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Common α-Fuzzy Fixed Point Results for F-Contractions with Applications

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Abstract: F-contractions have inspired a branch of metric fixed point theory committed to the generalization of the classical Banach contraction principle. The study of these contractions and α-fuzzy mappings in b-metric spaces was attempted timidly and was not successful. In this article, the main objective is to obtain common α-fuzzy fixed point results for F-contractions in b-metric spaces. Some multivalued fixed point results in the literature are derived as consequences of our main results. We also provide a non-trivial example to show the validity of our results. As applications, we investigate the solution for fuzzy initial value problems in the context of a generalized Hukuhara derivative. Our results generalize, improve and complement several developments from the existing literature.

Keywords: complete b-metric spaces; F-contractions; α-fuzzy mappings; multivalued mappings; Hukuhara derivative

MSC: 47H10; 46S40; 54H25

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1. Introduction

Zadeh [1] introduced the notion of fuzzy set as a means of dealing with unpredictability that is induced due to inaccuracy or obscurity in preference to haphazardness in 1960. Heilpern [2] introduced a class of fuzzy mappings by using the notion of fuzzy sets. He obtained fixed point results for fuzzy mappings in metric linear space and generalized various results for multivalaued mappings. Estruch et al. [3] established the existence of a fuzzy fixed point for fuzzy contraction mappings in the context of complete metric space. Several mathematicians [4–12] extended the work of Estruch et al. [3] in different metric spaces under generalized contractions. In 2014, Rashid et al. [13] introduced the notion of β-admissible for a pair of fuzzy mappings by utilizing the concept of β-admissible, which was first given by Samet et al. [14] in 2012.

On the other hand, Czerwik [15] initiated the notion of b-metric space to generalize metric space in 1993. Later on, Czerwik [16,17] defined the Hausdorff b-metric induced by the b-metric and obtained fixed point theorems for multivalued mappings.

In 2012, Wardowski [18] initiated a new notion of F-contraction and established a generalized theorem regarding F-contractions in the context of complete metric spaces. Many researchers [19–21] established several types of fixed point results by using and extending the F-contraction. Recently, Cosentino et al. [22] utilized the concept of F-contraction in the framework of b-metric space and proved fixed point theorems for multivalued mappings. Ali et al. [23,24] used the notion of β-admissible mappings and F-contractions to obtain Feng and Liu type fixed point results in the context of b-metric space.

In this paper, we establish some common α-fuzzy fixed point theorems for β-admissible mappings and F-contractions in the setting of complete b-metric space to generalize the main results of Ahmad et al. [7], Wardowski [18] and Cosentino et al. [22] and some familiar theorems of the literature.

2. Background and Preliminaries

In this section, we collect some basic definitions, lemmas and notation which will be used throughout the paper (see [1,2,13–16,18,22–24] and the references therein). Let $\mathbb{R}^+$ represent the set of all positive real numbers and $\mathbb{R}_0^+$ represent the set of nonnegative real numbers.

A fuzzy set in $\mathcal{R}$ is a function with domain $\mathcal{R}$ and values in $[0,1]$. If $\mathcal{L}$ is a fuzzy set and $\omega \in \mathcal{R}$, then the function values $\mathcal{L}(\omega)$ are called the grade of membership of ω in $\mathcal{L}$. The α-level set of $\mathcal{L}$ is represented by $[\mathcal{L}]_\alpha$ and is given as follows:

$$[\mathcal{L}]_\alpha = \{\omega : \mathcal{L}(\omega) \geq \alpha\} \text{ if } \alpha \in (0,1],$$

$$[\mathcal{L}]_0 = \overline{\{\omega : \mathcal{L}(\omega) > 0\}}$$

where $\overline{\mathcal{L}}$ represents the closure of $\mathcal{L}$. If V is a metric linear space, then a fuzzy set $\mathcal{L}$ in V is said to be an approximate quantity if and only if $[\mathcal{L}]_\alpha$ is compact and convex in V for each $\alpha \in [0,1]$ and $\sup\limits_{\omega \in V} \mathcal{L}(\omega) = 1$. Let $\Im(\mathcal{R})$ be the collection of all fuzzy sets in $\mathcal{R}$. Suppose $\mathcal{R}_1$ is any set, $\mathcal{R}_2$ is a metric space. A mapping $\mathcal{O} : \mathcal{R}_1 \to \Im(\mathcal{R}_2)$ is a fuzzy subset on $\mathcal{R}_1 \times \mathcal{R}_2$ with membership $\mathcal{O}(\omega)(\varpi)$, where $\mathcal{O}(\omega)(\varpi)$ denotes the grade of membership of ϖ in $\mathcal{O}(\omega)$.

Definition 1. *(see [2]) Let $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$; then a point $\omega^* \in \mathcal{R}$ is called an α-fuzzy fixed point of $\mathcal{O}_1$ if there exists $\alpha \in [0,1]$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_\alpha$ and $\omega^* \in \mathcal{R}$ is called a common α-fuzzy fixed point of $\mathcal{O}_1$ and $\mathcal{O}_2$ if there exists $\alpha \in [0,1]$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_\alpha \cap [\mathcal{O}_2\omega^*]_\alpha$. Whenever $\alpha = 1$, then ω^* becomes a common fixed point of $\mathcal{O}_1$ and $\mathcal{O}_2$.*

Samet et al. [14] initiated the notion of β-admissible mapping in 2012.

Definition 2. *(see [14]) Let $\mathcal{O} : \mathcal{R} \to \mathcal{R}$ and $\alpha : \mathcal{R} \times \mathcal{R} \to [0, +\infty)$. Then the mapping $\mathcal{O}$ is called β-admissible if*

$$\omega, \varpi \in \mathcal{W}, \quad \beta(\omega, \varpi) \geq 1 \quad \Longrightarrow \quad \beta(\mathcal{O}\omega, \mathcal{O}\varpi) \geq 1.$$

In 2014, Rashid et al. [13] extended the concept of β-admissible for fuzzy mappings and introduced the notion of β-admissible in this way.

