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Higher-Order Symmetries of a Time-Fractional Anomalous Diffusion Equation

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Abstract: Higher-order symmetries are constructed for a linear anomalous diffusion equation with the Riemann–Liouville time-fractional derivative of order $\alpha \in (0,1) \cup (1,2)$. It is proved that the equation in question has infinite sequences of nontrivial higher-order symmetries that are generated by two local recursion operators. It is also shown that some of the obtained higher-order symmetries can be rewritten as fractional-order symmetries, and corresponding fractional-order recursion operators are presented. The proposed approach for finding higher-order symmetries is applicable for a wide class of linear fractional differential equations.

Keywords: anomalous diffusion; Riemann–Liouville fractional derivative; Lie–Bäcklund transformation; higher-order symmetry; recursion operator

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1. Introduction

The theory of higher-order symmetries is an important branch of modern group analysis of differential and integro-differential equations [1–4]. In this theory, it is assumed that for a given differential equation or for a system of such equations, the coordinates of infinitesimal generators of symmetry groups depend on a finite number of derivatives of all dependent variables. Such group generator corresponds to a one-parameter local transformation group, and corresponding transformations are known as higher-order tangent local transformations or as Lie–Bäcklund transformations [1,4–6].

In general case, Lie–Bäcklund transformations are invertible infinite-order tangent transformations acting in infinite-dimensional space [7]. The infinitesimal description of such transformations leads to an infinite system of first-order ordinary differential equations that are similar to the classical Lie equations. It had been proved that this system is reduced to a finite-dimensional system only for Lie point transformations and for contact transformations [5]. As a result, applications of the theory of infinite-order tangent transformation groups to differential equations lead to more complex calculations than the classical theory of Lie point transformation groups. However, higher-order symmetries play an important role in practical applications of symmetry analysis because they give an opportunity to find new exact solutions and conservation laws for differential equations. At present, there are some techniques for finding higher-order symmetries, and a lot of sequences of such symmetries have been found for numerous integer-order differential equations that are of great importance for many fields of science and technology (see, e.g., [1,2,4,8–14] therein).

Nevertheless, the theory of higher-order symmetries has not been yet extended to fractional differential equations (FDEs) [15]. At present, the basic methods of classical Lie group analysis have been successfully adopted to investigation of symmetry properties of FDEs with different types of fractional derivatives. A detailed discussion of this branch of Lie group analysis can be found in [16–18]. By using these methods, numerous Lie point symmetries, invariant solutions and conservation laws had been found for different classes of FDEs (see, e.g., the overview given in the last section of [17]). However, to the best of our knowledge, there are not examples of finding higher-order symmetries for FDEs. In this paper, we overcome this drawback.

We illustrate the possibility of finding higher-order symmetries for fractional differential equations by a simple example of the linear one-dimensional anomalous diffusion equation given by

$$ {}_0D_t^{\alpha} u = u_{xx}, \quad \alpha \in (0,1) \cup (1,2), \tag{1} $$

where

$$ {}_0D_t^{\alpha} u = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\tau, x)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n = [\alpha] + 1 \tag{2} $$

is the Riemann–Liouville time-fractional derivative (see, e.g., [15]). The Equation (1) is known as the subdiffusion equation for $\alpha \in (0,1)$, and as the diffusion-wave equation for $\alpha \in (1,2)$. In the limiting case of $\alpha = 1$, Equation (1) coincides with the linear diffusion (heat) equation. For this equation, the corresponding higher-order symmetries and recursion operators had been firstly calculated by Ibragimov [19]. If $\alpha = 2$, Equation (1) coincides with the linear wave equation which admits an infinite-dimensional algebra of Lie–Bäcklund symmetries [1].

We prove that Equation (1) has infinite sequences of nontrivial higher-order symmetries that are generated by two local recursion operators. We also show that for the considered equation, some higher-order symmetries can be rewritten as fractional-order symmetries, and corresponding fractional recursion operators are presented in an explicit form.

The paper is organised as follows. In Section 2, we give a brief overview on higher-order symmetries. Section 3 is devoted to construction of second-order symmetries for the Equation (1). Recursion operators for the considered equation are derived in Section 4. A brief discussion about generalization of the obtained results is given in the Conclusions.

2. Brief Preliminary

In this section, we recall some necessary definitions of the theory of higher-order symmetries for integer-order differential equations [1,14]. For simplicity, we restrict our attention to the case of two independent variables t and x, and one dependent variable $u = u(t, x)$.

