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A New Family of Zeta Type Functions Involving the Hurwitz Zeta Function and the Alternating Hurwitz Zeta Function

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Abstract: In this paper, we further study the generating function involving a variety of special numbers and polynomials constructed by the second author. Applying the Mellin transformation to this generating function, we define a new class of zeta type functions, which is related to the interpolation functions of the Apostol–Bernoulli polynomials, the Bernoulli polynomials, and the Euler polynomials. This new class of zeta type functions is related to the Hurwitz zeta function, the alternating Hurwitz zeta function, and the Lerch zeta function. Furthermore, by using these functions, we derive some identities and combinatorial sums involving the Bernoulli numbers and polynomials and the Euler numbers and polynomials.

Keywords: Bernoulli numbers and polynomials; Euler numbers and polynomials; Apostol–Bernoulli and Apostol–Euler numbers and polynomials; Hurwitz–Lerch zeta function; Hurwitz zeta function; alternating Hurwitz zeta function; generating function; Mellin transformation

MSC: 05A15; 11B68; 26C0; 11M35

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1. Introduction

The families of zeta functions and special numbers and polynomials have been studied widely in many areas. They have also been used to model real-world problems. In addition to these functions, generating functions for special numbers and polynomials have also been used commonly in variety of studies (cf. [1–34]).

The motivation of this paper is to apply the Mellin transformation to the following generating function:

$$F_Y(w, k, a) = \sum_{n=0}^{\infty} \mathcal{Y}_n(k, a) \frac{w^n}{n!} = \frac{aw}{e^{(k+1)w} - e^{-(k+1)w} + e^w - e^{-w}}, \quad (1)$$

where $k \in \mathbb{Z}$ and a , and $w \in \mathbb{R}$ (or $\mathbb{C}$) (cf. [25]). This yields a new family of zeta type functions involving the Hurwitz zeta function, the alternating Hurwitz zeta function, and the Lerch zeta function. After investigating some properties of these functions, we found that these functions can interpolate the Apostol–Bernoulli polynomials, the Bernoulli numbers and polynomials, and the Euler numbers and polynomials at negative integers.

The following notations, definitions and relations are used in this paper: Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively.

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$, $\mathbb{Z}^- := \{-1, -2, -3, \dots\}$.

Moreover, the falling factorial is given by:

$$(x)_n = \begin{cases} x(x-1)(x-2) \dots (x-n+1) & \text{if } n \in \mathbb{N}, \\ 1 & \text{if } n = 0, \end{cases}$$

where $x \in \mathbb{R}$.

The Apostol–Bernoulli polynomials $\mathcal{B}_n(y; \lambda)$ are defined by means of the following generating function:

$$F_A(w, y; \lambda) = \frac{we^{wy}}{\lambda e^w - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(y; \lambda) \frac{w^n}{n!}, \quad (2)$$

(cf. [1]).

Substituting $y = 0$ into (2), we have:

$$\mathcal{B}_0(0; \lambda) = 0$$

and

$$\lambda \mathcal{B}_1(1; \lambda) = 1 + \mathcal{B}_1(\lambda)$$

and for $n \geq 2$,

$$\lambda \mathcal{B}_n(1; \lambda) = \mathcal{B}_n(\lambda),$$

(cf. [1]).

By using (2), we have:

$$\mathcal{B}_n(y; \lambda) = \sum_{j=0}^n \binom{n}{j} y^{n-j} \mathcal{B}_j(\lambda) \quad (3)$$

(cf. [1]). By using (3), the first few values of the Apostol–Bernoulli polynomials are given as follows:

$$\begin{aligned} \mathcal{B}_1(y; \lambda) &= \frac{1}{\lambda - 1}, \mathcal{B}_2(y; \lambda) = \frac{(\lambda - 1)y - 2\lambda}{(\lambda - 1)^2}, \\ \mathcal{B}_3(y; \lambda) &= \frac{3(\lambda - 1)^2 y^2 - 6\lambda(\lambda - 1)y + 3\lambda(\lambda + 1)}{(\lambda - 1)^3}, \end{aligned}$$

and so on.