Definition 3. *(see [13]) Let $(\mathcal{R}, d)$ be a metric space, $\beta : \mathcal{R} \times \mathcal{R} \to [0, +\infty)$ and let $\mathcal{O}_1, \mathcal{O}_2$ be fuzzy mapping from $\mathcal{R}$ into $\Im(\mathcal{R})$. The pair $(\mathcal{O}_1, \mathcal{O}_2)$ is said to be β-admissible if these conditions hold:*

(i) For each $\omega \in \mathcal{R}$ and $\varpi \in [\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, where $\alpha_{\mathcal{O}_1}(\omega) \in (0,1]$, with $\beta(\omega, \varpi) \geq 1$, we have $\beta(\varpi, z) \geq 1$ for all $z \in [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \neq \emptyset$, where $\alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$,

(ii) For each $\omega \in \mathcal{R}$ and $\varpi \in [\mathcal{O}_2\omega]_{\alpha_{\mathcal{O}_2}(\omega)}$, where $\alpha_{\mathcal{O}_2}(\omega) \in (0,1]$, with $\beta(\omega, \varpi) \geq 1$, we have $\beta(\varpi, z) \geq 1$ for all $z \in [\mathcal{O}_1\varpi]_{\alpha_{\mathcal{O}_1}(\varpi)} \neq \emptyset$, where $\alpha_{\mathcal{O}_1}(\varpi) \in (0,1]$.

Later on, many researchers [4–12] used this notion of fuzzy mapping and established various fuzzy fixed point results.

On the other hand, Czerwik [15] introduced the notion of b-metric space to generalize metric space in 1993 in this way:

Definition 4. *Let $\mathcal{R} \neq \emptyset$ and $s \geq 1$. A function d_b: $\mathcal{R} \times \mathcal{R} \to \mathbb{R}_0^+$ is called b-metric if these conditions hold:*

(b_1) $d_b(\omega, \varpi) = 0 \Leftrightarrow \omega = \varpi$,
(b_2) $d_b(\omega, \varpi) = d_b(\varpi, \omega)$,
(b_3) $d_b(\omega, v) \leq s(d_b(\omega, \varpi) + d_b(\varpi, v))$
for all $\omega, \varpi, v \in \mathcal{R}$.
Then $(\mathcal{R}, d_b, s)$ is called a b-metric space.

Czerwik [16] defined the Hausdorff b-metric H_b induced by the b-metric d_b in this way.

Let $P_{cb}(\mathcal{R})$ represent the class of all non-empty, closed and bounded subsets of $\mathcal{R}$. For $\Omega_1, \Omega_2 \in P_{cb}(\mathcal{R})$, Hausdorff b-metric $H_b : P_{cb}(\mathcal{R}) \times P_{cb}(\mathcal{R}) \to \mathbb{R}^+$ is defined as follows

$$H_b(\Omega_1, \Omega_2) = \max\left\{ \sup_{\omega \in \Omega_1} d_b(\omega, \Omega_2), \sup_{\varpi \in \Omega_2} d_b(\varpi, \Omega_1) \right\}$$

where

$$d_b(\varpi, \Omega_1) = \inf_{\omega \in \Omega_1} d_b(\varpi, \omega).$$

We remember these properties from [15–17].

Lemma 1. *(see. [16,17]) Let $(\mathcal{R}, d_b, s)$ be a b-metric space. For any $\Omega_1, \Omega_2, \Omega_3 \in P_{cb}(\mathcal{R})$ and any $\omega, \varpi \in \mathcal{R}$, these hold:*

(i) $d_b(\omega, \Omega_2) \leq d_b(\omega, \varpi)$ for any $\varpi \in \Omega_2$.
(ii) $d_b(\omega, \Omega_2) \leq H_b(\Omega_1, \Omega_2)$ for any $\omega \in \Omega_1$,
(iii) $H_b(\Omega_1, \Omega_1) = 0$,
(iv) $H_b(\Omega_1, \Omega_2) = H_b(\Omega_2, \Omega_1)$
(v) $H_b(\Omega_1, \Omega_3) \leq s[H_b(\Omega_1, \Omega_2) + H_b(\Omega_2, \Omega_3)]$
(vi) $d_b(\omega, \Omega_1) \leq s[d_b(\omega, \varpi) + d_b(\varpi, \Omega_1)]$.
(vii) d_b is continuous in its variables.

In 2012, Wardowski [18] initiated a new notion of F-contraction and established a generalized theorem regarding F-contractions in the context of complete metric spaces.

Definition 5. *(see [18]) Let $(\mathcal{R}, d)$ be a metric space and $\mathcal{O} : \mathcal{R} \to \mathcal{R}$. Then $\mathcal{O}$ is called an F-contraction if there exists $\tau > 0$ such that ;*

$$d(\mathcal{O}\omega, \mathcal{O}\varpi) > 0 \Longrightarrow \tau + F\big(d(\mathcal{O}\omega, \mathcal{O}\varpi)\big) \leq F\big(d(\omega, \varpi)\big) \tag{1}$$

for $\omega, \varpi \in \mathcal{R}$, where $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following assertions:

(F_1) $F(\omega) < F(\varpi)$ for $\omega < \varpi$;
(F_2) For all $\{\omega_n\} \subseteq R^+$, $\lim_{n\to\infty} \omega_n = 0 \Longleftrightarrow \lim_{n\to\infty} F(\omega_n) = -\infty$;
(F_3) There exists $0 < r < 1$ such that $\lim_{\omega\to 0^+} \omega^r F(\omega) = 0$.

Many researchers [19–21,25] established several types of fixed point results by using and extending the F-contraction. In the framework of b-metric space, Cosentino et al. [22] added a new condition (F_4) and opened a new area of research in this way:

(F_4) For $s \geq 1$ and each sequence $\{\omega_n\} \subseteq \mathbb{R}^+$ such that $\tau + F(s\omega_n) \leq F(\omega_{n-1})$, $\forall\, n \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^n\omega_n) \leq F(s^{n-1}\omega_{n-1})$, for all $n \in \mathbb{N}$.

