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Study on Implicit-Type Fractional Coupled System with Integral Boundary Conditions

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Abstract: This paper is concerned with a class of implicit-type coupled system with integral boundary conditions involving Caputo fractional derivatives. First, the existence result of solutions for the considered system is obtained by means of topological degree theory. Next, Ulam–Hyers stability and generalized Ulam–Hyers stability are studied under some suitable assumptions. Finally, one example is worked out to illustrate the main results.

Keywords: integral boundary conditions; fractional differential equations; Ulam–Hyers stability; implicit type

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1. Introduction

This paper aims to establish the sufficient conditions for the existence and stability results of solutions to a class of coupled systems of implicit-type fractional differential equations (IFDEs) involving Caputo fractional derivatives under the integral-type boundary conditions

$$\begin{cases} \mathcal{D}^\alpha x(t) = f_1(t, y(t), \mathcal{D}^\alpha x(t)), 0 < t < 1; \\ \mathcal{D}^\beta y(t) = f_2(t, x(t), \mathcal{D}^\beta y(t)), 0 < t < 1; \\ x(0) = h_1(x), \quad x(1) = \int_0^1 g_1(t)x(t)dt; \\ y(0) = h_2(y), \quad y(1) = \int_0^1 g_2(t)y(t)dt, \end{cases} \quad (1)$$

where $\alpha, \beta \in (1, 2]$, $\mathcal{D}$ denotes the standard Caputo fractional derivative; $f_1, f_2 \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h_1, h_2 \in C(C[0, 1], \mathbb{R})$, and $g_1, g_2 \in L^1[0, 1]$.

It is well known that the subject of fractional differential equations (FDEs) has gained considerable popularity and importance due to its wide range of applications in describing the real word problems. FDEs are powerful tools to model the many phenomena of different fields of scientific disciplines and engineering. The study of FDEs is found to be of great value and interest in view of the occurrence of such systems in a variety of problems of applied nature. For example, there are the Lorenz system [1], chaotic synchronization system [2], Chua circuit [3], impulsive problems [4,5], quantum theory [6], stability and controllability problems [7,8], diffusion phenomenon [9–12], and systems of thermoelasticity [13,14], etc. Compared with the classical integral order derivatives, the fractional derivatives provide significant tools for the analysis of memory and hereditary properties. In consequence, many meaningful results in these fields have been obtained. For more details, please see Kilbas et al. [15], Lakshmikantham et al. [16], Podlubny [17], Samko et al. [18], and references therein.

Another topic which has increasingly received interest in recent years is stability theory, which plays an important role in optimization and numerical point of view. In order to deal with the difficulties that arise from exponential Mittag–Leffler and Lyapunov stability theory, Ulam [19] introduced another kind of stability, that is Ulam–Hyers stability, which was further studied by Hyers [20] in the subsequent years. Since then, many

achievements in various kinds of differential equations with integer order in this field have been investigated ([21,22], etc.). Nowadays, researchers are focusing on the study of Ulam stability for FDEs—for more details about Ulam stability with FDEs, please see [23–26] and the references therein.

Integral boundary value conditions (IBCs, for short) of various types of differential equations have also been regarded as an important research field in recent years. IBCs play an important role in the study of computational fluid dynamics studies related to blood flow problems. In order to solve the unreasonable problem that the cross-section of blood vessels is always assumed to be circular, IBCs provide an effective and applicable approach. In addition, IBCs also have an important applications in the dynamic behavior of bioprocesses, such as HIV infection. More details can be found in [27,28]. Additionally, IBCs have useful applications in regularizing ill-posed parabolic backward problems in time partial differential equations—see, for example, mathematical models for bacterial self-regularization [29]. Some recent investigations on coupled systems of FDEs with IBCs can be found in [23,27,30–33] and the references cited therein.

As a branch of the research on FDEs, the IFDEs have been given widespread attention, due to the fact that many of the problems around managerial and economics are modeled by IFDEs. For more details on IFDEs, one can see [34–36], and references therein. Recently, IFDEs have been properly investigated by many researchers, such as Z. Ali et al. [25], M. Benchohra et al. [24], J. J. Nieto et al. [37], and Samina et al. [38]. They focus on the existence and stability theory. For example, J. J. Nieto et al., in [37], investigated the following implicit fractional differential equations via the Liouville–Caputo derivative:

$$\begin{cases} {}^c\mathcal{D}^\alpha y(t) = f(t, y(t), {}^c\mathcal{D}^\alpha y(t)), & t \in [0, b]; \\ y(0) = y_0, \end{cases}$$

where $\alpha \in (0, 1)$, $f : [0, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. The existence and uniqueness results are obtained by means of fixed point theory and an approximation method. In [38], Samina et al. studied the following IFDEs with antiperiodic boundary conditions involving Caputo fractional derivatives

$$\begin{cases} \mathcal{D}^\alpha x(t) = \vartheta(t, y(t), \mathcal{D}^\alpha x(t)), & \alpha \in (1, 2], t \in J; \\ \mathcal{D}^\beta y(t) = \theta(t, x(t), \mathcal{D}^\beta y(t)), & \beta \in (1, 2], t \in J; \\ x(0) = -x(\eta), & x'(0) = -x'(\eta); \\ y(0) = -y(\eta), & y'(0) = -y'(\eta), \end{cases}$$

where $\eta \in (0, +\infty)$, $J = [0, \eta]$, and $\vartheta, \theta : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are nonlinear continuous functions. By applying Banach and Schauder's fixed point theorem, the existence and uniqueness results for the considered problem were obtained under some suitable assumptions.

From all of the above works, we find the fact that their nonlinear terms need to satisfy the Lipschitz condition for two variables. The implicit of the corresponding problem can be overcome by this condition, which plays a key role in obtaining the existence results of solutions. Although this condition is convenient, it is very restrictive and excludes a lot of nonlinear IFDEs. Therefore, we want to investigate whether it works when one “certain variable” satisfies the Lipschitz condition in nonlinear terms.

As far as we know, there are few paper studied Ulam–Hyers stability for IFDEs and there is no paper considering such a problem for (1). Thus, the purpose of the present paper is to fill this gap. Moreover, our nonlinear terms only need to satisfy the Lipschitz condition for one “certain variable”, which relax the restriction of the nonlinear terms in the literatures [24,38].