Substituting $y = 1$ into (2), we have the following Apostol–Bernoulli numbers:

$$\mathcal{B}_n(1, \lambda) - \mathcal{B}_n(\lambda) = \sum_{j=0}^{n-1} \binom{n}{j} \mathcal{B}_j(\lambda),$$

where

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0; \lambda).$$

Using (2), one has the following well-known identity:

$$(1 - \lambda) \mathcal{B}_m(\lambda) - \lambda \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{B}_j(\lambda) = 0 \quad (4)$$

(cf. [24,29,31]).

By using the above identity, we have the following few values of the Apostol–Bernoulli numbers:

$$\mathcal{B}_1(\lambda) = \frac{1}{\lambda - 1}, \mathcal{B}_2(\lambda) = \frac{-2\lambda}{(\lambda - 1)^2}, \mathcal{B}_3(\lambda) = \frac{3\lambda(\lambda + 1)}{(\lambda - 1)^3}, \mathcal{B}_4(\lambda) = \frac{-4\lambda(\lambda^2 + 4\lambda + 1)}{(\lambda - 1)^4},$$

and so on (cf. [1,13,15,17–34]).

The Bernoulli polynomials $B_n(y)$ are defined by means of the following generating function:

$$F_B(w, y) = \frac{we^{wy}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(y) \frac{w^n}{n!} \quad (5)$$

(cf. [1,4–6,8,9,13–34]).

Hence, a few values of the Bernoulli polynomials are given as follows:

$$\begin{aligned} B_0(x) &= 1, B_1(x) = \frac{x-1}{2}, B_2(x) = \frac{x^2-x+1}{6}, B_3(x) = \frac{2x^3-3x^2+1x}{2}, \\ B_4(x) &= \frac{30x^4-60x^3+30x^2-1}{30}, B_5(x) = \frac{6x^5-15x^4+10x^3-x}{6}, \end{aligned}$$

and so on.

Since $B_n = B_n(0)$ denotes the Bernoulli numbers, a few values of these numbers are given as follows:

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0,$$

and so on, here note that $B_{2n+1} = 0$ for $n \geq 2$ (cf. [1,4–6,8–10,13–32]).

The Apostol–Euler polynomials of the first kind $\mathcal{E}_n(x, \lambda)$ are defined by means of the following generating function:

$$F_{P1}(t, x; k, \lambda) = \frac{2}{\lambda e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \lambda) \frac{t^n}{n!}, \quad (6)$$

and by using (6), we have:

$$\mathcal{E}_n(x; \lambda) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{E}_j(\lambda) \quad (7)$$

(cf. [13,15,17–34]).

By combining (6) with (2), we have the following well-known relation:

$$(n+1)\mathcal{E}_n(x; \lambda) + 2\mathcal{B}_{n+1}(x; -\lambda) = 0 \quad (8)$$

(cf. [29,30]).

Substituting $\lambda = 1$ into (6), we have the Euler polynomials of the first kind; that is:

$$E_n(x) = \mathcal{E}_n(x; 1).$$

With the help of the Equation (7), a few values of the Euler polynomials of the first kind are given as follows:

$$\begin{aligned} E_0(x) &= 1, E_1(x) = \frac{2x-1}{2}, E_2(x) = x^2 - x, E_3(x) = \frac{4x^3 - 6x^2 + 1}{4}, \\ E_4(x) &= x^4 - 2x^3 + x, E_5(x) = \frac{2x^5 - 5x^4 + 5x^2 - 1}{2}, \end{aligned}$$

and so on (cf. [1,4–6,8,9,13–34]).

Putting $x = 0$ in (6), we have the Apostol–Euler numbers:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0, \lambda),$$

where

$$\mathcal{E}_0(\lambda) = \frac{2}{\lambda + 1}.$$

By using (6), one has the following well-known identity:

$$\mathcal{E}_m(\lambda) = -\frac{\lambda}{1+\lambda} \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{E}_j(\lambda) \quad (9)$$

(cf. [24,29,31]).