We represent by $\digamma_s$ the set of all functions continuous from the right, $F : \mathbb{R}^+ \to \mathbb{R}$, satisfying (1) and ($F_1$)–($F_4$).

Example 1. *The following functions $F : \mathbb{R}^+ \to \mathbb{R}$ are the elements of $\digamma_s$:*

(1) $F(\iota) = \ln(\iota)$;
(2) $F(\iota) = \iota + \ln(\iota)$;
(3) $F(\iota) = -\frac{1}{\sqrt{\iota}}$;
(4) $F(\iota) = \ln(\iota^2 + \iota)$
for $\iota > 0$.

3. Results and Discussion

We present our main theorem as follows:

Theorem 1. *Let $(\mathcal{R}, d_b, s)$ be a complete b-metric space with coefficient $s \geq 1$, $\beta : \mathcal{R} \times \mathcal{R} \to [0, \infty)$ and $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ satisfying the following conditions.*

(a) For each $\omega, \varpi \in \mathcal{R}$, there exists $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$,

(b) For $\omega_0 \in \mathcal{R}$, there exists $\omega_1 \in [\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}$ with $\beta(\omega_0, \omega_1) \geq 1$,

(c) There exist $F \in \digamma_s$ and $\tau > 0$ such that

$$2\tau + \max\{\beta(\omega,\varpi), \beta(\varpi,\omega)\}F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \leq F(d_b(\omega,\varpi)) \tag{2}$$

for all $\omega, \varpi \in \mathcal{R}$ with $H_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$,

(d) $(\mathcal{O}_1, \mathcal{O}_1)$ is β-admissible,

(e) If $\{\omega_n\} \in \mathcal{R}$ such that $\beta(\omega_n, \omega_{n+1}) \geq 1$ and $\omega_n \to \omega$, then $\beta(\omega_n, \omega) \geq 1$.

Then there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

Proof. For $\omega_0 \in \mathcal{R}$. Then by supposition (a), there exists $\alpha_{\mathcal{O}_1}(\omega_0) \in (0,1]$ such that $[\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)} \in P_{cb}(\mathcal{R})$ and $\omega_1 \in [\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}$ such that $\beta(\omega_0,\omega_1) \geq 1$. For this ω_1, there exists $\alpha_{\mathcal{O}_2}(\omega_1) \in (0,1]$ such that $[\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)} \in P_{cb}(\mathcal{R})$. Since $F \in \digamma_s$ is continuous from the right function, there exists $h > 1$ such that

$$F\Big(hsH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) < F\Big(sH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau. \tag{3}$$

Next, as $d\Big(\omega_1, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big) < hH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)$, we deduce that there exists $\omega_2 \in [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}$ (obviously, $\omega_2 \neq \omega_1$) such that $d_b(\omega_1,\omega_2) \leq hH_b \Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)$. Thus, we have

$$F(sd_b(\omega_1,\omega_2)) \leq F\Big(hsH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) < F\Big(sH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau$$

which implies by (2) that

$$\begin{aligned} 2\tau + F(sd_b(\omega_1,\omega_2)) &\leq 2\tau + F\Big(sH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau \\ &\leq 2\tau + \max\{\beta(\omega_0,\omega_1), \beta(\omega_1,\omega_0)\}F\Big(sH_b\Big([\mathcal{O}_1\omega_0]_{\alpha_{\mathcal{O}_1}(\omega_0)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau \\ &\leq F(d_b(\omega_0,\omega_1)) + \tau \end{aligned}$$

Thus we have

$$\tau + F(sd_b(\omega_1,\omega_2)) \leq F(d_b(\omega_1,\omega_2)). \tag{4}$$

Now, $\beta(\omega_0,\omega_1) \geq 1$ and $(\mathcal{O}_1, \mathcal{O}_1)$ is β-admissible, so $\beta(\omega_1,\omega_2) \geq 1$. For this ω_2, there exists $\alpha_{\mathcal{O}_1}(\omega_2) \in (0,1]$ such that $[\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)} \in P_{cb}(\mathcal{R})$. Since $F \in \digamma_s$ is continuous from the right function, there exists $h > 1$ such that

$$\begin{aligned} F\Big(hsH_b\Big([\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}, [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\Big)\Big) &= F\Big(hsH_b\Big([\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) \\ &< F\Big(sH_b\Big([\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau. \end{aligned} \tag{5}$$

Next, as $d_b\Big(\omega_2, [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\Big) < hH_b\Big([\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}, [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\Big)$, we deduce that there exists $\omega_3 \in [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}$ (obviously, $\omega_3 \neq \omega_2$) such that $d_b(\omega_2,\omega_3) \leq hH_b \Big([\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}, [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\Big)$. Thus, we have

$$\begin{aligned} F(sd_b(\omega_2,\omega_3)) &\leq F\Big(hsH_b\Big([\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}, [\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}\Big)\Big) \\ &< F\Big(sH_b\Big([\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau. \end{aligned}$$

which implies by (2) that

$$\begin{aligned} 2\tau + F(sd_b(\omega_2, \omega_3)) &\leq 2\tau + F\Big(sH_b\Big([\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau \\ &\leq 2\tau + \max\{\beta(\omega_1, \omega_2), \beta(\omega_2, \omega_1)\}F\Big(sH_b\Big([\mathcal{O}_1\omega_2]_{\alpha_{\mathcal{O}_1}(\omega_2)}, [\mathcal{O}_2\omega_1]_{\alpha_{\mathcal{O}_2}(\omega_1)}\Big)\Big) + \tau \\ &\leq F(d_b(\omega_2, \omega_1)) + \tau = F(d_b(\omega_1, \omega_2)) + \tau. \end{aligned}$$

