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Some Important Criteria for Oscillation of Non-Linear Differential Equations with Middle Term

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Abstract: In this work, we present new oscillation conditions for the oscillation of the higher-order differential equations with the middle term. We obtain some oscillation criteria by a comparison method with first-order equations. The obtained results extend and simplify known conditions in the literature. Furthermore, examining the validity of the proposed criteria is demonstrated via particular examples.

Keywords: higher-order; neutral delay; oscillation

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1. Introduction

Neutral equations contribute to many applications in physics, engineering, biology, non-Newtonian fluid theory, and the turbulent flow of a polytrophic gas in a porous medium. Also, oscillation of neutral equations contribute to many applications of problems dealing with vibrating masses attached to an elastic bar, see [1].

In this paper, we investigate the oscillatory properties of solutions of the higher-order neutral differential equation

$$\left(\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + \alpha_2(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)} + \zeta(x)\delta^{(p-1)}(\beta_2(x)) = 0,\ x \geq x_0, \quad (1)$$

where

$$\varpi(x) := \delta(x) + c(x)\delta(\beta_1(x)). \quad (2)$$

The main results are obtained under the following conditions:

$$\begin{cases} \alpha_1 \in C^1([x_0,\infty)), \alpha_1(x) > 0, \alpha_1'(x) \geq 0,\ 1 < p < \infty, \\ c, \alpha_2, \zeta \in C([x_0,\infty)), \alpha_2(x) > 0, \zeta(x) > 0, 0 \leq c(x) < c_0 < 1, \\ \beta_1 \in C^1([x_0,\infty)), \beta_2 \in C([x_0,\infty)), \beta_1'(x) > 0, \beta_1(x) \leq x, \lim_{x\to\infty}\beta_1(x) = \lim_{x\to\infty}\beta_2(x) = \infty, \\ \ell \geq 4 \text{ is an even natural number},\ \zeta \text{ is not identically zero for large } x. \end{cases}$$

Moreover, we establish the oscillatory behavior of (1) under the conditions

$$\beta_2(x) < \beta_1(x),\ \beta_2'(x) \geq 0 \text{ and } \left(\beta_1^{-1}(x)\right)' > 0 \quad (3)$$

and

$$\int_{x_0}^{\infty}\left(\frac{1}{\alpha_1(s)}\exp\left(-\int_{x_0}^{s}\frac{\alpha_2(\varpi)}{\alpha_1(\varpi)}d\varpi\right)\right)^{1/(p-1)} ds = \infty. \quad (4)$$

Over the past few years, there has been much research activity concerning the oscillation and asymptotic behavior of various classes of differential equations; see [2–11]. In particular, the study of the oscillation of neutral delay differential equations is of great interest in the last three decades; see [12–23].

Bazighifan et al. [2] examined the oscillation of higher-order delay differential equations with damping of the form

$$\begin{cases} \left(\alpha_1(x)\Phi_p[\varpi^{(\ell-1)}(x)]\right)' + \alpha_2(x)\Phi_p[f\left(\varpi^{(\ell-1)}(x)\right)] + \sum_{i=1}^{j} \zeta_i(x)\Phi_p[g(\varpi(\beta_i(x)))] = 0, \\ \Phi_p[s] = |s|^{p-2}s,\ j \geq 1,\ x \geq x_0 > 0. \end{cases}$$

This time, the authors used the Riccati technique.

Zhang et al. in [3] considered a higher-order differential equation

$$\begin{cases} L'_\varpi + \alpha_2(x)\left|\varpi^{(\ell-1)}(x)\right|^{p-2}\varpi^{(\ell-1)}(x) + \zeta(x)|\delta(\beta_2(x))|^{p-2}\delta(\beta_2(x)) = 0, \\ 1 < p < \infty,\ x \geq x_0 > 0,\ \varpi(x) = \delta(x) + c(x)\delta(\beta_1(x)), \end{cases}$$

where

$$L_\varpi = \left|\varpi^{(\ell-1)}(x)\right|^{p-2}\varpi^{(\ell-1)}(x).$$

Bazighifan and Ramos [4] considered the oscillation of the even-order nonlinear differential equation with middle term of the form

$$\begin{cases} \left(\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{p-1}\right)' + \alpha_2(x)\left(\varpi^{(\ell-1)}(x)\right)^{p-1} + \zeta(x)\varpi(\beta(x)) = 0, \\ x \geq x_0 > 0, \end{cases}$$

where $1 < p < \infty$.