By using the above identity, we have the following few values of the Apostol–Euler numbers:

$$\mathcal{E}_1(\lambda) = -\frac{2\lambda}{(\lambda+1)^2}, \mathcal{E}_2(\lambda) = \frac{2\lambda(\lambda-1)}{(\lambda+1)^3}, \mathcal{E}_3(\lambda) = -\frac{2\lambda(\lambda^2-4\lambda+1)}{(\lambda+1)^4},$$

and so on. Setting $\lambda = 1$ into (6), we have the Euler numbers of the first kind:

$$E_n = \mathcal{E}_n^{(1)}(1) = E_n(0),$$

which are given by:

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

(cf. [1,4–6,8,9,13–34]). Hence a few of the values of the Euler numbers of the first kind are given as follows:

$$E_0 = 1, E_1 = -\frac{1}{2}, E_2 = 0, E_3 = \frac{1}{4}, E_5 = -\frac{1}{2}, E_7 = \frac{17}{8}, E_9 = -\frac{31}{2},$$

and so on, here noting that $E_{2n} = 0$ for $n \geq 1$ (cf. [1,4–6,8–11,13–34]).

The Euler numbers of the second kind E_j^* are given by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{j=0}^{\infty} E_j^* \frac{t^j}{j!}.$$

We now give the following well-known relation between the numbers E_n^* and E_n :

$$\begin{aligned} E_n^* &= 2^n E_n \left(\frac{1}{2} \right) \\ &= \sum_{j=0}^n \binom{n}{j} 2^j E_j. \end{aligned}$$

Using the above equation, we have:

$$E_0^* = 1, E_2^* = -1, E_3^* = 0, E_4^* = 5, E_5^* = 0, E_6^* = -61, E_7^* = 0, E_8^* = 1385,$$

and so on, and here noting that $E_{2n+1}^* = 0$ for $n \geq 0$ (cf. [1,4–6,8–10,13–34]).

In [21], Simsek gave many remarks and observations on the numbers E_n^* and E_n . Simsek [21] divided the Euler numbers into two groups, naming them as the Euler numbers of the first kind E_n and the Euler numbers of the second kind E_n^* . The alternating Hurwitz zeta function can be used to interpolate these numbers at negative integers.

By using Equation (1), we have:

$$F_Y(w, k, a) = \frac{awe^{(k+1)w}}{(e^{(k+2)w} - 1)(e^{kw} + 1)}, \quad (10)$$

(cf. [25]).

Equation (10) yields the following relations among the numbers $\mathcal{Y}_n(k, a)$, the Bernoulli polynomials and numbers, and the Euler polynomials and numbers:

$$\mathcal{Y}_n(0, 2) = 2^{n-1} B_n\left(\frac{1}{2}\right) = \frac{1}{2} \sum_{j=0}^n \binom{n}{j} B_j(1) E_{n-j},$$

$$\mathcal{Y}_n(1, 2) = \sum_{j=0}^n \binom{n}{j} 3^{n-j-1} B_{n-j}\left(\frac{2}{3}\right) E_j,$$

and

$$\mathcal{Y}_n(1, 2) = \sum_{j=0}^n \binom{n}{j} 3^{n-j-1} B_{n-j} E_j(2),$$

(cf. [25]). By using (1), we arrive at the following theorem:

Theorem 1. (cf. [25]) Let $n \in \mathbb{N}_0$. Then we have:

$$\begin{aligned} & (\mathcal{Y}(k, a) + k + 1)^n - (\mathcal{Y}(k, a) - k - 1)^n + (\mathcal{Y}(k, a) + 1)^n - (\mathcal{Y}(k, a) - 1)^n \\ &= \begin{cases} 0 & \text{if } n \neq 1, \\ a & \text{if } n = 1, \end{cases} \end{aligned} \quad (11)$$

where, after expansion, each index of $\mathcal{Y}^n(k, a)$ is to be replaced by the corresponding $a \in \mathbb{R}$ and $k \in \mathbb{Z}$ as in the usual umbral convention method and the finite difference methods.