We subdivide this paper into the following Sections. Section 2 contains some preliminaries. In Section 3, some transformations are introduced to convert the problem (1) into the corresponding explicit form and integral form. In Section 4, the existence results of solutions will be given and proved by topological degree theory together with the properties of Green function. In Section 5, we establish Ulam–Hyers stability and generalized

Ulam–Hyers stability under some suitable assumptions. Finally, in Section 6, one example is worked out to support our main results.

2. Preliminaries

In this section, we recall some basic definitions and fundamental results necessary for this work throughout this paper. First, we provide some fundamental material about fractional calculus.

Definition 1 ([17]). *The Riemann–Liouville standard fractional integral of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \rightarrow \mathbb{R}$ is given by*

$$\mathcal{I}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right side integral is pointwise defined on $\mathbb{R}^+ =: (0, +\infty)$.

Definition 2 ([17]). *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \rightarrow \mathbb{R}$ is given by*

$$\mathcal{D}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α , and provided that the right side integral is pointwise defined on $[0, +\infty)$.

Lemma 1 ([17]). *Let $n-1 < \alpha < n$ ($n \in \mathbb{N}$). Then,*

$$\mathcal{I}^\alpha \mathcal{D}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \dots + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Next, we list some basic definitions and fundamental results about topological degree theory. For this sake, let E be a Banach space with norm $\|\cdot\|$, $D \subset E$, and the operator $\mathcal{A} : D \rightarrow E$. $\chi(\cdot)$ represents the Kuratowski measure of noncompactness.

Definition 3 ([39]). *Assume that $\mathcal{A}$ is continuous and bounded.*

(1) *If there is a constant $\sigma \geq 0$, such that $\chi(\mathcal{A}(S)) \leq \sigma \chi(S)$ for any bounded $S \subset E$, then $\mathcal{A}$ is called a σ -set contraction; a σ -set contraction with $\sigma < 1$ is called a strict set contraction.*

(2) *$\mathcal{A}$ is said to be condensing if $\chi(\mathcal{A}(S)) \leq \chi(S)$ for any bounded $S \subset E$, which is not relatively compact.*

Proposition 1 ([39]). *If $\mathcal{A}$ is χ -Lipschitz having constants h , namely*

$$\|\mathcal{A}(x) - \mathcal{A}(y)\| \leq h\|x - y\|, \quad \forall x, y \in D.$$

Then $\mathcal{A}$ is h -set contraction in D .

Proposition 2 ([39]). *If $\mathcal{A}$ is compact, then $\mathcal{A}$ is 0-set contraction.*

Proposition 3 ([39]). *If $\mathcal{A}_1 : D \rightarrow E$ is σ_1 -set contraction and $\mathcal{A}_2 : D \rightarrow E$ is σ_2 -set contraction, then $\mathcal{A}$ is $(\sigma_1 + \sigma_2)$ -set contraction.*

Theorem 1 ([39]). *Let $\mathcal{T} : E \rightarrow E$ be condensing and*

$$\Omega = \{x \in E : \exists 0 \leq \xi \leq 1 \text{ with } x = \xi \mathcal{T}(x)\}.$$

If Ω is a bounded set in E , then $\mathcal{T}$ has at least one fixed point.

3. Conversion of the Considered System

In this section, we convert problem (1) into explicit form to overcome the difficulties that arise from implicitity. $\mathcal{X} = C[0, 1]$ represents a Banach space with norm $\|x\| = \max\{|x(t)| : t \in [0, 1]\}$. The product $\mathcal{X} \times \mathcal{X}$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$. For simplicity and convenience, set

$$\begin{aligned}\delta_1 &= 1 - \int_0^1 g_1(\tau) d\tau, \quad \delta_2 = 1 - \int_0^1 \tau g_1(\tau) d\tau, \quad \delta = \frac{|\delta_2|}{|\delta_1| + |\delta_2|}; \\ \gamma_1 &= 1 - \int_0^1 g_2(\tau) d\tau, \quad \gamma_2 = 1 - \int_0^1 \tau g_2(\tau) d\tau, \quad \gamma = \frac{|\gamma_2|}{|\gamma_1| + |\gamma_2|}.\end{aligned}\quad (2)$$

Now let us list the following assumptions satisfied throughout the paper.

Hypothesis 1. $f_1, f_2 \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $h_1, h_2 \in C(\mathcal{X}, \mathbb{R})$, $g_1, g_2 \in L^1[0, 1]$, and $\delta_2, \gamma_2 \neq 0$.

Hypothesis 2. There exist constants $k_1 \in [0, \delta)$ and $k_2 \in [0, \gamma)$ such that

$$\begin{aligned}|h_1(x) - h_1(x_0)| &\leq k_1|x - x_0|, \quad \forall x, x_0 \in \mathcal{X}; \\ |h_2(y) - h_2(y_0)| &\leq k_2|y - y_0|, \quad \forall y, y_0 \in \mathcal{X}.\end{aligned}$$

Hypothesis 3. There exist constants $\sigma_1, \sigma_2 \in (0, 1)$, such that, for each $u, u_0, v, v_0 \in \mathbb{R}$,

$$\begin{aligned}|f_1(t, y, u) - f_1(t, y, u_0)| &\leq \sigma_1|u - u_0|; \\ |f_2(t, x, v) - f_2(t, x, v_0)| &\leq \sigma_2|v - v_0|.\end{aligned}$$

For convenience, set

$$\begin{aligned}\kappa_1 &:= (1 + 2\sigma_1)^{-1}, \quad \kappa_2 := (1 + 2\sigma_2)^{-1}; \\ F_1(t, y, u) &:= u - f_1(t, y, u), \quad F_2(t, x, v) := v - f_2(t, x, v).\end{aligned}\quad (3)$$

In order to convert system (1) into explicit form, we also need to list the following two lemmas.

Lemma 2. Assume that Hypothesis 3 holds, Then, there exist two continuous functions $u(t) = \varphi_1(t, y(t))$ and $v(t) = \varphi_2(t, x(t))$, satisfying

$$\begin{cases} F_1(t, y(t), u(t)) = 0, & 0 < t < 1; \\ F_2(t, x(t), v(t)) = 0, & 0 < t < 1, \end{cases}$$

for each pair of $(x, y) \in \mathcal{X} \times \mathcal{X}$.