Let τ, ξ and η be the functions of t, x, u, and any finite number of derivatives u_t, u_x, u_{tt}, u_{tx}, u_{xx}, etc. Then a differential operator

$$ X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \cdots, \tag{3} $$

where

$$ \zeta_i = D_i(\eta - \tau u_t - \xi u_x) + \tau D_i(u_t) + \xi D_i(u_x), \quad i \in \{t, x\}; $$

$$ \zeta_{ij} = D_i D_j(\eta - \tau u_t - \xi u_x) + \tau D_i D_j(u_t) + \xi D_i D_j(u_x), \quad i, j \in \{t, x\}; $$

$$ \cdots $$

is called a *Lie-Bäcklund operator* . Here, D_i is the total derivative operator with respect to the variable $i \in \{t, x\}$.

It is proved that any operator of the form (3) is equivalent to the operator

$$ X - \tau D_t - \xi D_x = (\eta - \tau u_t - \xi u_x) \frac{\partial}{\partial u} + \cdots . $$

An operator of the form

$$X = f\frac{\partial}{\partial u} + D_t(f)\frac{\partial}{\partial u_t} + D_x(f)\frac{\partial}{\partial u_x} + \cdots, \tag{4}$$

where $f = f(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots)$, is called a *canonical Lie-Bäcklund operator*.

A Lie–Bäcklund operator generates a one-parameter transformation group, which is called a *Lie–Bäcklund transformation group*, and corresponding transformations are called *Lie–Bäcklund transformations* or infinite-order tangent transformations.

The operator X is called *an infinitesimal Lie–Bäcklund symmetry* or *a higher-order symmetry* for a differential equation $F = 0$ if

$$XF\big|_{[F=0]} = 0, \tag{5}$$

where $[F = 0]$ is the so-called extended frame of the equation $F = 0$ defined by

$$F = 0, \quad D_i F = 0, \quad D_i D_j F = 0, \ldots.$$

The Equation (5) is called the *determining equation* for higher-order symmetries. Note that since the operator (4) is fully defined by a function f, this function is also often called a higher-order symmetry.

3. Second-Order Symmetries of the Linear Anomalous Diffusion Equation

First of all, we note that similarly to the case of Lie point symmetry groups, the Lie–Bäcklund operator (3) can be prolonged to the fractional-order dependent variable ${}_0D_t^\alpha u$ as

$$\tilde{X} = X + \zeta_\alpha \frac{\partial}{\partial({}_0D_t^\alpha u)}.$$

Here, the infinitesimal ζ_α is defined by using the prolongation formula

$$\zeta_\alpha = {}_0D_t^\alpha(\eta - \tau u_t - \xi u_x) + \tau\, {}_0D_t^{\alpha+1} u + \xi D_x({}_0D_t^\alpha u),$$

which has been obtained earlier for the Lie point transformation groups (see, e.g., [16,17]). Then, the prolonged canonical Lie–Bäcklund operator (4) has the form

$$\tilde{X} = f\frac{\partial}{\partial u} + D_t(f)\frac{\partial}{\partial u_t} + D_x(f)\frac{\partial}{\partial u_x} + {}_0D_t^\alpha(f)\frac{\partial}{\partial({}_0D_t^\alpha u)} + \cdots. \tag{6}$$

Similarly to the integer-order case, the operator (4) will be called *an infinitesimal Lie–Bäcklund symmetry* for a time-fractional differential equation $F(t, x, u, D_t^\alpha u, u_x, u_{xx}, \ldots) = 0$ if

$$\tilde{X}F\big|_{\{F=0\}} = 0, \tag{7}$$

where the time-fractional extended frame $\{F = 0\}$ of FDE $F = 0$ is defined by

$$F = 0, \quad D_x F = 0, \quad D_t^\alpha F = 0, \quad D_x^2 F = 0, \quad D_t^\alpha D_x F = 0, \ \ldots.$$

For the considered Equation (1), the determining Equation (7) takes the form

$$\left({}_0D_t^\alpha f - D_x^2 f\right)\big|_{\{{}_0D_t^\alpha u - u_{xx} = 0\}} = 0. \tag{8}$$

Note that this equation can be obtained as a consistency condition of Equation (1) with the equation

$$\frac{\partial u}{\partial a} = f(t, x, u, u_t, u_x, \ldots),$$

where a is a group parameter.