Using (11), we have the following few values of the numbers $\mathcal{Y}_n(k, a)$:

$$\mathcal{Y}_0(k, a) = \frac{a}{2(k+1)}, \mathcal{Y}_1(k, a) = 0, \mathcal{Y}_2(k, a) = -\frac{a(k+1)^3 + a}{6(k+1)^2},$$

and so on.

Theorem 2. (cf. [25]) Let $n \in \mathbb{N}_0$. Then we have:

$$\mathcal{Y}_n(k, a) = \frac{a}{2} \sum_{v=0}^n \binom{n}{v} k^{n-v} (k+2)^{v-1} B_v E_{n-v} \left(\frac{k+1}{k} \right). \quad (12)$$

Theorem 3. (cf. [25]) Let $n \in \mathbb{N}_0$. Then we have:

$$\mathcal{Y}_n(k, a) = \frac{a}{2} \sum_{v=0}^n \binom{n}{v} k^{n-v} (k+2)^{v-1} E_{n-v} B_v \left(\frac{k+1}{k+2} \right). \quad (13)$$

Identities given by Equations (12) and (13) will be used in the next section.

Family of Zeta Functions

In order to present the results of this paper, we also need the following Lerch zeta function (the Hurwitz–Lerch zeta function) $\Phi(\lambda, z, b)$, which is a generalization of the Hurwitz zeta function, the polylogarithm, and the Riemann zeta function. This function was discovered by the Czech mathematician Mathias Lerch (20 February 1860–3 August 1922) (cf. [1,4,6,27,28,30,31]).

Let $z \in \mathbb{C}$. $z = x + iy$ with $x, y \in \mathbb{R}$.

The Hurwitz–Lerch zeta function $\Phi(\lambda, z, b)$ is defined by:

$$\Phi(\lambda, z, b) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j+b)^z},$$

where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\lambda, z \in \mathbb{C}$ when $|\lambda| < 1$; and $x > 1$ when $|\lambda| = 1$ (cf. [1,4,6,27,28,30,31]).

The function $\Phi(\lambda, z, b)$ interpolates the Apostol–Bernoulli polynomials at negative integers:

$$\Phi(\lambda, 1 - n, b) = -\frac{1}{n} \mathcal{B}_n(b; \lambda), \quad (14)$$

where $n \in \mathbb{N}$, $|\lambda| < 1$ (cf. [1,11,27,28,30,31,33]).

By combining (14) with (8), we have the following known result:

$$\Phi(-\lambda, 1 - n, b) = \frac{1}{2} \mathcal{E}_{n-1}(b; \lambda), \quad (15)$$

where $n \in \mathbb{N}$.

Putting $|\lambda| = 1$ in (15), the function $\Phi(\lambda, z, b)$ reduces to the following Hurwitz zeta function $\zeta(z, b)$:

$$\zeta(z, b) = \sum_{j=0}^{\infty} \frac{1}{(j+b)^z}, \quad (16)$$

where $(z \in \mathbb{C}$ and $z = x + iy$ with $x, y \in \mathbb{R})$ $x > 1$.

The function $\zeta(z, b)$ interpolates the Bernoulli polynomials at negative integers:

$$\zeta(1 - n, b) = -\frac{1}{n} B_n(b), \quad (17)$$

where $n \in \mathbb{N}$ (cf. [4,7,8,15,18,26–32,34]).

Substituting $\lambda = -1$ into the function $\Phi(\lambda, z, b)$, we have the following well-known alternating Hurwitz zeta function $\zeta_E(z, b)$:

$$\zeta_E(z, b) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+b)^z}, \quad (18)$$

where $(z \in \mathbb{C}$ and $z = x + iy$ with $x, y \in \mathbb{R})$ $x > 1$ and $b > 0$.