Liu et al. [5] investigated the higher-order differential equations

$$\begin{cases} \left(\alpha_1(x)\Phi\left(\varpi^{(\ell-1)}(x)\right)\right)' + \alpha_2(x)\Phi\left(\varpi^{(\ell-1)}(x)\right) + \zeta(x)\Phi(\varpi(\beta(x))) = 0, \\ \Phi = |s|^{p-2}s,\ x \geq x_0 > 0,\ \ell \text{ is even}, \end{cases}$$

where n is even and used integral averaging technique.

The authors in [6,7] discussed oscillation criteria for the equations

$$\begin{cases} \left(\alpha_1(x)\left|\varpi^{(\ell-1)}(x)\right|^{p-2}\varpi^{(\ell-1)}(x)\right)' + \sum_{i=1}^{j} \zeta_i(x)g(\varpi(\beta_i(x))) = 0, \\ j \geq 1,\ x \geq x_0 > 0, \end{cases}$$

where ℓ is even and $p > 1$ is a real number, the authors used comparison method with first and second-order equations.

Li et al. [8] studied the oscillation of fourth-order neutral differential equations

$$\begin{cases} \left(\alpha_1(x)|\varpi'''(x)|^{p-2}\varpi'''(x)\right)' + \zeta(x)|\delta(\beta_2(x))|^{p-2}\delta(\beta_2(x)) = 0, \\ 1 < p < \infty,\ x \geq x_0 > 0, \end{cases}$$

where $\varpi(x) = \delta(x) + c(x)\delta(\beta_1(x))$.

In [9,10], the authors considered the equation

$$\varpi^{(\ell)}(x) + \zeta(x)\delta(\beta_2(x)) = 0 \tag{5}$$

by using the Riccati method, they proved that this equation is oscillatory if

$$\liminf_{x\to\infty} \int_{\beta_2(x)}^{x} z(s)\mathrm{d}s > \frac{(\ell-1)2^{(\ell-1)(\ell-2)}}{\mathrm{e}} \tag{6}$$

and

$$\liminf_{x\to\infty} \int_{\beta_2(x)}^{x} z(s)\mathrm{d}s > \frac{(\ell-1)!}{\mathrm{e}}, \text{ respectively,} \tag{7}$$

where $z(x) := \beta_2^{\ell-1}(x)(1-\alpha_2(\beta_2(x)))\zeta(x)$.

We can easily apply conditions (6) and (7) to the equation

$$\left(\delta(x) + \frac{1}{2}\delta\left(\frac{1}{2}x\right)\right)^{(4)} + \frac{\zeta_0}{x^4}\delta\left(\frac{9}{10}x\right) = 0,\ x \geq 1, \tag{8}$$

then we get that (8) is oscillatory if

The condition	(6)	(7)
The criterion	$\zeta_0 > 1839.2$	$\zeta_0 > 59.5$

Hence, [10] improved the results in [9].

Thus, the main purpose of this article is to extend the results in [9,10,23]. An example is considered to illustrate the main results.

We mention some important lemmas:

Lemma 1 ([11])**.** *Let* $\delta \in C^{\ell}([x_0,\infty),(0,\infty))$, $\delta^{(\ell-1)}(x)\delta^{(\ell)}(x) \leq 0$ *and* $\lim_{x\to\infty}\delta(x) \neq 0$, *then*

$$\delta(x) \geq \frac{\mu}{(\ell-1)!}x^{\ell-1}\left|\delta^{(\ell-1)}(x)\right| \text{ for } x \geq x_\mu,\ \mu \in (0,1).$$

Lemma 2 ([16])**.** *If* $\delta^{(i)}(x) > 0$, $i = 0,1,...,\ell$, *and* $\delta^{(\ell+1)}(x) < 0$, *then*

$$\frac{\delta(x)}{x^{\ell}/\ell!} \geq \frac{\delta'(x)}{x^{\ell-1}/(\ell-1)!}.$$