Proof of Lemma 2. For each $y \in \mathcal{X}$, define a mapping $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$\mathcal{F}\varphi = \varphi - \kappa_1 F_1(\cdot, y, \varphi). \quad (4)$$

Then, we claim that the mapping $\mathcal{F}$ is contractive.

In fact, by Hypothesis 3, for each $\varphi, \varphi_0 \in \mathcal{X}$, by (3), (4), and Hypothesis 3, one can get that

$$\begin{aligned}
|\mathcal{F}\varphi(t) - \mathcal{F}\varphi_0(t)| &= |\varphi(t) - \varphi_0(t) - \kappa_1[F_1(t, y(t), \varphi(t)) - F_1(t, y(t), \varphi_0(t))]| \\
&= |\varphi(t) - \varphi_0(t) - \kappa_1[\varphi(t) - f_1(t, y(t), \varphi(t)) - \varphi_0(t) + f_1(t, y(t), \varphi_0(t))]| \\
&= |(1 - \kappa_1)[\varphi(t) - \varphi_0(t)] - \kappa_1[f_1(t, y(t), \varphi(t)) - f_1(t, y(t), \varphi_0(t))]| \\
&\leq (1 - \kappa_1)|\varphi(t) - \varphi_0(t)| + \sigma_1 \kappa_1 |\varphi(t) - \varphi_0(t)| \\
&= 3\sigma_1 \cdot \kappa_1 |\varphi(t) - \varphi_0(t)|.
\end{aligned} \tag{5}$$

By (5), we have

$$\|\mathcal{F}\varphi - \mathcal{F}\varphi_0\| \leq \varrho \|\varphi - \varphi_0\|,$$

where $\varrho := 3\sigma_1 \cdot \kappa_1 \in (0, 1)$. Namely, $\mathcal{F}$ is a contractive mapping in $\mathcal{X}$. Applying the contraction mapping principle, one can get that there exists continuous function $u(t) = \varphi_1(t, y(t))$, satisfying $F_1(t, y(t), u(t)) = 0$, $t \in [0, 1]$, for $y \in \mathcal{X}$.

From a process similar to the above, there exists the continuous function $v(t) = \varphi_2(t, x(t))$ satisfying $F_2(t, x(t), v(t)) = 0$, $t \in [0, 1]$, for $x \in \mathcal{X}$.

To sum up, the proof is complete. $\square$

Lemma 3. Define the mapping $G : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ by $G = (\varphi_1, \varphi_2)$. Assume that Hypothesis 1 and Hypothesis 3 hold. Then, the mapping G is continuous.

Proof of Lemma 3. Let $(x_n, y_n) \rightarrow (x, y) \in \mathcal{X} \times \mathcal{X}$ as $n \rightarrow \infty$. For convenience, set $u_n := \varphi_1(\cdot, y_n)$, $v_n := \varphi_2(\cdot, x_n)$. Then, we need only to claim that

$$G(u_n, v_n) \rightarrow G(u, v), \text{ as } n \rightarrow \infty.$$

To this end, by Hypothesis 1, Hypothesis 3, and Lemma 2, one can get that, for each $t \in [0, 1]$,

$$\begin{aligned}
|u_n - u| &= |\varphi_1(t, y_n) - \varphi_1(t, y)| \\
&= |f_1(t, y_n, u_n) - f_1(t, y, u)| \\
&\leq |f_1(t, y_n, u_n) - f_1(t, y_n, u)| + |f_1(t, y_n, u) - f_1(t, y, u)| \\
&\leq \sigma_1 |u_n - u| + |f_1(t, y_n, u) - f_1(t, y, u)|.
\end{aligned}$$

Taking the maximum for both sides of the above inequality in $t \in [0, 1]$, we immediately get that

$$\|u_n - u\| \leq \frac{1}{1 - \sigma_1} \max_{t \in [0, 1]} |f_1(t, y_n, u) - f_1(t, y, u)|.$$

So,

$$\|u_n - u\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Similarly,

$$\|v_n - v\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As a result, the conclusion of this lemma follows. $\square$

Based on the Lemmas 2 and 3, we know that the existence of nontrivial solutions for problem (1) is equivalent to the existence of the following explicit fractional equations, as

$$\begin{cases} \mathcal{D}^\alpha x(t) = \varphi_1(t, y(t)), & 0 < t < 1; \\ \mathcal{D}^\beta y(t) = \varphi_2(t, x(t)), & 0 < t < 1; \\ x(0) = h_1(x), \quad x(1) = \int_0^1 g_1(t)x(t)dt; \\ y(0) = h_2(y), \quad y(1) = \int_0^1 g_2(t)y(t)dt. \end{cases} \tag{6}$$

Therefore, we need only to investigate the existence of nontrivial solutions for (6).

Lemma 4. Let $h \in \mathcal{X}$ be a given function. Then, the unique solution of the following boundary value problem

$$\begin{cases} \mathcal{D}^\alpha x(t) = h(t), & 0 < t < 1; \\ x(0) = h_1(x), & x(1) = \int_0^1 g_1(t)x(t)dt, \end{cases} \quad (7)$$

is given by

$$x(t) = (1 - \frac{\delta_1}{\delta_2})h_1(x) + \int_0^1 G_1(t,s)h(s)ds,$$

where

$$\begin{aligned} G_1(t,s) &= G_{1\alpha}(t,s) + G_{2\alpha}(t,s), \\ G_{1\alpha}(t,s) &= \begin{cases} \frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{-t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1; \end{cases} \\ G_{2\alpha}(t,s) &= \frac{t}{\delta_2} \int_0^1 g_1(\tau)G_{1\alpha}(\tau,s)d\tau. \end{aligned} \quad (8)$$

Proof of Lemma 4. By using Lemma 1, we can reduce (6) to the following equivalent equation

$$\begin{aligned} x(t) &= c_0 + c_1 t + \mathcal{I}^\alpha h(t) \\ &= c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \end{aligned} \quad (9)$$

for some $c_i \in \mathbb{R}$, $i = 0, 1$.

