\ \to&0,\ \text{as } n\to\infty.\end{aligned} \tag{30}$$

Noting that

$$\begin{aligned}&\langle\Phi'_\mu(u_n)-\Phi'_\mu(u_0),u_n-u_0\rangle\\ =&\langle\varphi'_{\lambda,\mu}(u_n)-\varphi'_{\lambda,\mu}(u_0),u_n-u_0\rangle\\ &+\lambda\int_\Omega(f(x,u_n)-f(x,u_0))(u_n-u_0)dx.\end{aligned} \tag{31}$$

Thus, in view of (27), (29)–(31), we see that (28) holds. Finally, since Φ'_μ is of type $(S)_+$, we obtain $u_n\to u_0$ in E. The proof of Lemma 6 is complete. □

Lemma 7. *Suppose that the assumptions (h_1)–(h_2) and (h_4) are satisfied. Then, for every $\mu\in[0,\frac{pS_H}{q})$, the following limit holds*

$$\lim_{\rho\to0^+}\frac{\sup\limits_{u\in\Phi_\mu^{-1}((-\infty,\rho))}\Psi(u)}{\rho}=0.$$

Proof. Fix $\mu\in[0,\frac{pS_H}{q})$. In view of conditions (h_2), for an arbitrarily small $\varepsilon>0$, there is $\delta=\delta(\varepsilon)>0$ such that

$$|f(x,t)|\leq\frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\frac{1}{c_q^q}|t|^{q-1},\ \forall x\in\Omega,\ |t|<\delta. \tag{32}$$

Additionally, by (h_1), for all $x\in\Omega$ and $|t|\geq\delta$, we obtain

$$\begin{aligned}|f(x,t)|\leq&C(1+|t|^{r-1})\\ \leq&C(\frac{|t|}{\delta}+|t|^{r-1})\\ \leq&C\Big((\frac{|t|}{\delta})^{r-1}+|t|^{r-1}\Big)\\ =&C(1+\delta^{1-r})|t|^{r-1}.\end{aligned} \tag{33}$$

Combining (32) and (33), we obtain

$$|f(x,t)| \leq \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\frac{1}{c_q^q}|t|^{q-1}+C(1+\delta^{1-r})|t|^{r-1},\ \forall (x,t)\in\Omega\times\mathbb{R}.$$

After integration, we obtain

$$|F(x,t)| \leq \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\frac{1}{c_q^q}|t|^{q}+C(1+\delta^{1-r})|t|^{r}, \tag{34}$$

for all $(x,t)\in\Omega\times\mathbb{R}$.

For $\rho>0$, we define the following sets

$$\begin{aligned} A_\rho &= \{u\in E:\Phi_\mu(u)<\rho\},\\ B_\rho &= \{u\in E:\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\int_\Omega(|\nabla u(x)|^p+a(x)|\nabla u(x)|^q)dx<\rho\}. \end{aligned} \tag{35}$$

Owing to (3), it is clear that $A_\rho\subseteq B_\rho$. Moreover, by using (34), (35) and (2), for every $u\in B_\rho$ we have

$$\begin{aligned} \Psi(u) \leq& \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\frac{1}{c_q^q}\int_\Omega|u|^q dx+C(1+\delta^{1-r})\int_\Omega|u|^r dx\\ \leq& \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\|u\|^q+C(1+\delta^{1-r})c_r^r\|u\|^r\\ \leq& \begin{cases} \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\Big(\int_\Omega(|\nabla u(x)|^p+a(x)|\nabla u(x)|^q)dx\Big)^{\frac{q}{p}}\\ +C(1+\delta^{1-r})c_r^r\Big(\int_\Omega(|\nabla u(x)|^p+a(x)|\nabla u(x)|^q)dx\Big)^{\frac{r}{p}}, & \|u\|\geq 1,\\ \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\Big(\int_\Omega(|\nabla u(x)|^p+a(x)|\nabla u(x)|^q)dx\Big)\\ +C(1+\delta^{1-r})c_r^r\Big(\int_\Omega(|\nabla u(x)|^p+a(x)|\nabla u(x)|^q)dx\Big)^{\frac{r}{q}}, & \|u\|\leq 1, \end{cases}\\ \leq& \begin{cases} \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{1-\frac{q}{p}}\rho^{\frac{q}{p}}\\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{p}}\rho^{\frac{r}{p}}, & \|u\|\geq 1,\\ \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)\rho\\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{q}}\rho^{\frac{r}{q}}, & \|u\|\leq 1. \end{cases} \end{aligned} \tag{36}$$