Lemma 3 ([13])**.** *Let (30) hold and*

$$\delta \text{ be an eventually positive solution of (1).} \tag{9}$$

Then, we have these cases:

$$\begin{aligned}
(\mathbf{I}_1): &\quad \varpi(x) > 0,\ \varpi'(x) > 0, \varpi''(x) > 0,\ \varpi^{(\ell-1)}(x) > 0 \text{ and } \varpi^{(\ell)}(x) < 0,\\
(\mathbf{I}_2): &\quad \varpi(x) > 0, \varpi^{(j)}(x) > 0, \varpi^{(j+1)}(x) < 0 \text{ for all odd integer}\\
&\quad j \in \{1,2,...,\ell-3\}, \varpi^{(\ell-1)}(x) > 0 \text{ and } \varpi^{(\ell)}(x) < 0,
\end{aligned}$$

for $x \geq x_1$, *where* $x_1 \geq x_0$ *is sufficiently large.*

2. Oscillation Criteria

Theorem 1. *If the differential equation*

$$\phi'(x) + (1-c(\beta_2(x)))^{(p-1)}\zeta(x)\frac{y_{x_0}(x)}{y_{x_0}(\beta_2(x))}\left(\frac{\mu\beta_2^{\ell-1}(x)}{(\ell-1)!\alpha_1^{1/(p-1)}(\beta_2(x))}\right)^{(p-1)}\phi(\beta_2(x)) = 0 \tag{10}$$

is oscillatory for some constant $\mu \in (0,1)$, where

$$y_{x_0}(x) := \exp\left(\int_{x_0}^{x} \frac{\alpha_2(t)}{t_1(t)}\mathrm{d}t\right),$$

then (1) is oscillatory.

Proof. Let (9) hold. Then, we see that $\delta(x)$, $\delta(\beta_1(x))$ and $\delta(\beta_2(x))$ are positive for all $x \geq x_1$ sufficiently large. It is not difficult to see that

$$\begin{aligned} &\tfrac{1}{y_{x_0}(x)}\tfrac{\mathrm{d}}{\mathrm{d}x}\left(y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right) \\ &\quad = \tfrac{1}{y_{x_0}(x)}\left(y_{x_0}(x)\left(\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + y'_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right) \\ &\quad = \left(\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + \tfrac{y'_{x_0}(x)}{y_{x_0}(x)}\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)} \\ &\quad = \left(\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + \alpha_2(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}. \end{aligned} \tag{11}$$

Taking into account (2) and $\varpi'(x) > 0$, we get that $\delta(x) \geq (1 - c(x))\varpi(x)$. Thus, from (1) and (11), we have that

$$\left(y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + y_{x_0}(x)\zeta(x)(1 - c(\beta_2(x)))^{(p-1)}\varpi^{(p-1)}(\beta_2(x)) \leq 0, \tag{12}$$

for $c_0 < 1$.

Using Lemma 1, we get that

$$\varpi(x) \geq \frac{\mu}{(\ell-1)!}x^{\ell-1}\varpi^{(\ell-1)}(x), \tag{13}$$

for some $\mu \in (0,1)$. From (1), (12) and (13), we see that

$$\left(y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + y_{x_0}(x)\zeta(x)(1 - c(\beta_2(x)))^{(p-1)}\left(\frac{\mu\beta_2^{\ell-1}(x)}{(\ell-1)!}\right)^{(p-1)}\left(\varpi^{(\ell-1)}(\beta_2(x))\right)^{(p-1)} \leq 0.$$

Then, if we set $\phi(x) = y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}$, then we have that $\phi > 0$ is a solution of the delay inequality

$$\phi'(x) + (1 - c(\beta_2(x)))^{(p-1)}\zeta(x)\frac{y_{x_0}(x)}{y_{x_0}(\beta_2(x))}\left(\frac{\mu\beta_2^{\ell-1}(x)}{(\ell-1)!\alpha_1^{1/(p-1)}(\beta_2(x))}\right)^{(p-1)}\phi(\beta_2(x)) \leq 0.$$

It is clear that the equation (10) has a positive solution (see [17], Theorem 1), this is a contradiction. The proof is complete. □