Consequently, we get

$$\tau + F(sd_b(\omega_2, \omega_3)) \leq F(d_b(\omega_1, \omega_2)). \tag{6}$$

By pursuing a solution in this way, we obtain a sequence $\{\omega_n\}$ in $\mathcal{R}$ such that $\omega_{2n+1} \in [\mathcal{O}_1\omega_{2n}]_{\alpha_{\mathcal{O}_1}(\omega_{2n})}$, $\omega_{2n+2} \in [\mathcal{O}_2\omega_{2n+1}]_{\alpha_{\mathcal{O}_2}(\omega_{2n+1})}$, $\beta(\omega_{n-1}, \omega_n) \geq 1$ and

$$\tau + F(sd_b(\omega_{2n+1}, \omega_{2n+2})) \leq F(d_b(\omega_{2n}, \omega_{2n+1})) \tag{7}$$

and

$$\tau + F(sd_b(\omega_{2n+2}, \omega_{2n+3})) \leq F(d_b(\omega_{2n+1}, \omega_{2n+2})) \tag{8}$$

for all $n \in \mathbb{N}$. From (7) and (8), we get

$$\tau + F(sd_b(\omega_n, \omega_{n+1})) \leq F(d_b(\omega_{n-1}, \omega_n)) \tag{9}$$

for all $n \in \mathbb{N}$. It follows by (9) and property (F_4) that

$$\tau + F(s^n d_b(\omega_n, \omega_{n+1})) \leq F(s^{n-1} d_b(\omega_{n-1}, \omega_n)). \tag{10}$$

Therefore by (10), we have

$$\begin{aligned} F(s^n d_b(\omega_n, \omega_{n+1})) &\leq F\Big(s^{n-1} d_b(\omega_{n-1}, \omega_n)\Big) - \tau \leq F\Big(s^{n-2} d_b(\omega_{n-2}, \omega_{n-1})\Big) - 2\tau \\ &\leq \ldots \leq F(d_b(\omega_0, \omega_1)) - n\tau. \end{aligned} \tag{11}$$

Taking $n \to \infty$, we get $\lim\limits_{n\to\infty} F(s^n d_b(\omega_n, \omega_{n+1})) = -\infty$ that together with (F_2) gives

$$\lim_{n\to\infty} s^n d_b(\omega_n, \omega_{n+1}) = 0.$$

By (F_3), there exists $r \in (0, 1)$ such that

$$\lim_{n\to\infty} [s^n d_b(\omega_n, \omega_{n+1})]^r F(s^n d_b(\omega_n, \omega_{n+1})) = 0.$$

From (11) we have

$$\begin{aligned} &[s^n d_b(\omega_n, \omega_{n+1})]^r F(s^n d_b(\omega_n, \omega_{n+1})) - [s^n d_b(\omega_n, \omega_{n+1})]^r F(d_b(\omega_0, \omega_1)) \\ \leq\ &[s^n d_b(\omega_n, \omega_{n+1})]^r [F(d_b(\omega_0, \omega_1)) - n\tau] - [s^n d_b(\omega_n, \omega_{n+1})]^r F(d_b(\omega_0, \omega_1)) \\ \leq\ &-n\tau[s^n d_b(\omega_n, \omega_{n+1})]^r \leq 0. \end{aligned}$$

Taking $n \to \infty$, we get

$$\lim_{n\to\infty} n[s^n d_b(\omega_n, \omega_{n+1})]^r = 0. \tag{12}$$

Thus $\lim\limits_{n\to\infty} n^{\frac{1}{r}} s^n d_b(\omega_n, \omega_{n+1}) = 0$. Hence $\sum_{n=1}^{\infty} s^n d_b(\omega_n, \omega_{n+1})$ is convergent and thus $\{\omega_n\}$ is a Cauchy sequence in $\mathcal{R}$. Since $(\mathcal{R}, d_b, s)$ is complete, there exists $\omega^* \in \mathcal{R}$ such that

$$\lim_{n\to\infty} \omega_n = \omega^*. \tag{13}$$

By condition (e), we have $\beta(\omega_n, \omega^*) \geq 1$ for all $n \in \mathbb{N}$. Now, we prove that $\omega^* \in [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$. We assume on the contrary that $\omega^* \notin [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$, so there exist $n_0 \in \mathbb{N}$ and $\{\omega_{n_k}\}$ of $\{\omega_n\}$ such that $d_b(\omega_{2n_k+1}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}) > 0$, $\forall\, n_k \geq n_0$. Now, using (2) with $\omega = \omega_{2n_k+1}$ and $\varpi = \omega^*$, we obtain

$$2\tau + F\left[sH_b\big([\mathcal{O}_1\omega_{2n_k}]_{\alpha_{\mathcal{O}_1}(\omega_{2n_k})}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}\big)\right] \leq F(d(\omega_{2n_k}, \omega^*)).$$

This implies that

$$\begin{aligned} & 2\tau + F\left[d_b(\omega_{2n_k+1}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)})\right] \\ \leq\ & 2\tau + F\left[sH_b\big([\mathcal{O}_1\omega_{2n_k}]_{\alpha_{\mathcal{O}_1}(\omega_{2n_k})}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}\big)\right] \\ \leq\ & 2\tau + \max\{\beta(\omega_{2n_k}, \omega^*), \beta(\omega^*, \omega_{2n_k})\} F\left[sH_b\big([\mathcal{O}_1\omega_{2n_k}]_{\alpha_{\mathcal{O}_1}(\omega_{2n_k})}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}\big)\right] \\ \leq\ & F(d_b(\omega_{2n_k}, \omega^*)). \end{aligned}$$

As $\tau > 0$, by (F_1) we obtain

$$d_b(\omega_{2n_k+1}, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}) < d_b(\omega_{2n_k}, \omega^*).$$

Letting $n \to \infty$, we have

$$d_b(\omega^*, [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}) \leq 0.$$

Hence $\omega^* \in [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$. Similarly, one can easily prove that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)}$. Thus $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$. □

From now on, we denote a complete b-metric space with coefficient $s \geq 1$ as $(\mathcal{R}, d_b, s)$.

Theorem 2. *Let $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega, \varpi \in \mathcal{R}$, there exists $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$. Assume that there exist $F \in \digamma_s$ and $\tau > 0$ such that*

$$2\tau + F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \leq F(d_b(\omega, \varpi)) \tag{14}$$

for all $\omega, \varpi \in \mathcal{R}$ with $H_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$.