The function $\zeta_E(z, b)$ interpolates the Euler polynomials at negative integers:

$$\zeta_E(-n, b) = \frac{(-1)^n}{2} E_n(1 - b),$$

where $b \in \mathbb{N}_0$ (cf. [4]) [Equation (3.20)].

In this study, we find it to be convenient to make use of the notation $\zeta_E(z, b)$, instead of their traditionally used notations $\eta(z, b)$ given by various studies (see, for example, [4,6,8,27,28,30,31]):

$$\zeta_E(z, b) := \eta(z, b) := \Phi(-1, z, b).$$

Here we note that the alternating Hurwitz zeta function is also the so-called the Hurwitz–Euler eta function (cf. [4,8,31,34]).

2. A New Family of Zeta Type Functions

There are many well-known methods to construct zeta type functions (cf. [1–34]). Here, our method relies on an application the Mellin transform to the generating function for the numbers $\mathcal{Y}_n(k, a)$.

In order to give this new family of zeta type functions, we assume that $s \in \mathbb{C}$ with $s = c + id$, $i^2 = -1$, and $c > 1$.

By applying the Mellin transformation to the generating function (10), we arrive at the following integral representation of a new family of zeta type functions $\mathcal{Z}(s; k, a)$:

$$\mathcal{Z}(s; k, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} w^{s-1} F_Y(-w, k, a) dw.$$

Hence,

$$\mathcal{Z}(s; k, a) = \frac{a}{\Gamma(s)} \int_0^\infty \frac{w^s e^{-w(k+1)}}{(e^{-w(k+2)} - 1)(e^{-kw} + 1)} dw.$$

where $(s \in \mathbb{C}$ with $s = c + id$, $c, d \in \mathbb{R}$) $c > 1$. Assuming that $|e^{-w(k+2)}| < 1$ and $|e^{-kw}| < 1$, the above equation reduces to the following relation:

$$\mathcal{Z}(s; k, a) = \frac{a}{\Gamma(s)} \sum_{m=0}^\infty \sum_{n=0}^\infty (-1)^n \int_0^\infty w^s e^{-w(kn + (k+2)m + k+1)} dw \quad (19)$$

with:

$$\min\{c, kn + (k+2)m + k+1\} > 0,$$

where the supplementary restriction $kn + (k+2)m + k+1 > 0$ is required for the convergence of the infinite integral occurring on the right-hand side at its upper terminal of Equation (19). After some elementary calculation, we get:

$$\mathcal{Z}(s; k, a) = a \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^n}{(k+1 + kn + (k+2)m)^{s+1}} \frac{1}{\Gamma(s)} \int_0^\infty z^s e^{-z} dz,$$

where $(s \in \mathbb{C}$ with $s = c + id$, $c, d \in \mathbb{R}$) $c > 1$, and $k+1 + kn + (k+2)m > 0$.

Therefore, with the aid of the Euler gamma function, the following equation is easily reached applying the partial integral to the integral on the right-hand side of the above equation:

$$\mathcal{Z}(s; k, a) = a \frac{\Gamma(s+1)}{\Gamma(s)} \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^n}{(k+1 + kn + (k+2)m)^{s+1}}.$$

Since

$$\Gamma(s+1) = s\Gamma(s),$$

we arrive at the following definition:

Definition 1. Let $s \in \mathbb{C}$ with $s = c + id$ and let $k \in \mathbb{N}$, $a, c, d \in \mathbb{R}$ with $c > 1$. A new family of zeta functions involving the alternating Hurwitz zeta function and the Hurwitz zeta function is defined by:

$$\mathcal{Z}(s; k, a) = as \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{(-1)^n}{(k+1 + kn + (k+2)m)^{s+1}}. \quad (20)$$

Here, we show that the function $\mathcal{Z}(s; k, a)$ is related to not only to the Hurwitz zeta function and the alternating Hurwitz zeta function, but also to the Lerch zeta function.