It is obvious that for a classical evolution equation of the form $u_t = F(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots)$, all time derivatives in the function f can be excluded by this equation. Therefore, in this case, the function f depends only on spatial derivatives of u, i.e., $f = f(t, x, u, u_x, u_{xx}, u_{xxx}, \ldots)$. However, it is not valid for time-fractional evolution equation such as Equation (1). Nevertheless, since Equation (1) is linear, we can exclude all even-order spatial derivatives from the function f and rewrite it as a function of fractional differential variables (note that such variables are nonlocal ones). Thus, we have

$$f = f(t, x, u, u_t, u_x, u_{tt}, u_{tx}, {}_0D_t^{\alpha} u, u_{ttt}, u_{ttx}, {}_0D_t^{\alpha+1} u, {}_0D_t^{\alpha} u_x, \ldots).$$

At first, we will find second-order symmetries of Equation (1) in the form

$$f = f(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}),$$

or in the equivalent form

$$f = f(t, x, u, u_t, u_x, u_{tt}, u_{tx}, {}_0D_t^{\alpha} u).$$

It is easy to show by direct calculations that in this case, the function $D_x^2 f$ is a linear function with respect to variables $u_{ttxx} = {}_0D_t^{\alpha+2} u$, $u_{txxx} = {}_0D_t^{\alpha+1} u_x$ and $u_{xxxx} = {}_0D_t^{\alpha}\,{}_0D_t^{\alpha} u$. Then, from the determining Equation (8), it follows that ${}_0D_t^{\alpha} f$ also has to be a linear function of these variables. It is possible if and only if the function f is linear with respect to u_{tt}, u_{tx} and ${}_0D_t^{\alpha} u$. Thus, we can write

$$f = \varphi u_{tt} + \psi u_{tx} + \theta\, {}_0D_t^{\alpha} u + \omega, \tag{9}$$

where φ, ψ, θ, and ω are functions of variables t, x, u, u_t, u_x.

Now, we consider the structure of ${}_0D_t^{\alpha} f$. It follows from solvability conditions of the Cauchy-type problem for Equation (1) (see Theorem 6.1 in [15]) that the initial condition for this equation should have the form ${}_0I_t^{n-\alpha} u|_{t=0} = u_0(x)$, where ${}_0I_t^{n-\alpha} u$ is a fractional integral of order $n - \alpha$ with $n = [\alpha] + 1$. It means that the function u has a singularity at the point $t = 0$, and the main term of its asymptotic expansion $\sim t^{\alpha-n}$ when $t \to 0$. In this case, the fractional derivatives ${}_0D_t^{\alpha} u_{tx}$ and ${}_0D_t^{\alpha} u_{tt}$ do not exist and, therefore, we cannot use the generalized Leibniz rule

$${}_0D_t^{\alpha}(yz) = \sum_{k=0}^{\infty} \binom{\alpha}{k} {}_0D_t^{\alpha-k} y\, D_t^k z \tag{10}$$

for the terms in the right-hand side of Equation (9). Moreover, ${}_0D_t^{\alpha}(\varphi u_{tt})$ exists only if

$$\varphi|_{t=0} = 0, \quad \varphi'|_{t=0} = 0, \tag{11}$$

and ${}_0D_t^{\alpha}(\psi u_{tx})$ exists only if

$$\psi|_{t=0} = 0. \tag{12}$$

We will assume that these conditions are satisfied and that the functions φ and ψ are analytic as functions of t. For such functions, the conditions given above lead to more strict conditions

$$(\varphi u)|_{t=0} = 0, \quad (u D_t \varphi)|_{t=0} = 0, \quad (\psi u)|_{t=0} = 0. \tag{13}$$

The right-hand side of (9) can be rewritten in a more convenient form by using the following obvious equalities:

$$\varphi u_{tt} = D_t^2(u\varphi) - 2D_t(u D_t \varphi) + u D_t^2 \varphi, \qquad \psi u_{tx} = D_t(u_x \psi) - u_x D_t \psi.$$

In view of (13), we have

$$ {}_0D_t^\alpha D_t^2(u\varphi) = {}_0D_t^{\alpha+2}(u\varphi), \quad {}_0D_t^\alpha D_t(uD_t\varphi) = {}_0D_t^{\alpha+1}(uD_t\varphi), \quad {}_0D_t^\alpha D_t(u_x\psi) = {}_0D_t^{\alpha+1}(u_x\psi). $$

Thus, we obtain

$$ {}_0D_t^\alpha f = {}_0D_t^{\alpha+2}(u\varphi) - 2{}_0D_t^{\alpha+1}(uD_t\varphi) + {}_0D_t^\alpha(uD_t^2\varphi) \\ + {}_0D_t^{\alpha+1}(u_x\psi) - {}_0D_t^\alpha(u_xD_t\psi) + {}_0D_t^\alpha(\theta\,{}_0D_t^\alpha u) + {}_0D_t^\alpha(\omega). $$