Theorem 2. *Assume that (3) and (30) hold. If the differential equations*

$$z'(x) + \zeta(x)\frac{y_{x_0}(x)}{y_{x_0}\left(\beta_1^{-1}(\beta_2(x))\right)}\left(\frac{\mu\left(\beta_1^{-1}(\beta_2(x))\right)^{\ell-1}c_\ell(\beta_2(x))}{(\ell-1)!\alpha_1^{1/(p-1)}\left(\beta_1^{-1}(\beta_2(x))\right)}\right)^{(p-1)} z\left(\beta_1^{-1}(\beta_2(x))\right) = 0 \tag{14}$$

and

$$\omega'(x) + \beta_1^{-1}(\beta_2(x))\widetilde{y}_{\ell-3}(x)\omega\left(\beta_1^{-1}(\beta_2(x))\right) = 0 \tag{15}$$

are oscillatory, where

$$\begin{aligned}\widetilde{y}_0(x) \quad &: \quad = \left(\frac{1}{y_{x_1}(x)\alpha_1(x)}\int_x^\infty \zeta(s)y_{x_1}(s)c_2^{(p-1)}(\beta_2(s))\mathrm{d}s\right)^{1/(p-1)}, \\ \widetilde{y}_k(x) \quad &: \quad = \int_x^\infty \widetilde{y}_{k-1}(s)\mathrm{d}s, \;\; k = 1,2,...,\ell-2\end{aligned}$$

and

$$c_m(x) := \frac{1}{c\left(\beta_1^{-1}(x)\right)}\left(1 - \frac{\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)^{m-1}}{\left(\beta_1^{-1}(x)\right)^{m-1}c\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)}\right), \; m = 2, \ell,$$

then (1) is oscillatory.

Proof. Let (9) hold. Then, we see that $\delta(x)$, $\delta(\beta_1(x))$ and $\delta(\beta_2(x))$ are positive.

Let $(\mathbf{I}_1)$ hold, from Lemma 2, we find $\varpi(x) \geq \frac{1}{(\ell-1)}x\varpi'(x)$ and then $\left(x^{1-\ell}\varpi(x)\right)' \leq 0$. Hence, since $\beta_1^{-1}(x) \leq \beta_1^{-1}\left(\beta_1^{-1}(x)\right)$, we obtain

$$\varpi\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right) \leq \frac{\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)^{\ell-1}}{\left(\beta_1^{-1}(x)\right)^{\ell-1}}\varpi\left(\beta_1^{-1}(x)\right). \tag{16}$$

From (2), we obtain

$$\begin{aligned} c\left(\beta_1^{-1}(x)\right)\delta(x) &= \varpi\left(\beta_1^{-1}(x)\right) - \delta\left(\beta_1^{-1}(x)\right) \\ &= \varpi\left(\beta_1^{-1}(x)\right) - \left(\frac{\varpi\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)}{c\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)} - \frac{\delta\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)}{c\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)}\right) \\ &\geq \varpi\left(\beta_1^{-1}(x)\right) - \frac{1}{c\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right)}\varpi\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right), \end{aligned} \tag{17}$$

which with (1), (11) and (17) give

$$\begin{aligned}&\left(y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' \\ &\quad + \frac{y_{x_0}\zeta(x)}{c^{(p-1)}\left(\beta_1^{-1}(\beta_2(x))\right)}\left(\varpi\left(\beta_1^{-1}(\beta_2(x))\right) - \frac{\varpi\left(\beta_1^{-1}\left(\beta_1^{-1}(\beta_2(x))\right)\right)}{c\left(\beta_1^{-1}\left(\beta_1^{-1}(\beta_2(x))\right)\right)}\right)^{(p-1)} \leq 0. \end{aligned} \tag{18}$$

We have that (18), which (16) gives

$$\left(y_{x_1}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + y_{x_1}(x)\zeta(x)c_\ell^{(p-1)}(\beta_2(x))\varpi^{(p-1)}\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0. \tag{19}$$

From Lemma 1, we get (13). Therefore, from (19), we obtain

$$\begin{aligned}&\left(y_{x_1}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' \\ \leq\;& -y_{x_1}(x)\zeta(x)\left(\frac{\mu c_\ell(\beta_2(x))}{(\ell-1)!}\left(\beta_1^{-1}(\beta_2(x))\right)^{\ell-1}\right)^{(p-1)}\left(\varpi^{(\ell-1)}\left(\beta_1^{-1}(\beta_2(x))\right)\right)^{(p-1)}.\end{aligned}$$