Then there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

Proof. Set $\beta(\omega, \varpi) = 1$ for all $\omega, \varpi \in \mathcal{R}$ in Theorem 1. □

Theorem 3. *Let $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega, \varpi \in \mathcal{R}$, there exists $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$. Suppose that there exists $\lambda \in (0,1)$ such that*

$$sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) \leq \lambda d_b(\omega, \varpi) \tag{15}$$

for all $\omega, \varpi \in \mathcal{R}$. Then there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

Proof. Let $0 < \lambda < 1$ be in this way that $\lambda = e^{-2\tau}$ where $\tau > 0$ and $F(\iota) = ln(\iota)$ for $\iota > 0$. By (15), with $H_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$, we get

$$F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \leq -2\tau + F(d_b(\omega, \varpi))$$

that is
$$2\tau + F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \le F(d_b(\omega,\varpi))$$
for all $\omega,\varpi \in \mathcal{R}$. Thus by Theorem 2, there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$. □

Corollary 1. *Let $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega,\varpi \in \mathcal{R}$, there exists $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$. Assume that there exist $\lambda \in (0,1)$ such that*
$$sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})e^{sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) - d_b(\omega,\varpi)} \le \lambda d_b(\omega,\varpi) \tag{16}$$
for all $\omega,\varpi \in \mathcal{R}$. Then there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

Proof. Let $0 < \lambda < 1$ be such that $\lambda = e^{-2\tau}$ where $\tau > 0$ and $F(\iota) = \iota + ln(\iota)$ for $\iota > 0$. By (16), with $H_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$, we get
$$F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \le -2\tau + F(d_b(\omega,\varpi))$$
for all $\omega,\varpi \in \mathcal{R}$, that is, $2\tau + F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \le F(d_b(\omega,\varpi))$. Thus by Theorem 2, we get $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)}$. □

Corollary 2. *Let $\mathcal{O}_1, \mathcal{O}_2$: $\mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega,\varpi \in \mathcal{R}$, there exist $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$. Assume that there exists $\lambda \in (0,1)$ such that*
$$\begin{aligned} sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) + 1) \\ \le \lambda d_b(\omega,\varpi)(d_b(\omega,\varpi) + 1) \end{aligned} \tag{17}$$
for all $\omega,\varpi \in \mathcal{R}$. Then there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

Proof. Let $0 < \lambda < 1$ be such that $\lambda = e^{-2\tau}$ where $\tau > 0$ and $F(\iota) = ln(\iota^2 + \iota)$ for $\iota > 0$. By (17), with $H_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$; we get
$$F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \le -2\tau + F(d_b(\omega,\varpi));$$
that is,
$$2\tau + F(sH_b([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \le F(d_b(\omega,\varpi))$$
for all $\omega,\varpi \in \mathcal{R}$. Thus by Theorem 2, there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$. □

Corollary 3. *Let $\mathcal{O} : \mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega,\varpi \in \mathcal{R}$, $\exists$ $\alpha_{\mathcal{O}}(\omega), \alpha_{\mathcal{O}}(\varpi) \in (0,1]$ such that $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}$, $[\mathcal{O}\varpi]_{\alpha_{\mathcal{O}}(\varpi)} \in CB(\mathcal{R})$. Assume that there exist $F \in \digamma_s$ and $\tau > 0$ such that*
$$2\tau + F(sH_b([\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}, [\mathcal{O}\varpi]_{\alpha_{\mathcal{O}}(\varpi)})) \le F(d_b(\omega,\varpi)) \tag{18}$$
for all $\omega,\varpi \in \mathcal{R}$ with $H_b([\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}}(\varpi)}) > 0$. Then there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}\omega^*]_{\alpha_{\mathcal{O}}(\omega^*)}$.

Example 2. *Let $\mathcal{R} = \{0,1,2\}$. Define $d_b : \mathcal{R} \times \mathcal{R} \to \mathbb{R}_0^+$ by*

$$d_b(\omega,\varpi) = \begin{cases} 0, \text{ if } \omega = \varpi \\ 1, \text{ if } \omega \neq \varpi \text{ and } \omega,\varpi \in \{1,2\} \\ \frac{1}{6}, \text{ if } \omega \neq \varpi \text{ and } \omega,\varpi \in \{0,1\} \\ \frac{1}{2}, \text{ if } \omega \neq \varpi \text{ and } \omega,\varpi \in \{0,2\}. \end{cases}$$

It is very simple to show that $(\mathcal{R}, d)$ is a complete b-metric space with coefficient $s = \frac{3}{2}$. Define

$$(\mathcal{O}0)(\iota) = (\mathcal{O}_2 1)(\iota) = \begin{cases} \frac{1}{2}, \text{ if } \iota = 0 \\ 0, \text{ if } \iota = 1,2 \end{cases}$$

and

$$(\mathcal{O}2)(\iota) = \begin{cases} 0, \text{ if } \iota = 0,2 \\ \frac{1}{2}, \text{ if } \iota = 1. \end{cases}$$

Define $\alpha : \mathcal{R} \to (0,1]$ by $\alpha(\omega) = \frac{1}{2}$ for all $\omega \in \mathcal{R}$. Now we obtain that

$$[\mathcal{O}\omega]_{\frac{1}{2}} = \begin{cases} \{0\}, \text{ if } \omega = 0,1 \\ \{1\}, \text{ if } \omega = 2. \end{cases}$$

For $\omega, \varpi \in \mathcal{R}$, we get

$$H_b([\mathcal{O}0]_{\frac{1}{2}}, [\mathcal{O}2]_{\frac{1}{2}}) = H_b([\mathcal{O}1]_{\frac{1}{2}}, [\mathcal{O}2]_{\frac{1}{2}}) = H_b(\{0\},\{1\}) = \frac{1}{6}.$$