Applying boundary conditions to (9), one can obtain that

$$c_0 = h_1(x), \quad c_1 = -h_1(x) - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \int_0^1 g_1(t)x(t)dt. \quad (10)$$

Putting (10) into (9), we have

$$\begin{aligned} x(t) &= h_1(x) + [-h_1(x) - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \int_0^1 g_1(t)x(t)dt] \cdot t \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &= (1-t)h_1(x) - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\ &\quad + t \int_0^1 g_1(t)x(t)dt \\ &= (1-t)h_1(x) + \int_0^t \left[\frac{(t-s)^{\alpha-1} - t(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right] h(s)ds - \int_t^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds \\ &\quad + t \int_0^1 g_1(\tau)x(\tau)d\tau \\ &= (1-t)h_1(x) + \int_0^1 G_{1\alpha}(t,s)h(s)ds + t \int_0^1 g_1(\tau)x(\tau)d\tau. \end{aligned} \quad (11)$$

Moreover, by (11) one can derive

$$\begin{aligned} \int_0^1 g_1(\tau)x(\tau)d\tau &= \int_0^1 g_1(\tau) \left[(1-\tau)h_1(x) + \int_0^1 G_{1\alpha}(\tau,s)h(s)ds + \tau \int_0^1 g_1(\tau)x(\tau)d\tau \right] d\tau \\ &= h_1(x) \cdot \int_0^1 (1-\tau)g_1(\tau)d\tau + \int_0^1 g_1(\tau) \cdot \left[\int_0^1 G_{1\alpha}(\tau,s)h(s)ds \right] d\tau \\ &\quad + \left(\int_0^1 g_1(\tau)x(\tau)d\tau \right) \cdot \left(\int_0^1 g_1(\tau)\tau d\tau \right), \end{aligned} \quad (12)$$

which indicates that

$$\int_0^1 g_1(\tau)x(\tau)d\tau = \frac{1}{\delta_2} \cdot [h_1(x) \cdot \int_0^1 (1-\tau)g_1(\tau)d\tau + \int_0^1 g_1(\tau) \cdot \left[\int_0^1 G_{1\alpha}(\tau,s)h(s)ds \right] d\tau]. \quad (13)$$

Substituting (13) into (11), we obtain

$$\begin{aligned} x(t) &= [1 - t + \frac{t}{\delta_2} \int_0^1 (1 - \tau) g_1(\tau) d\tau] \cdot h_1(x) + \int_0^1 G_{1\alpha}(t, s) h(s) ds \\ &\quad + \frac{t}{\delta_2} \int_0^1 g_1(\tau) \int_0^1 G_{1\alpha}(\tau, s) h(s) ds d\tau \\ &= (1 - \frac{\delta_1}{\delta_2} t) \cdot h_1(x) + \int_0^1 G_{1\alpha}(t, s) h(s) ds + \int_0^1 G_{2\alpha}(t, s) h(s) ds \\ &= (1 - \frac{\delta_1}{\delta_2} t) \cdot h_1(x) + \int_0^1 G_1(t, s) h(s) ds. \end{aligned} \quad (14)$$

This completes the proof. $\square$

From Lemmas 2–4, and Hypothesis 1, we need to only solve the following system of integral equations:

$$\begin{cases} x(t) = (1 - \frac{\delta_1}{\delta_2} t) h_1(x) + \int_0^1 G_1(t, s) \varphi_1(s, y(s)) ds, \\ y(t) = (1 - \frac{\gamma_1}{\gamma_2} t) h_2(y) + \int_0^1 G_2(t, s) \varphi_2(s, x(s)) ds. \end{cases} \quad (15)$$

Now, define operators $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, and $\mathcal{T} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}_1, \mathcal{A}_2 : \mathcal{X} \rightarrow \mathcal{X}$ and $\mathcal{B}_1, \mathcal{B}_2 : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ are defined by

$$\mathcal{A}_1(x)(t) = (1 - \frac{\delta_1}{\delta_2} t) h_1(x), \quad \mathcal{A}_2(y)(t) = (1 - \frac{\gamma_1}{\gamma_2} t) h_2(y);$$

$$\mathcal{B}_1(x, y)(t) = \int_0^1 G_1(t, s) \varphi_1(s, y(s)) ds,$$

$$\mathcal{B}_2(x, y)(t) = \int_0^1 G_2(t, s) \varphi_2(s, x(s)) ds;$$

and

$$\mathcal{A}(x, y) = (\mathcal{A}_1(x), \mathcal{A}_2(y)),$$

$$\mathcal{B}(x, y) = (\mathcal{B}_1(x, y), \mathcal{B}_2(x, y)),$$

$$\mathcal{T}(x, y) = \mathcal{A}(x, y) + \mathcal{B}(x, y).$$

Then (15) in operator form becomes

$$(u, v) = \mathcal{T}(u, v). \quad (16)$$

Obviously, (16) guarantees that the existence of solutions for (15) is equivalent to the existence of the fixed point of $\mathcal{T}$ in $\mathcal{X} \times \mathcal{X}$. Therefore, we just need to find the fixed point of $\mathcal{T}$ in the following work.

4. Existence Results

In this section, we will present the existence result of solutions for problem (1). To do this, first we give some properties of mapping $\mathcal{T}$.

Lemma 5. Assume that Hypothesis 1 and Hypothesis 2 hold. Then, mapping $\mathcal{A}$ satisfies the Lipschitz condition. Namely, there exists $k \in (0, 1)$, such that

$$\|\mathcal{A}(x, y) - \mathcal{A}(x_0, y_0)\| \leq k \|(x, y) - (x_0, y_0)\|, \quad \forall (x, y), (x_0, y_0) \in \mathcal{X} \times \mathcal{X}.$$

Consequently, it is a k -set contraction.