Then, if we set $z(x) = y_{x_0}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}$, then we have that $z > 0$ is a solution of the delay inequality

$$z'(x) + \zeta(x)\frac{y_{x_1}(x)}{y_{x_1}\left(\beta_1^{-1}(\beta_2(x))\right)}\left(\frac{\mu\left(\beta_1^{-1}(\beta_2(x))\right)^{\ell-1}c_\ell(\beta_2(x))}{(\ell-1)!\alpha_1^{1/(p-1)}\left(\beta_1^{-1}(\beta_2(x))\right)}\right)^{(p-1)} z\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0.$$

It is clear (see [17] Theorem 1) that the Equation (14) also has a positive solution, this is a contradiction.

Let $(\mathbf{I}_2)$ hold, from Lemma 2, we obtain

$$\varpi(x) \geq x\varpi'(x) \tag{20}$$

and then $\left(x^{-1}\varpi(x)\right)' \leq 0$. Hence, since $\beta_1^{-1}(x) \leq \beta_1^{-1}\left(\beta_1^{-1}(x)\right)$, we get

$$\varpi\left(\beta_1^{-1}\left(\beta_1^{-1}(x)\right)\right) \leq \frac{\beta_1^{-1}\left(\beta_1^{-1}(x)\right)}{\beta_1^{-1}(x)}\varpi\left(\beta_1^{-1}(x)\right), \tag{21}$$

which with (18) yields

$$\left(y_{x_1}(x)\alpha_1(x)\left(\varpi^{(\ell-1)}(x)\right)^{(p-1)}\right)' + \zeta(x)y_{x_1}(x)c_2^{(p-1)}(\beta_2(x))\varpi^{(p-1)}\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0. \tag{22}$$

Integrating (22) from x to ∞, we obtain

$$\begin{aligned} -\varpi^{(\ell-1)}(x) &\leq -\left(\frac{1}{y_{x_1}(x)\alpha_1(x)}\int_x^\infty \zeta(s)y_{x_1}(s)c_2^{(p-1)}(\beta_2(s))\varpi^{(p-1)}\left(\beta_1^{-1}(\beta_2(s))\right)\mathrm{d}s\right)^{1/(p-1)} \\ &\leq -\widetilde{y}_0(x)\varpi\left(\beta_1^{-1}(\beta_2(x))\right). \end{aligned}$$

Integrating this inequality $\ell-3$ times from x to ∞, we find

$$\varpi''(x) + \widetilde{y}_{\ell-3}(x)\varpi\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0, \tag{23}$$

which with (20) gives

$$\varpi''(x) + \beta_1^{-1}(\beta_2(x))\widetilde{y}_{\ell-3}(x)\varpi'\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0.$$

Thus, if we put $\omega(x) := \varpi'(x)$, then we conclude that $\omega > 0$ is a solution of

$$\omega'(x) + \beta_1^{-1}(\beta_2(x))\widetilde{y}_{\ell-3}(x)\omega\left(\beta_1^{-1}(\beta_2(x))\right) \leq 0. \tag{24}$$

It is clear (see [17] Theorem 1) that the equation (15) also has a positive solution, this is a contradiction. The proof is complete. □

Next, we establish new oscillation conditions for Equation (1) according to the results obtained some related contributions to the subject.

Corollary 1. *Assume that $c_0 < 1$ and (30) hold. If*

$$\liminf_{x\to\infty}\int_{\beta_2(x)}^x (1-c(\beta_2(s)))^{(p-1)}\zeta(s)\frac{y_{x_0}(s)}{y_{x_0}(\beta_2(s))}\left(\frac{\mu\beta_2^{\ell-1}(s)}{\alpha_1^{1/(p-1)}(\beta_2(s))}\right)^{(p-1)}\mathrm{d}s > \frac{((\ell-1)!)^{(p-1)}}{\mathrm{e}} \tag{25}$$

is oscillatory, then (1) is oscillatory.