Taking $F(\iota) = \iota + \ln(\iota)$, for $\iota > 0$ and $\tau = \frac{1}{100} > 0$. Then

$$2\tau + F(sH_b([\mathcal{O}0]_{\frac{1}{2}}, [\mathcal{O}2]_{\frac{1}{2}})) = \frac{1}{50} + \frac{1}{4} + \ln(\frac{3}{2}.\frac{1}{6}) \leq \frac{1}{2} + \ln(\frac{1}{2}) = F(d_b(0,2))$$

also

$$2\tau + F(sH_b([\mathcal{O}1]_{\frac{1}{2}}, [\mathcal{O}2]_{\frac{1}{2}})) = \frac{1}{50} + \frac{1}{4} + \ln(\frac{3}{2}.\frac{1}{6}) \leq 1 + \ln(1) = F(d_b(1,2))$$

for all $\omega, \varpi \in \mathcal{R}$. Therefore, all conditions of Corollary 3 hold and there exists a point $0 \in \mathcal{R}$ such that $0 \in [\mathcal{O}0]_{\frac{1}{2}}$ is an α-fuzzy fixed point of $\mathcal{O}$.

Now we derive some multivalued mappings fixed point results from our main result.

Theorem 4. *Let $G_1, G_2 : \mathcal{R} \to P_{cb}(\mathcal{R})$. If there exist $F \in F_s$ and $\tau > 0$ such that*

$$2\tau + F(sH_b(G_1\omega, G_2\varpi)) \leq F(d_b(\omega,\varpi))$$

for all $\omega, \varpi \in \mathcal{R}$ with $H_b(G_1\omega, G_2\varpi) > 0$, then there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in G_1\omega^* \cap G_2\omega^*$.

Proof. Consider $\alpha : \mathcal{R} \to (0,1]$ and $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ defined by

$$\mathcal{O}_1(\omega)(\iota) = \begin{cases} \alpha(\omega), \text{ if } \iota \in G_1\omega, \\ 0, \text{ if } \iota \notin G_1\omega \end{cases}$$

and

$$\mathcal{O}_2(\omega)(\iota) = \begin{cases} \alpha(\omega), \text{ if } \iota \in G_2\omega, \\ 0, \text{ if } \iota \notin G_2\omega. \end{cases}$$

Then

$$[\mathcal{O}_1\omega]_{\alpha(\omega)} = \{\iota : \mathcal{O}_1(\omega)(\iota) \geq \alpha(\omega)\} = G_1\omega \quad \text{and} \quad [\mathcal{O}_2\omega]_{\alpha(\omega)} = \{\iota : \mathcal{O}_2(\omega)(\iota) \geq \alpha(\omega)\} = G_2\omega.$$

Thus by Theorem 2 there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)} = G_1\omega^* \cap G_2\omega^*$. $\square$

The main result of Cosentino et al. [22] can be derived by taking a single multivalued mapping in the above Corollary.

Corollary 4. *[22] Let $G : \mathcal{R} \to P_{cb}(\mathcal{R})$. If there exist $F \in \digamma_s$ and $\tau > 0$ such that*

$$2\tau + F(sH_b(G\omega, G\varpi)) \leq F(d_b(\omega, \varpi))$$

for all $\omega, \varpi \in \mathcal{R}$ with $H_b(G\omega, G\varpi) > 0$, then there exist $\omega^* \in \mathcal{R}$ such that $\omega^* \in G\omega^*$.

Remark 1. *If we put $s = 1$ in the above result and consider $G : \mathcal{R} \to \mathcal{R}$, then we get the main result of Wardowski [7] .*

Now we state some fuzzy fixed point results in the context of metric spaces.

Theorem 5. *Let $(\mathcal{R}, d)$ be a complete metric space and let $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im(\mathcal{R})$ and for each $\omega, \varpi \in \mathcal{R}$, there exist $\alpha_{\mathcal{O}_1}(\omega), \alpha_{\mathcal{O}_2}(\varpi) \in (0,1]$ such that $[\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}$, $[\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)} \in P_{cb}(\mathcal{R})$. Assume that there exist $F \in \digamma$ and $\tau > 0$ such that*

$$2\tau + F(H([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)})) \leq F(d(\omega, \varpi))$$

for all $\omega, \varpi \in \mathcal{R}$ with $H([\mathcal{O}_1\omega]_{\alpha_{\mathcal{O}_1}(\omega)}, [\mathcal{O}_2\varpi]_{\alpha_{\mathcal{O}_2}(\varpi)}) > 0$. Then there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}_1\omega^*]_{\alpha_{\mathcal{O}_1}(\omega^*)} \cap [\mathcal{O}_2\omega^*]_{\alpha_{\mathcal{O}_2}(\omega^*)}$.

For the single fuzzy mapping, we have the following result.

Corollary 5. *[7] Let $(\mathcal{R}, d)$ be a complete metric space and let $\mathcal{O} : \mathcal{R} \to \Im(\mathcal{R})$, and for each $\omega, \varpi \in \mathcal{R}$, there exist $\alpha_{\mathcal{O}}(\omega), \alpha_{\mathcal{O}}(\varpi) \in (0,1]$ such that $[\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}$, $[\mathcal{O}\varpi]_{\alpha_{\mathcal{O}}(\varpi)} \in P_{cb}(\mathcal{R})$. Assume that there exist $F \in \digamma$ and $\tau > 0$ such that*

$$2\tau + F(H([\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}, [\mathcal{O}\varpi]_{\alpha_{\mathcal{O}}(\varpi)})) \leq F(d(\omega, \varpi))$$

for all $\omega, \varpi \in \mathcal{R}$ with $H([\mathcal{O}\omega]_{\alpha_{\mathcal{O}}(\omega)}, [\mathcal{O}\varpi]_{\alpha_{\mathcal{O}}(\varpi)}) > 0$. Then there exists $\omega^* \in \mathcal{R}$ such that $\omega^* \in [\mathcal{O}\omega^*]_{\alpha_{\mathcal{O}}(\omega^*)} \cap [\mathcal{O}\omega^*]_{\alpha_{\mathcal{O}}(\omega^*)}$.