Proof of Lemma 5. For $(x, y), (x_0, y_0) \in \mathcal{X} \times \mathcal{X}$, by virtue of Hypothesis 2, we can obtain

$$\begin{aligned}
\|\mathcal{A}(x, y) - \mathcal{A}(x_0, y_0)\| &= \|(\mathcal{A}_1(x) - \mathcal{A}_1(x_0), \mathcal{A}_2(y) - \mathcal{A}_2(y_0))\| \\
&= \|\mathcal{A}_1(x) - \mathcal{A}_1(x_0)\| + \|\mathcal{A}_2(y) - \mathcal{A}_2(y_0)\| \\
&= \max_{t \in [0,1]} |(1 - \frac{\delta_1}{\delta_2}t)(h_1(x) - h_1(x_0))| + \max_{t \in [0,1]} |(1 - \frac{\gamma_1}{\gamma_2}t)(h_2(y) - h_2(y_0))| \\
&\leq (1 + \frac{\delta_1}{\delta_2})k_1 \max_{t \in [0,1]} |x - x_0| + (1 + \frac{\gamma_1}{\gamma_2})k_2 \max_{t \in [0,1]} |y - y_0| \\
&= (1 + \frac{\delta_1}{\delta_2})k_1 \|x - x_0\| + (1 + \frac{\gamma_1}{\gamma_2})k_2 \|y - y_0\| \\
&\leq k(\|x - x_0\| + \|y - y_0\|) \\
&:= k\|(x, y) - (x_0, y_0)\|,
\end{aligned}$$

where $k = \max\{(1 + \frac{\delta_1}{\delta_2})k_1, (1 + \frac{\gamma_1}{\gamma_2})k_2\} < 1$.

Therefore, mapping $\mathcal{A}$ satisfies the Lipschitz condition. By Proposition 1, we immediately get that mapping $\mathcal{A}$ is a k -set contraction. $\square$

Lemma 6. Assume that Hypothesis 1 and Hypothesis 3 hold. Then, mapping $\mathcal{B} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is completely continuous.

Proof of Lemma 6. This proof will be carried out in two steps.

Claim 1. Mapping $\mathcal{B}$ is continuous.

In fact, for each $r > 0$, set $U_r(0) := \{(x, y) \in \mathcal{X} \times \mathcal{X} : \|(x, y)\| < r\}$. Let $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ be a sequence in $U_r(0)$, such that

$$(x_n, y_n) \rightarrow (x_0, y_0), \text{ as } n \rightarrow +\infty. \quad (17)$$

Now, we show

$$\|\mathcal{B}(x_n, y_n) - \mathcal{B}(x_0, y_0)\| \rightarrow 0, \text{ as } n \rightarrow +\infty. \quad (18)$$

To this end, notice that, for each, $t \in [0, 1]$,

$$\begin{aligned}
|\mathcal{B}_1(x_n, y_n)(t) - \mathcal{B}_1(x_0, y_0)(t)| &= \left| \int_0^1 G_1(t, s) \varphi_1(s, y_n(s)) ds - \int_0^1 G_1(t, s) \varphi_1(s, y_0(s)) ds \right| \\
&= \left| \int_0^1 G_1(t, s) [\varphi_1(s, y_n(s)) - \varphi_1(s, y_0(s))] ds \right| \\
&\leq \left(\max_{s \in [0,1]} |G_1(t, s)| \right) \cdot \left(\int_0^1 |\varphi_1(s, y_n(s)) - \varphi_1(s, y_0(s))| ds \right).
\end{aligned}$$

Taking the maximum for both sides of the above inequality in $t \in [0, 1]$, one can find that

$$\|\mathcal{B}_1(x_n, y_n) - \mathcal{B}_1(x_0, y_0)\| \leq \left(\max_{t, s \in [0,1]} |G_1(t, s)| \right) \cdot \left(\int_0^1 |\varphi_1(s, y_n(s)) - \varphi_1(s, y_0(s))| ds \right).$$

This, together with Lemmas 2 and 3, implies that

$$\|\mathcal{B}_1(x_n, y_n) - \mathcal{B}_1(x_0, y_0)\| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Similarly,

$$\|\mathcal{B}_2(x_n, y_n) - \mathcal{B}_2(x_0, y_0)\| \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Claim 2. Mapping $\mathcal{B}$ is compact.

To this end, let $\{(u_n, v_n)\}_{n \in \mathbb{N}}$ be a sequence in $U_r(0)$. Then, one can see that $\mathcal{B}(\{(u_n, v_n)\}_{n \in \mathbb{N}})$ is bounded. So, we need to only show that $\mathcal{B}(\{(u_n, v_n)\}_{n \in \mathbb{N}})$ is equicontinuous.

In fact, the definition of $\{(u_n, v_n)\}_{n \in \mathbb{N}}$, together with Lemma 3, guarantees that there exists $M_1 > 0$ such that

$$|\varphi_1(t, v_n(t))| \leq M_1, \quad \forall n \in \mathbb{N}, t \in [0, 1].$$

Thus, for $t, \ell \in [0, 1]$, we can have that

$$\begin{aligned} |\mathcal{B}_1(u_n, v_n)(t) - \mathcal{B}_1(u_n, v_n)(\ell)| &= \left| \int_0^1 G_1(t, s) \varphi_1(s, v_n(s)) ds - \int_0^1 G_1(\ell, s) \varphi_1(s, v_n(s)) ds \right| \\ &\leq \int_0^1 |G_1(t, s) - G_1(\ell, s)| \cdot |\varphi_1(s, v_n(s))| ds \\ &\leq M_1 \int_0^1 |G_1(t, s) - G_1(\ell, s)| ds. \end{aligned}$$

Thus, $\mathcal{B}_1(\{(u_n, v_n)\}_{n \in \mathbb{N}})$ is equicontinuous. Similarly, $\mathcal{B}_2(\{(u_n, v_n)\}_{n \in \mathbb{N}})$ is equicontinuous. As a result, $\mathcal{B}(E)$ is equicontinuous. Therefore, by Arzela–Ascoli theorem, $\mathcal{B}(E)$ is compact.

To sum up, the conclusion of this lemma follows. $\square$

Now we are in a position to prove our existence result of solutions in the following.

Theorem 2. Assume that Hypotheses 1–3 hold. Then problem (1) has at least one solution in $\mathcal{X} \times \mathcal{X}$.

Proof of Theorem 2. From Lemmas 5 and 6, we know that $\mathcal{A}$ is a k -set contraction and $\mathcal{B}$ is a 0-set contraction, respectively. Thus, Definition 3 and Proposition 3 indicate that $\mathcal{T}$ is condensing. Define

$$\Lambda := \{(x, y) \in \mathcal{X} \times \mathcal{X} : \exists \xi \in [0, 1] \text{ with } (x, y) = \xi \mathcal{T}(x, y)\} \subset \mathcal{X} \times \mathcal{X}.$$

Now, we claim that Λ is bounded in $\mathcal{X} \times \mathcal{X}$.