Set

$$\rho(\varepsilon)=\min\Big\{1,[k(\varepsilon)]^{\frac{p}{r-p}}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{\frac{r+p-q}{r-p}},[k(\varepsilon)]^{\frac{q}{r-q}}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{\frac{r+q}{r-q}}\Big\},$$

where $k(\varepsilon) = \frac{\varepsilon}{2C(1+\delta^{1-r})c_r^r}$. Then for every $0 < \rho < \rho(\varepsilon)$, we deduce

$$
\begin{aligned}
0 \leq & \frac{\sup\limits_{u\in A_\rho}\Psi(u)}{\rho} \leq \frac{\sup\limits_{u\in B_\rho}\Psi(u)}{\rho} \\
\leq & \begin{cases} \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{1-\frac{q}{p}}\rho^{\frac{q}{p}-1} \\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{p}}\rho^{\frac{r}{p}-1}, & \|u\|\geq 1, \\ \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big) \\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{q}}\rho^{\frac{r}{q}-1}, & \|u\|\leq 1. \end{cases} \\
\leq & \begin{cases} \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{1-\frac{q}{p}} \\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{p}}\rho^{\frac{r}{p}-1}, & \|u\|\geq 1, \\ \frac{\varepsilon}{2}\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big) \\ +C(1+\delta^{1-r})c_r^r\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{-\frac{r}{q}}\rho^{\frac{r}{q}-1}, & \|u\|\leq 1. \end{cases} \\
\leq & \begin{cases} \varepsilon\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big)^{1-\frac{q}{p}}, & \|u\|\geq 1, \\ \varepsilon\Big(\frac{1}{q}-\frac{\mu}{pS_H}\Big), & \|u\|\leq 1. \end{cases}
\end{aligned} \tag{37}
$$

Let $\varepsilon \to 0$ in (37), and so the proof of lemma 7 is completed. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Let $X = E$, $G = \Phi_\mu$ and $H = \Psi$. Fix $\mu \in [0, \frac{pS_H}{q})$. Clearly, $\Phi_\mu(0) = \Psi(0) = 0$, and $\Phi_\mu(u) \geq 0$ for all $u \in E$. So, the condition (T_1) of Lemma 2 are verified.

In the following, we verify the condition (T_2). Due to (h_4), there exists a $s_0 \in \mathbb{R}$ such that $F(x, s_0) > 0$ for all $x \in \Omega$. Additionally, choose $R_0 > 0$ in such a way that $R_0 < \mathrm{dist}(0, \partial\Omega)$. For $\tau \in (0, 1)$ define

$$
\eta_\tau(x) = \begin{cases} 0, & x \in \mathbb{R}^N \backslash B_{R_0}(0), \\ s_0, & x \in B_{\tau R_0}(0), \\ \frac{s_0}{R_0(1-\tau)}(R_0 - |x|), & x \in B_{R_0}(0)\backslash B_{\tau R_0}(0), \end{cases}
$$

where $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$. Moreover, denoting by w_N the volume of the N-dimensional unit ball, one has

$$
\begin{aligned}
& \int_\Omega (|\nabla\eta_\tau(x)|^p + a(x)|\nabla\eta_\tau(x)|^q)dx \\
= & \int_{B_{R_0}(0)\backslash B_{\tau R_0}(0)} (|\nabla\eta_\tau(x)|^p + a(x)|\nabla\eta_\tau(x)|^q)dx \\
= & N w_N R_0^N (1-\tau^N)\Big[\Big(\frac{s_0}{R_0(1-\tau)}\Big)^p + |a|_1\Big(\frac{s_0}{R_0(1-\tau)}\Big)^q\Big].
\end{aligned}
$$