4. Applications

Fuzzy differential equations and fuzzy integral equations play significant roles in modeling dynamic systems in which uncertainties or vague notions of flourishing. These notions have been set up in distinct theoretical directions, and many applications [26–30] in practical problems have been investigated. Various frameworks for investigating fuzzy differential equations have been presented. The primary and most attractive approach is using the Hukuhara differntiability (H-differentiability) for fuzzy valued functions (see [31,32]). Consequently, the theory of fuzzy integral equations was introduced by Kaleva [33] and Seikkala [34]. In the study of existence and uniqueness conditions for solutions of fuzzy differential equations and fuzzy integral equations, numerous researchers have applied distinct fixed point results. Subrahmanyam et al. [35] established an existence and uniqueness theorem for some Volterra integral equations regarding fuzzy set-valued mappings by using the classical Banach's fixed point theorem. Villamizar-Roa et al. [36] studied the existence and uniqueness of solution of fuzzy initial value problem in the context of a generalized Hukuhara derivatives. For more details in this direction, we refer the readers to [32,33,37].

We represent $K_c(\mathbb{R})$ the set of all nonempty compact and convex subsets of $\mathbb{R}$. The Hausdorff metric H in $K_c(\mathbb{R})$ is defined as follows:

$$H(A,B) = \max\left\{ \sup_{a\in A}\inf_{b\in B} \|a-b\|_{\mathbb{R}}, \sup_{b\in B}\inf_{a\in A} \|a-b\|_{\mathbb{R}} \right\}$$

where $A, B \in K_c(\mathbb{R})$. Then $(K_c(\mathbb{R}), H)$ is a complete metric space (see [38]).

Definition 6. *A function $x : \mathbb{R} \to [0,1]$ is said to be a fuzzy number in $\mathbb{R}$ if it satisfies:*
(i) x is normal, i.e, $\exists\ t_0 \in \mathbb{R}$ such that $x(t_0) = 1$.
(ii) x is fuzzy convex; i.e, for $0 \le \lambda \le 1$,

$$x(\lambda t_1 + (1-\lambda)t_2) \ge \min\{x(t_1), x(t_2)\}$$

$\forall t_1, t_2 \in \mathbb{R}$.
(iii) x is upper semicontinuous.
(iv) $[x]^0 = cl\{t \in \mathbb{R} : x(t) > 0\}$ is compact.

Consequently, the symbol E^1 will be used to represent the collection of fuzzy number in $\mathbb{R}$ satisfying the above properties.

For $\alpha \in (0,1]$, $[x]^\alpha = \{t \in \mathbb{R} : x(t) > \alpha\} = [x_l^\alpha, x_r^\alpha]$ represents α-cut of the fuzzy set x. For $x \in E^1$ one has that $[x]^\alpha \in K_c(\mathbb{R})$ for every $\alpha \in [0,1]$. The supremum on E^1 is defined by

$$d_\infty(x_1, x_2) = \sup_{\alpha\in[0,1]} \max\left\{ \left|x_{1,l}^\alpha - x_{2,l}^\alpha\right|, \left|x_{1,r}^\alpha - x_{2,r}^\alpha\right| \right\}$$

for every $x_1, x_2 \in E^1$, where $x_r^\alpha - x_l^\alpha = diam([x])$ is said to be the diameter of $[x]$. We represent the set of all continuous fuzzy functions defined on $[0,\Lambda]$, for $\Lambda > 0$ as $C([0,\Lambda], E^1)$.

From [39], it is well-known that $C([0,\Lambda], E^1)$ is a complete metric space with respect to the metric

$$d(x_1, x_2) = \sup_{t\in[0,1]} d_\infty(x_1(t), x_2(t)), \;\; x_1, x_2 \in C([0,\Lambda], E^1).$$

Lemma 2. *([33]) Let $x_1, x_2 : [0,\Lambda] \to E^1$ and $\eta \in \mathbb{R}$. Then*
(i) ${}_0^t(x_1 + x_2)(t)dt = {}_0^t x_1(t)dt + {}_0^t x_2(t)dt$;
(ii) ${}_0^t \eta x_1(t)dt = \eta {}_0^1 x_1(t)dt$;
(iii) $d_\infty(x_1(t), x_2(t))$ is integrable;
(iv) $d_\infty({}_0^t x_1(t)dt, {}_0^t x_2(t)dt) \le {}_0^t d_\infty(x_1(t), x_2(t))dt$
for $t \in [0,\Lambda]$.

Definition 7. *([36]) Let E^n denote the set of all fuzzy numbers in $\mathbb{R}^n$ and $x, y, z \in E^n$. A point z is said to be the Hukuhara difference of x and y, if the equation $x = y + z$ holds. If the Hukuhara difference of x and y exists, then it is represented by $x\, \Theta_H\, y$ (or $x - y$). It is very simple to see that $x\, \Theta_H\, x = \{0\}$, and if $x\, \Theta_H\, y$ exists, it is unique.*

Definition 8. *([36]) Let $f : (0,\Lambda) \to E^n$. The function f is said to be strongly generalized differentiable (or GH-differentiable) at $t_0 \in (0,\Lambda)$, if there exists an element $f_G^{/}(t_0) \in E^n$ such that there exist the Hukuhara differences:*

$$f(t_0 + \delta)\Theta_H f(t_0), f(t_0)\Theta_H f(t_0 - \delta)$$

and

$$\lim_{\delta \to 0^+} \frac{f(t_0+\delta)\Theta_H f(t_0)}{\delta} = \lim_{\delta \to 0^+} \frac{f(t_0)\Theta_H f(t_0-\delta)}{\delta} = f_G^{/}(t_0).$$

Consider the following fuzzy initial value problem

$$\begin{cases} x^{/}(t) = f(t, x(t)), \quad t \in J = [0, \Lambda] \\ x(0) = x_0, \end{cases} \tag{19}$$

where $x^{/}$ is taken as GH-differentiable and the fuzzy function $f : J \times E^1 \to E^1$ is continuous. The initial datum x_0 is assumed in E^1. We represent the set of all continuous fuzzy functions $f : J \to E^1$ with continuous derivatives as $C^1(J, E^1)$.