To this end, by Hypothesis 2, one can get that for $(x, y) \in \Lambda$,

$$\begin{aligned} \|\mathcal{A}(x, y)\| &= \|\mathcal{A}_1(x)\| + \|\mathcal{A}_2(y)\| \\ &= \max_{t \in [0, 1]} \left| \left(1 - \frac{\delta_1}{\delta_2} t\right) h_1(x) \right| + \max_{t \in [0, 1]} \left| \left(1 - \frac{\gamma_1}{\gamma_2} t\right) h_2(y) \right| \\ &\leq \left(1 + \frac{\delta_1}{\delta_2}\right) (\|h_1(0)\| + k_1 \|x\|) + \left(1 + \frac{\gamma_1}{\gamma_2}\right) (\|h_2(0)\| + k_2 \|y\|) \\ &= \left\{ \left(1 + \frac{\delta_1}{\delta_2}\right) \|h_1(0)\| + \left(1 + \frac{\gamma_1}{\gamma_2}\right) \|h_2(0)\| \right\} + k \|(x, y)\| \\ &:= Y_a + k \|(x, y)\|, \end{aligned} \quad (19)$$

where $Y_a = \left(1 + \frac{\delta_1}{\delta_2}\right) \|h_1(0)\| + \left(1 + \frac{\gamma_1}{\gamma_2}\right) \|h_2(0)\|$ and

$$k = \max \left\{ \left(1 + \frac{\delta_1}{\delta_2}\right) k_1, \left(1 + \frac{\gamma_1}{\gamma_2}\right) k_2 \right\} < 1.$$

$$\begin{aligned} \|\mathcal{B}(x, y)\| &= \|\mathcal{B}_1(x, y)\| + \|\mathcal{B}_2(x, y)\| \\ &= \left| \int_0^1 G_1(t, s) \varphi_1(s, y(s)) ds \right| + \left| \int_0^1 G_2(t, s) \varphi_2(s, x(s)) ds \right| \\ &\leq \left(\int_0^1 \max_{t \in [0, 1]} |G_1(t, s)| ds + \int_0^1 |\varphi_1(s, y(s))| ds \right) \\ &\quad + \left(\int_0^1 \max_{t \in [0, 1]} |G_2(t, s)| ds + \int_0^1 |\varphi_2(s, x(s))| ds \right) \\ &:= Y_b. \end{aligned} \quad (20)$$

From (19) and (20), it follows that

$$\|(x, y)\| \leq \|\mathcal{T}(x, y)\| = \|\mathcal{A}(x, y)\| + \|\mathcal{B}(x, y)\| \leq Y_a + Y_b + k\|(x, y)\|.$$

Thus,

$$\|(x, y)\| \leq \frac{Y_a + Y_b}{1 - k}.$$

Then, Λ is bounded in $\mathcal{X} \times \mathcal{X}$. Hence, by Theorem 1, $\mathcal{T}$ has at least one fixed point. The conclusion of this theorem follows. $\square$

5. Stable Results

In this section, we investigate the Ulam Stability for problem (1). First, let us list the following assumption.

Hypothesis 4. *There exist constants $c_\alpha, c_\beta \in (0, +\infty)$ and $d_\alpha, d_\beta \in (0, 1)$, such that*

$$|f_1(t, y, u) - f_1(t, y_0, u_0)| \leq c_\alpha |y - y_0| + d_\alpha |u - u_0|, \quad \forall y, y_0, u, u_0 \in \mathbb{R},$$

$$|f_2(t, x, v) - f_2(t, x_0, v_0)| \leq c_\beta |x - x_0| + d_\beta |v - v_0|, \quad \forall x, x_0, v, v_0 \in \mathbb{R}.$$

Next, for some $\varepsilon = (\varepsilon_1, \varepsilon_2) > 0$, consider the following differential inequalities:

$$\begin{cases} |\mathcal{D}^\alpha x(t) - f_1(t, y(t), \mathcal{D}^\alpha x(t))| \leq \varepsilon_1, & 0 < t < 1, \\ |\mathcal{D}^\beta y(t) - f_2(t, x(t), \mathcal{D}^\beta y(t))| \leq \varepsilon_2, & 0 < t < 1, \\ x(0) = h_1(x), \quad x(1) = \int_0^1 g_1(t)x(t)dt, \\ y(0) = h_2(y), \quad y(1) = \int_0^1 g_2(t)y(t)dt. \end{cases} \quad (21)$$

Definition 4. ([40]) *Problem (1) is Ulam–Hyers stable, if there exists $L_{(1,2)} = (L_1, L_2) > 0$, such that for each solution $(x, y) \in \mathcal{X} \times \mathcal{X}$ of (21), there is a unique solution $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{X}$ of (1) with*

$$|(x, y)(t) - (\bar{x}, \bar{y})(t)| \leq L_{(1,2)}\varepsilon, \quad t \in [0, 1].$$

Definition 5. ([40]) *Problem (1) is generalized Ulam–Hyers stable, if there exists $\Phi_{(1,2)} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi_{(1,2)}(0) = 0$, such that for each solution $(x, y) \in \mathcal{X} \times \mathcal{X}$ of (21), there is a unique solution $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{X}$ of (1) with*

$$|(x, y)(t) - (\bar{x}, \bar{y})(t)| \leq \Phi_{(1,2)}(\varepsilon), \quad t \in [0, 1].$$

Moreover, we also need the following lemma in stability analysis.