We now give the following two alternative forms of the function $\mathcal{Z}(s; k, a)$:

I. Combining (20) with (16), we obtain:

$$\mathcal{Z}(s; k, a) = \frac{as}{(k+2)^{s+1}} \sum_{n=0}^\infty (-1)^n \zeta\left(s+1, \frac{k(n+1)+1}{k+2}\right). \quad (21)$$

II. Combining (20) with (18), we get:

$$\mathcal{Z}(s; k, a) = \frac{as}{k^{s+1}} \sum_{m=0}^\infty \zeta_E\left(s+1, \frac{k+1+(k+2)m}{k}\right). \quad (22)$$

By applying the principle of analytic continuation together with the Cauchy Residue Theorem to (19) and making use of the same method as that used earlier by Srivastava et al. (see, for details, [32]), we arrive at the following theorems:

Theorem 4. Let $v \in \mathbb{N}$. Then we have:

$$\mathcal{Z}(-v; k, a) = -a \sum_{j=0}^v \binom{v}{j} \frac{k^j (k+2)^{v-j-1}}{j+1} B_{v-j} \mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right). \quad (23)$$

Proof. Substituting $s = -v$, ($v \in \mathbb{N}$) into (21), we get:

$$\mathcal{Z}(-v; k, a) = -va(k+2)^{v-1} \sum_{n=0}^{\infty} (-1)^n \zeta \left(1-v, \frac{k(n+1)+1}{k+2} \right).$$

Combining the above equation with (17), we obtain:

$$\mathcal{Z}(-v; k, a) = a(k+2)^{v-1} \sum_{n=0}^{\infty} (-1)^n B_v \left(\frac{k(n+1)+1}{k+2} \right).$$

After some elementary calculations in the above equation, we have:

$$\mathcal{Z}(-v; k, a) = a(k+2)^{v-1} \sum_{j=0}^v \binom{v}{j} \left(\frac{k}{k+2} \right)^j B_{v-j} \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{k+1}{k} \right)^j.$$

Combining the above equation with Equation (14), which modified as follows:

$$\Phi \left(-1, -j, \frac{k+1}{k} \right) = \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{k+1}{k} \right)^j = -\frac{1}{j+1} \mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right), \quad (24)$$

we obtain:

$$\mathcal{Z}(-v; k, a) = -a \sum_{j=0}^v \binom{v}{j} \frac{k^j (k+2)^{v-j-1}}{j+1} B_{v-j} \mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right).$$

Thus the proof of theorem is completed. $\square$

Substituting $x = \frac{k+1}{k}$ into (8), we have:

$$\mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right) = -\frac{j+1}{2} \mathcal{E}_j \left(\frac{k+1}{k}; 1 \right).$$

Since

$$\mathcal{E}_j \left(\frac{k+1}{k}; 1 \right) = E_j \left(\frac{k+1}{k} \right),$$

we get:

$$\mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right) = -\frac{j+1}{2} E_j \left(\frac{k+1}{k} \right). \quad (25)$$

Combining the above equation with (23), we arrive at the following theorems:

Theorem 5. Let $v \in \mathbb{N}$. Then we have:

$$\mathcal{Z}(-v; k, a) = \frac{a}{2} \sum_{j=0}^v \binom{v}{j} k^j (k+2)^{v-j-1} B_{v-j} E_j \left(\frac{k+1}{k} \right). \quad (26)$$

Theorem 6. Let $v \in \mathbb{N}$. Then we have:

$$\mathcal{Z}(-v; k, a) = \frac{av}{2} \sum_{j=0}^{v-1} \binom{v-1}{j} \frac{k^{v-j-1} (k+2)^j}{j+1} E_{v-j-1} B_{j+1} \left(\frac{k+1}{k+2} \right). \quad (27)$$

Proof. Substituting $s = -v$, ($v \in \mathbb{N}$) into (22), we get:

$$\mathcal{Z}(-v; k, a) = -avk^{v-1} \sum_{m=0}^{\infty} \zeta_E \left(1 - v, \frac{k+1 + (k+2)m}{k} \right).$$