Lemma 3. *A function $x \in C^1(J, E^1)$ is a solution of the fuzzy initial value problem (19) if and only if it satisfies the fuzzy Volterra integral equation:*

$$x(t) = x_0 \Theta_H (-1)_0^t f(s, x(s))ds, \quad t \in J = [0, \Lambda].$$

Theorem 6. *Let $f : J \times E^1 \to E^1$ be continuous such that*

(i) The function f is strictly increasing in the second variable; that is, if $x < y$,then $f(t, x) < f(t, y)$,

(ii) There exists $\tau \in [1, +\infty)$ such that

$$\|f(t, x(t)) - f(t, y(t))\| \leq \tau e^{-2\tau} \max_{t \in J}\{d_\infty(x, y)e^{-\tau t}\}.$$

if $x < y$ for each $t \in J$ and $x, y \in E^1$, where $d_\infty(x, y)$ is the supremum on E^1. Then the fuzzy initial valued problem (19) has a fuzzy solution in $C^1(J, E^1)$.

Proof. Let $\tau > 0$. We consider the space $C^1(J, E^1)$ endowed with the weighted metric

$$d_\tau(x, y) = \sup_{t \in J}\{d_\infty(x(t), y(t))e^{-\tau t}\},$$

$x, y \in C^1(J, E^1)$. Let $M, Q : X \to (0, 1]$ be any two mappings. For $x \in X$, take

$$L_x(t) = x_0 \Theta_H (-1)_0^t f(s, x(s))ds.$$

Assume $x < y$. Then by hypothesis (i),

$$\begin{aligned} L_x(t) &= x_0 \Theta_H (-1)_0^t f(s, x(s))ds \\ &< x_0 \Theta_H (-1)_0^t f(s, y(s))ds \\ &= R_y(t). \end{aligned}$$

Hence $L_x(t) \neq R_y(t)$. Consider two fuzzy mappings $\mathcal{O}_1, \mathcal{O}_2 : X \to E^X$ defined by

$$\mu_{\mathcal{O}_1 x}(r) = \begin{cases} M(x), \text{ if } r(t) = L_x(t) \\ 0, \text{ otherwise.} \end{cases}$$

$$\mu_{\mathcal{O}_2 y}(r) = \begin{cases} Q(y), \text{ if } r(t) = R_y(t) \\ 0, \text{ otherwise.} \end{cases}$$

By taking $\alpha_{\mathcal{O}_1}(x) = M(x)$ and $\alpha_{\mathcal{O}_2}(y) = Q(y)$, we have

$$\begin{aligned} [\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)} &= \{r \in X : (\mathcal{O}_1 x)(t) \geq M(x)\} \\ &= \{L_x(t)\} \end{aligned}$$

and similarly $[\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)} = \{R_y(t)\}$. Therefore

$$\begin{aligned}
H\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big) &= \max\left\{\begin{array}{c} \sup_{x\in[\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, y\in[\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}} \inf\|x-y\|_{\mathbb{R}}, \\ \sup_{y\in[\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}, x\in[\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}} \inf\|x-y\|_{\mathbb{R}} \end{array}\right\} \\
&\leq \max\left\{\sup_{t\in J}\big\|L_x(t) - R_y(t)\big\|_{\mathbb{R}}\right\} \\
&= \sup_{t\in J}\big\|L_x(t) - R_y(t)\big\|_{\mathbb{R}} \\
&= \sup_{t\in J}\big\|{}_0^t f(s, x(s))ds - {}_0^t f(s, y(s))ds\big\|_{\mathbb{R}} \\
&\leq \sup_{t\in J}\{{}_0^t \|f(s, x(s)) - f(s, y(s))\|ds\} \\
&\leq \sup_{t\in J}\Big\{{}_0^t \tau e^{-2\tau}\|x(s) - y(s)\|e^{-\tau s}e^{\tau s}ds\Big\} \\
&= \tau e^{-2\tau}\frac{1}{\tau}d_\tau(x, y)e^{\tau t}.
\end{aligned}$$

This implies that

$$H\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big)e^{-\tau t} \leq e^{-2\tau}d_\tau(x, y)$$

or equivalently

$$H_\tau\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big) \leq e^{-2\tau}d_\tau(x, y).$$

By passing to logarithms, we can write this as

$$\ln\Big(H_\tau\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big)\Big) \leq \ln\Big(e^{-2\tau}d_\tau(x, y)\Big)$$

and, after routine calculations, we get

$$2\tau + \ln\Big(H_\tau\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big)\Big) \leq \ln(d_\tau(x, y)).$$

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(t) = \ln t$, for each $t \in C^1(J, E^1)$ is in $\digamma_s$. Thus

$$2\tau + F\Big(H_\tau\Big([\mathcal{O}_1 x]_{\alpha_{\mathcal{O}_1}(x)}, [\mathcal{O}_2 y]_{\alpha_{\mathcal{O}_1}(y)}\Big)\Big) \leq F(d_\tau(x, y)).$$

It follows that there exists $x^* \in C^1(J, E^1)$ such that $x^* \in [\mathcal{O}_1 x^*]_{\alpha_{\mathcal{O}_1}(x^*)} \cap [\mathcal{O}_2 x^*]_{\alpha_{\mathcal{O}_1}(x^*)}$. Thus all the hypotheses of Theorem 5 are satisfied and consequently x^* is a fuzzy solution of the fuzzy initial valued problem (19). □

5. Conclusions

In this paper, we have obtained some generalized common α-fuzzy fixed point results for α-fuzzy mappings regarding F-contractions in the context of b-metric spaces. The results which we obtained improved and extended certain famous theorems in literature. As applications, we investigated the solution for fuzzy initial value problems in the context of a generalized Hukuhara derivative. Our results are up to date and contemporary contributions to the existing literature in the theory of fixed points. Some related extensions of these results for the L-fuzzy mappings $\mathcal{O}_1, \mathcal{O}_2 : \mathcal{R} \to \Im_L(\mathcal{R})$ would be a well defined subject for future work. One can use our theorems in the solution of fractional differential inclusions as a subsequent study.

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