Lemma 7. *Let $(x, y) \in \mathcal{X} \times \mathcal{X}$ be a solution of inequalities (21). Then, the following inequalities hold:*

$$\begin{cases} |x(t) - (1 - \frac{\delta_1}{\delta_2}t)h_1(x) - \int_0^1 G_1(t, s)f_1(s, y(s), \mathcal{D}^\alpha x(s))ds| \leq L_1\varepsilon_1, \\ |y(t) - (1 - \frac{\gamma_1}{\gamma_2}t)h_2(y) - \int_0^1 G_2(t, s)f_2(s, x(s), \mathcal{D}^\beta y(s))ds| \leq L_2\varepsilon_2, \end{cases}$$

$$\text{where } L_1 = \max_{t \in [0, 1]} \int_0^1 |G_1(t, s)|ds, \quad L_2 = \max_{t \in [0, 1]} \int_0^1 |G_2(t, s)|ds.$$

Proof of Lemma 7. Obviously, there exist $\Phi_1, \Phi_2 \in \mathcal{X}$, which depend on x, y , respectively, such that $|\Phi_1(t)| \leq \varepsilon_1, |\Phi_2(t)| \leq \varepsilon_2$;

and

$$\begin{cases} \mathcal{D}^\alpha x(t) = f_1(t, y(t), \mathcal{D}^\alpha x(t)) + \Phi_1(t), 0 < t < 1; \\ \mathcal{D}^\beta y(t) = f_2(t, x(t), \mathcal{D}^\beta y(t)) + \Phi_2(t), 0 < t < 1; \\ x(0) = h_1(x), \quad x(1) = \int_0^1 g_1(t)x(t)dt; \\ y(0) = h_2(y), \quad y(1) = \int_0^1 g_2(t)y(t)dt. \end{cases} \quad (22)$$

By using Lemma 4, one can see that

$$\begin{cases} x(t) = (1 - \frac{\delta_1}{\delta_2}t)h_1(x) + \int_0^1 G_1(t,s)f_1(s, y(s), \mathcal{D}^\alpha x(s))ds + \int_0^1 G_1(t,s)\Phi_1(s)ds, \\ y(t) = (1 - \frac{\gamma_1}{\gamma_2}t)h_2(y) + \int_0^1 G_2(t,s)f_2(s, x(s), \mathcal{D}^\beta y(s))ds + \int_0^1 G_2(t,s)\Phi_2(s)ds. \end{cases} \quad (23)$$

From (23), we have

$$\begin{aligned} & |x(t) - (1 - \frac{\delta_1}{\delta_2}t)h_1(x) - \int_0^1 G_1(t,s)f_1(s, y(s), \mathcal{D}^\alpha x(s))ds| \\ &= |\int_0^1 G_1(t,s)\Phi_1(s)ds| \\ &\leq \max_{t \in [0,1]} \int_0^1 |G_1(t,s)\Phi_1(s)|ds \leq L_1\varepsilon_1. \end{aligned} \quad (24)$$

Similarly,

$$|y(t) - (1 - \frac{\gamma_1}{\gamma_2}t)h_2(y) - \int_0^1 G_2(t,s)f_2(s, x(s), \mathcal{D}^\beta y(s))ds| \leq L_2\varepsilon_2.$$

The proof is complete. $\square$

Now, we are in a position to prove the main stable theorem of this section.

Theorem 3. Assume that Hypothesis 1 and (H3') hold. Then the solution of problem (1) is Ulam–Hyers stable if

$$\Delta = 1 - \frac{L_1 L_2 c_\alpha c_\beta}{(1 - d_\alpha)(1 - d_\beta)} > 0. \quad (25)$$

Furthermore, the solution of problem (1) is generalized Ulam–Hyers stable.

Proof of Lemma 3. Let $(x, y) \in \mathcal{X} \times \mathcal{X}$ be a solution of (21) and $(\bar{x}, \bar{y}) \in \mathcal{X} \times \mathcal{X}$ be the unique solution to (1). Then, in view of Lemma 4, we have that

$$\begin{cases} \bar{x}(t) = (1 - \frac{\delta_1}{\delta_2}t)h_1(\bar{x}) + \int_0^1 G_1(t,s)f_1(s, \bar{y}(s), \mathcal{D}^\alpha \bar{x}(s))ds, \\ \bar{y}(t) = (1 - \frac{\gamma_1}{\gamma_2}t)h_2(\bar{y}) + \int_0^1 G_2(t,s)f_2(s, \bar{x}(s), \mathcal{D}^\beta \bar{y}(s))ds. \end{cases} \quad (26)$$

Now, consider

$$\begin{aligned} |x(t) - \bar{x}(t)| &= |x(t) - (1 - \frac{\delta_1}{\delta_2}t)h_1(\bar{x}) - \int_0^1 G_1(t,s)f_1(s, \bar{y}(s), \mathcal{D}^\alpha \bar{x}(s))ds| \\ &\leq |x(t) - (1 - \frac{\delta_1}{\delta_2}t)h_1(\bar{x}) - \int_0^1 G_1(t,s)f_1(s, y(s), \mathcal{D}^\alpha x(s))ds| \\ &\quad + |\int_0^1 G_1(t,s)f_1(s, y(s), \mathcal{D}^\alpha x(s))ds - \int_0^1 G_1(t,s)f_1(s, \bar{y}(s), \mathcal{D}^\alpha \bar{x}(s))ds| \\ &\leq L_1\varepsilon_1 + |\int_0^1 G_1(t,s)(f_1(s, y(s), \mathcal{D}^\alpha x(s)) - f_1(s, \bar{y}(s), \mathcal{D}^\alpha \bar{x}(s)))ds|. \end{aligned}$$

By (H3'), we have

$$|x(t) - \bar{x}(t)| \leq L_1 \varepsilon_1 + \max_{t \in [0,1]} \int_0^1 |G_1(t,s)| \cdot \left(\frac{c_\alpha}{1-d_\alpha} |y(s) - \bar{y}(s)| \right) ds.$$

Thus,

$$\|x - \bar{x}\| \leq L_1 \varepsilon_1 + \frac{L_1 c_\alpha}{1-d_\alpha} \|y - \bar{y}\|. \quad (27)$$

Similarly, we also have

$$\|y - \bar{y}\| \leq L_2 \varepsilon_2 + \frac{L_2 c_\beta}{1-d_\beta} \|x - \bar{x}\|. \quad (28)$$

From (27) and (28), it follows that

$$\begin{cases} \|x - \bar{x}\| - \frac{L_1 c_\alpha}{1-d_\alpha} \|y - \bar{y}\| \leq L_1 \varepsilon_1, \\ \|y - \bar{y}\| - \frac{L_2 c_\beta}{1-d_\beta} \|x - \bar{x}\| \leq L_2 \varepsilon_2. \end{cases} \quad (29)$$