Corollary 2. *Let (3) and (30) hold. If*

$$\liminf_{x\to\infty}\int_{\beta_1^{-1}(\beta_2(x))}^{x}\zeta(s)\frac{y_{x_0}(s)}{y_{x_0}\left(\beta_1^{-1}(\beta_2(s))\right)}\left(\frac{\mu\left(\beta_1^{-1}(\beta_2(s))\right)^{\ell-1}c_\ell(\beta_2(s))}{\alpha_1^{1/(p-1)}\left(\beta_1^{-1}(\beta_2(s))\right)}\right)^{(p-1)}\mathrm{d}s>\frac{((\ell-1)!)^{(p-1)}}{\mathrm{e}} \tag{26}$$

and

$$\liminf_{x\to\infty}\int_{\beta_1^{-1}(\beta_2(x))}^{x}\beta_1^{-1}(\beta_2(s))\widetilde{y}_{\ell-3}(s)\mathrm{d}s>\frac{1}{\mathrm{e}} \tag{27}$$

are oscillatory, then (1) is oscillatory.

3. Applications

This section presents some interesting application which are addressed based on above hypothesis to show some interesting results in this paper.

Example 1. *Let the equation*

$$\left(\delta(x)+\frac{1}{2}\delta\left(\frac{x}{3}\right)\right)^{(4)}(x)+\frac{1}{x}\varpi^{(3)}(x)+\frac{\zeta_0}{x^4}\delta\left(\frac{x}{2}\right)=0, \tag{28}$$

where $\zeta_0>0$ *is a constant. Let* $p=2,\ \ell=4,\ \alpha_1(x)=1,\ \alpha_2(x)=1/x,\ \zeta(x)=\zeta_0/x^4,\ \beta_2(x)=x/2$ *and* $\beta_1(x)=x/3$*. So, we get*

$$y_{x_0}(x)=x,\ y_{x_0}(\beta_2(x))=x/2.$$

Thus, we find

$$\begin{aligned}&\liminf_{x\to\infty}\int_{\beta_2(x)}^{x}(1-c(\beta_2(s)))^{(p-1)}\zeta(s)\frac{y_{x_0}(s)}{y_{x_0}(\beta_2(s))}\left(\frac{\mu\beta_2^{\ell-1}(s)}{\alpha_1^{1/(p-1)}(\beta_2(s))}\right)^{(p-1)}\mathrm{d}s\\ =\ &\liminf_{x\to\infty}\int_{x/2}^{x}\frac{\zeta_0}{x^4}\left(\frac{x^3}{8}\right)\mathrm{d}s=\frac{\zeta_0}{8}\ln 2.\end{aligned}$$

Hence, the condition becomes

$$\zeta_0>\frac{48}{\mathrm{e}\ln 2}. \tag{29}$$

Therefore, by Corollary 1, every solution of (28) is oscillatory if $\zeta_0>25.5$*.*

Remark 1. *Consider the equation (8), by Corollary 1, all solution of (8) is oscillatory if* $\zeta_0>57.5$*. Whereas, the criterion obtained from the results of [9,10] are* $\zeta_0>1839.2$ *and* $\zeta_0>59.5$*. So, our results extend the results in [9].*

4. Conclusions

In this paper, we obtain sufficient criteria for oscillation of solutions of higher-order differential equation with middle term. We discussed the oscillation behavior of solutions for Equation (1). We obtain some oscillation criteria by comparison method with first order equations. Our results unify and improve some known results for differential equations with middle term. In future work, we will discuss the oscillatory behavior of these equations using integral averaging method and under condition

$$\int_{x_0}^{\infty}\left(\frac{1}{\alpha_1(s)}\exp\left(-\int_{x_0}^{s}\frac{\alpha_2(\varpi)}{\alpha_1(\varpi)}d\varpi\right)\right)^{1/(p-1)}ds<\infty. \tag{30}$$

For researchers interested in this field, and as part of our future research, there is a nice open problem which is finding new results in the following cases:

$$\begin{array}{ll}(\mathbf{F}_1) & \varpi(x) > 0,\ \varpi'(x) > 0, \varpi''(x) > 0,\ \varpi^{(\ell-1)}(x) > 0,\ \varpi^{(\ell)}(x) < 0, \\ (\mathbf{F}_2) & \varpi(x) > 0, \varpi^{(j)}(x) > 0, \varpi^{(j+1)}(x) < 0 \text{ for all odd integers} \\ & j \in \{1, 3, ..., \ell-3\}, \varpi^{(\ell-1)}(x) > 0, \varpi^{(\ell)}(x) < 0.\end{array}$$

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