Owing to assumption (h_4), we deduce that

$$\begin{aligned}\Psi(\eta_\tau) &= \int_\Omega F(x,\eta_\tau(x))dx \\ &= \int_{B_{\tau R_0}(0)} F(x,\eta_\tau(x))dx + \int_{B_{R_0}(0)\setminus B_{\tau R_0}(0)} F(x,\eta_\tau(x))dx \\ &\geq F(x,s_0)w_N R_0^N \tau^N - w_N R_0^N(1-\tau^N)\max_{|t|\leq|s_0|}|F(x,t)| \\ &= w_N R_0^N\Big[\tau^N F(x,s_0) - (1-\tau^N)\max_{|t|\leq|s_0|}|F(x,t)|\Big].\end{aligned}$$

As $\tau \to 1^-$, the first term on the right hand side of the above inequality tends to the positive constant $w_N R_0^N F(x,s_0)$, and the second term goes to zero. We thus pick up some τ_0 and u_{τ_0} such that $\Psi(u_{\tau_0}) > 0$. Thus, in view of (5), we see that

$$\Phi_\mu(\eta_{\tau_0}) \geq \begin{cases} \dfrac{pS_H - \mu q}{pqS_H}\|\eta_{\tau_0}\|^p, \ \|\eta_{\tau_0}\| \geq 1, \\ \dfrac{pS_H - \mu q}{pqS_H}\|\eta_{\tau_0}\|^q, \ \|\eta_{\tau_0}\| \leq 1. \end{cases}$$

Again from Lemma 7, we may choose $\rho_0 > 0$ such that

$$\rho_0 < \begin{cases} \dfrac{pS_H - \mu q}{pqS_H}\|\eta_{\tau_0}\|^p, \ \|\eta_{\tau_0}\| \geq 1, \\ \dfrac{pS_H - \mu q}{pqS_H}\|\eta_{\tau_0}\|^q, \ \|\eta_{\tau_0}\| \leq 1, \end{cases}$$

and

$$\begin{aligned}\frac{\sup\limits_{u\in\Phi_\mu^{-1}((-\infty,\rho_0))}\Psi(u)}{\rho_0} &< \frac{w_N R_0^N p\Big[\tau_0^N F(x,s_0) - (1-\tau_0^N)\max\limits_{|t|\leq|s_0|}|F(x,t)|\Big]}{\|\eta_{\tau_0}\|^p + \|\eta_{\tau_0}\|^q} \\ &\leq \frac{\Psi(\eta_{\tau_0})}{\Phi_\mu(\eta_{\tau_0})}.\end{aligned}$$

By choosing $u_1 = \eta_{\tau_0}$, the condition (T_2) of Lemma 2 is verified. Define

$$\gamma = \gamma_\mu =: \frac{1+\rho_0}{\dfrac{\Psi(\eta_{\tau_0})}{\Phi_\mu(\eta_{\tau_0})} - \dfrac{\sup\limits_{u\in\Phi_\mu^{-1}((-\infty,\rho_0))}\Psi(u)}{\rho_0}}.$$

In view of Lemmas 5 and 6, we testify all the conditions in Lemma 2. Hence, there exist an open interval $\Lambda_\mu \subset [0,\gamma_\mu]$ and a number $\sigma_\mu > 0$, such that, for any $\lambda \in \Lambda_\mu$, the equation $\Phi'_\mu(u) = \lambda\Psi'(u)$ has at least three solutions in E having E-norm less than σ_μ. Because one of them may be the trivial solution (since $f(x,0) = 0$, see (h_2)), so the problem $(P_{\lambda,\mu})$ still has at least two distinct nontrivial solutions. □

6. Conclusions

In this paper, we have discussed the double phase problems with Hardy type potential. Due to the presence of the term $\mu\frac{|u|^{p-2}u}{|x|^p}$, the embedding of the Sobolev space $W_0^{1,H}(\Omega)$ into the weight Lebesgue space $L^p(\Omega,|x|^{-p})$ is continuous but not compact, so the classical variational approach are not applicable. In view of this difficulty, few papers turn their attention to the existence of solutions of problem $(P_{\mu,\lambda})$. In order to overcome this difficulty, In the present paper, we use the Ricceri's variational principle to obtain the existence of at least one nontrivial solution for problem $(P_{\mu,\lambda})$, formulated in the paper as Theorem 1. Moreover, we use Bonann's three critical points theorem to obtain the existence of at least

two nontrivial solutions for problem $(P_{\mu,\lambda})$, formulated in the paper as Theorem 2. The main results in this paper extend and complement the previous research results.

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