We write (29) in matrix form as

$$\begin{bmatrix} 1 & -\frac{L_1 c_\alpha}{1-d_\alpha} \\ -\frac{L_2 c_\beta}{1-d_\beta} & 1 \end{bmatrix} \cdot \begin{bmatrix} \|x - \bar{x}\| \\ \|y - \bar{y}\| \end{bmatrix} = \begin{bmatrix} L_1 \varepsilon_1 \\ L_2 \varepsilon_2 \end{bmatrix}. \quad (30)$$

Then, we get

$$\begin{bmatrix} \|x - \bar{x}\| \\ \|y - \bar{y}\| \end{bmatrix} \leq \begin{bmatrix} \frac{1}{\Omega} & \frac{L_1 c_\beta}{\Omega(1-d_\beta)} \\ \frac{L_2 c_\alpha}{\Omega(1-d_\alpha)} & \frac{1}{\Omega} \end{bmatrix} \cdot \begin{bmatrix} L_1 \varepsilon_1 \\ L_2 \varepsilon_2 \end{bmatrix}, \quad (31)$$

where $\Omega = 1 - \frac{L_1 L_2 c_\alpha c_\beta}{(1-d_\alpha)(1-d_\beta)} \neq 0$.

Further simplification gives

$$\|x - \bar{x}\| \leq \frac{L_1 \varepsilon_1}{\Omega} + \frac{L_1 L_2 c_\beta \varepsilon_2}{\Omega(1-d_\beta)},$$

$$\|y - \bar{y}\| \leq \frac{L_2 \varepsilon_2}{\Omega} + \frac{L_1 L_2 c_\alpha \varepsilon_1}{\Omega(1-d_\alpha)}.$$

From above, we have

$$\|x - \bar{x}\| + \|y - \bar{y}\| \leq \frac{L_1 \varepsilon_1}{\Omega} + \frac{L_1 L_2 c_\beta \varepsilon_2}{\Omega(1-d_\beta)} + \frac{L_2 \varepsilon_2}{\Omega} + \frac{L_1 L_2 c_\alpha \varepsilon_1}{\Omega(1-d_\alpha)}. \quad (32)$$

Taking $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$, one can get that

$$\|(x, y) - (\bar{x}, \bar{y})\| \leq L_{(1,2)} \varepsilon,$$

where

$$L_{(1,2)} = \frac{L_1 \varepsilon_1}{\Omega} + \frac{L_1 L_2 c_\beta \varepsilon_2}{\Omega(1-d_\beta)} + \frac{L_2 \varepsilon_2}{\Omega} + \frac{L_1 L_2 c_\alpha \varepsilon_1}{\Omega(1-d_\alpha)}.$$

Therefore, the solution of system (1) is Ulam–Hyers stable.

In addition, set $\Phi_{(1,2)}(\varepsilon) = L_{(1,2)}\varepsilon$, then $\Phi_{(1,2)}(0) = 0$. By Definition 5, the problem (1) is generalized Ulam–Hyers stable. $\square$

6. Illustrative Example

Here, we give an example to show the effectiveness of the obtained results. Let us consider the following implicit-type fractional differential coupled system:

$$\begin{cases} \mathcal{D}^{\frac{3}{2}}x(t) = t + \sin(y(t)) + \frac{1}{2}\mathcal{D}^{\frac{3}{2}}x(t), & 0 < t < 1; \\ \mathcal{D}^{\frac{5}{4}}y(t) = t + x(t) + \frac{1}{3}\mathcal{D}^{\frac{5}{4}}y(t), & 0 < t < 1; \\ x(0) = \frac{x}{8-2e}, \quad x(1) = \int_0^1 \frac{e^t}{2}x(t)dt; \\ y(0) = \frac{y}{3}, \quad y(1) = \int_0^1 e^{-t}y(t)dt. \end{cases} \quad (33)$$

System (33) can be regarded as a system of the form (1). Here,

$$\alpha = \frac{3}{2}, \quad h_1(x) = \frac{x}{8-2e}, \quad g_1(t) = \frac{e^t}{2};$$

$$\beta = \frac{5}{4}, \quad h_2(y) = \frac{y}{3}, \quad g_2(t) = e^{-t};$$

$$f_1(t, y(t), \mathcal{D}^\alpha x(t)) = t + \sin(y(t)) + \frac{1}{2}\mathcal{D}^{\frac{3}{2}}x(t);$$

$$f_2(t, x(t), \mathcal{D}^\beta y(t)) = t + x(t) + \frac{1}{3}\mathcal{D}^{\frac{5}{4}}y(t).$$

By careful calculation and (2), one can obtain that

$$\delta_1 = \frac{3-e}{2}, \quad \delta_2 = \frac{1}{2}, \quad \delta = \frac{1}{4-e};$$

$$\gamma_1 = e^{-1}, \quad \gamma_2 = 2e^{-1}, \quad \gamma = \frac{2}{3}.$$

Obviously, we can see that Hypothesis 1 holds. Now assumptions Hypothesis 2–(H3') are satisfied if we set

$$k_1 = \frac{1}{8-2e}, \quad k_2 = \frac{1}{3}, \quad \sigma_1 = \frac{1}{2}, \quad \sigma_2 = \frac{1}{3}, \quad c_\alpha = c_\beta = 1, \quad d_\alpha = \frac{1}{2}, \quad d_\beta = \frac{1}{3}.$$

Consequently, system (33) has at least one solution by Theorem 2 and the solution of system (33) is Ulam–Hyers stable and generalized Ulam–Hyers stable by Theorem 3.

7. Discussion

In this paper, firstly by means of topological degree theory, some existence results are obtained for a class of implicit-type coupled systems with integral boundary conditions, involving Caputo fractional derivatives. As far as we know, there are no existence results available for considered systems (1). Furthermore, the nonlinear terms here need only to satisfy the Lipschitz condition for one variable, which relax the restriction of the nonlinear terms in the literatures. The main reason is that the contraction mapping principle is used to obtain the existence of implicit function (see Lemmas 2 and 3). Secondly, Ulam–Hyers stability and generalized Ulam–Hyers stability are investigated under some suitable assumptions. The method used in present paper can be applied in broadest context, such as many implicit-type systems with various boundary conditions involving integer or fractional derivatives.

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