Applying Equation (24) to the above equation, and using (25), we obtain:

$$\mathcal{Z}(-v; k, a) = -\frac{avk^{v-1}}{2} \sum_{m=0}^{\infty} E_v \left(\frac{k+1 + (k+2)m}{k} \right).$$

Therefore:

$$\mathcal{Z}(-v; k, a) = -\frac{av}{2} \sum_{j=0}^{v-1} \binom{v-1}{j} k^{v-j-1} (k+2)^j E_{v-j-1} \sum_{m=0}^{\infty} \left(m + \frac{k+1}{k+2} \right)^j. \quad (28)$$

By using Equation (17), we have:

$$\zeta \left(-j, \frac{k+1}{k+2} \right) = \sum_{m=0}^{\infty} \left(m + \frac{k+1}{k+2} \right)^j = -\frac{1}{j+1} B_{j+1} \left(\frac{k+1}{k+2} \right)$$

Combining the above equation with (28), after some calculations, we arrive at the desired result. $\square$

By using (27), for $v \in \mathbb{N}$, we also obtain:

$$\mathcal{Z}(-v; k, a) = \frac{a}{2} \sum_{j=0}^{v-1} \binom{v}{j+1} k^{v-j-1} (k+2)^j E_{v-j-1} B_{j+1} \left(\frac{k+1}{k+2} \right). \quad (29)$$

Remark 1. With the help of the theorems given above, we have examined the behavior of the function $\mathcal{Z}(s; k, a)$. In addition, it was seen that the numbers $B_{v-j} \mathcal{B}_{j+1} \left(\frac{k+1}{k}; -1 \right)$, $B_{v-j} E_j \left(\frac{k+1}{k} \right)$ and $E_{v-j} B_j \left(\frac{k+1}{k+2} \right)$ ($v, j \in \mathbb{N}_0$ with $v \geq j$) are also interpreted by the function $\mathcal{Z}(s; k, a)$, respectively, at negative integers.

Combining (29) with (12), we arrive at the following corollary:

Corollary 1. Let $v \in \mathbb{N}$. Then we have:

$$\mathcal{Z}(-v; k, a) = \mathcal{Y}_v(k, a). \quad (30)$$

By combining (30) with (23) and (29), we get the following result:

Theorem 7. Let $v \in \mathbb{N}$. Then we have:

$$\begin{aligned} & \sum_{j=0}^{v-1} \binom{v}{j+1} k^{v-j-1} (k+2)^j E_{v-j-1} B_{j+1} \left(\frac{k+1}{k+2} \right) \\ &= \sum_{j=0}^v \binom{v}{j} k^j (k+2)^{v-j-1} B_{v-j} E_j \left(\frac{k+1}{k} \right). \end{aligned}$$

3. Conclusions

In this paper, we defined a new class zeta type function. By applying the Mellin transformation to Equation (1), we presented an integral representation of this function. With the help of this integral representation, we also showed that this function interpolates the products of the Apostol–Bernoulli polynomials, the Bernoulli polynomials and numbers, and also Euler polynomials and numbers at negative integers. For the proof of the main

theorems of this paper, we used the Lerch zeta function, the Hurwitz zeta function, and the alternating Hurwitz zeta function. We also gave relations among the Apostol–Bernoulli polynomials, Apostol–Euler polynomials, a combinatorial sum involving the Bernoulli numbers and polynomials, and the Euler numbers and polynomials.

Here, we also note that in [12], Kucukoglu and Simsek gave some formulas involving

$$h(x; n, m) = B_n \mathcal{B}_m(x)$$

by using a zeta type function, the Dirichlet series, and Lambert-type series. The method of this paper is also very useful in order to obtain identities including the function $h(x; n, m)$.

The results given in this paper can be potentially used in physics and engineering applications, and especially in mathematics.

Our future plan is to study new types of L -functions, Dirichlet type series and their applications with the help of the function $\mathcal{Z}(s; k, a)$. We will also investigate relationships among this function, special polynomials, and splines containing Bernstein basis functions.

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