Now, we can use the generalized Leibniz rule (10) in this expression. After simple calculations, we get

$$ {}_0D_t^\alpha f = \varphi\,{}_0D_t^{\alpha+2}u + \alpha D_t(\varphi)\,{}_0D_t^{\alpha+1}u + \frac{\alpha(\alpha-1)}{2}D_t^2(\varphi)\,{}_0D_t^\alpha u + \sum_{k=1}^{\infty}\binom{\alpha}{k+2}D_t^{\alpha-k}u\,D_t^{k+2}\varphi \\ + \psi\,{}_0D_t^{\alpha+1}u_x + \alpha D_t\psi\,{}_0D_t^\alpha u_x + \sum_{k=1}^{\infty}\binom{\alpha}{k+1}{}_0D_t^{\alpha-k}u_x\,D_t^{k+1}\psi \\ + \theta\,{}_0D_t^\alpha\,{}_0D_t^\alpha u + \sum_{k=1}^{\infty}\binom{\alpha}{k}{}_0D_t^{\alpha-k}({}_0D_t^\alpha u)\,D_t^k\theta + {}_0D_t^\alpha\omega. $$

By using (9), we can also write

$$ D_x^2 f = \varphi_0D_t^{2+\alpha}u + 2D_x\varphi u_{ttx} + D_x^2\varphi u_{tt} + \psi_0D_t^{\alpha+1}u_x + 2D_x\psi_0D_t^\alpha u + D_x^2\psi u_{tx} + \\ \theta_0D_t^\alpha{}_0D_t^\alpha u + 2D_x\theta_0D_t^\alpha u_x + D_x^2\theta_0D_t^\alpha u + D_x^2\omega. $$

The functions φ, ψ and ω do not depend on fractional variables ${}_0D_t^{\alpha-k}u$, ${}_0D_t^{\alpha-k}u_x$, and ${}_0D_t^{\alpha-k}({}_0D_t^\alpha u)$. Moreover, for Equation (1) such variables cannot arise under differentiation by x. Thus, $D_x^2 f$ does not contain such variables. Nevertheless, such variables can be generated by ${}_0D_t^\alpha\omega$. We can use the chain rule for the Riemann–Liouville fractional derivative to get the expansion of this function, but this technique is very complex. For this reason, we will use a more simple approach. We represent the function ω in the form

$$ \omega(t,x,u,u_t,u_x) = \mu(t,x,u,u_t,u_x)u_x + \nu(t,x,u,u_t). \tag{14} $$

Then, by using the generalized Leibniz rule (10), we obtain

$$ {}_0D_t^\alpha\omega = \mu_0D_t^\alpha u_x + \sum_{k=1}^{\infty}\binom{\alpha}{k}D_t^{\alpha-k}u_xD_t^k\mu + {}_0D_t^\alpha\nu. $$

After substituting this representation into the determining equation, we can isolate the terms containing ${}_0D_t^{\alpha-k}u_x$ and set them equal to zero. As a result, we get an infinite chain of equations

$$ \binom{\alpha}{k}D_t^k\mu + \binom{\alpha}{k+1}D_t^{k+1}\psi = 0, \quad k = 1,2,\dots. $$

The solution of these equations can be written as

$$ \psi \equiv \psi(t,x) = \psi_2(x)t^2 + \psi_1(x)t + \psi_0(x), \quad \mu \equiv \mu(t,x) = (1-\alpha)\psi_2(x)t + \mu_0(x), \tag{15} $$

where ψ_0, ψ_1, ψ_2, μ_0 are arbitrary functions of x. The initial condition (12) yields $\psi_0(x) = 0$.

Next, we represent the function ν in the form

$$ \nu(t,x,u,u_t) = \rho(t,x,u,u_t)u_t + \sigma(t,x,u)u + \lambda(t,x). $$

Since ${}_0D_t^{\alpha}u_t$ does not exist in the general case, an additional condition

$$\rho|_{t=0} = 0 \tag{16}$$

have to be fulfilled. As earlier, we assume that the function ρ is an analytic function with respect to t. Then,

$$\nu = D_t(\rho u) + (\sigma - D_t\rho)u + \lambda,$$

and, by using (10), we obtain

$$\begin{aligned}
&{}_0D_t^{\alpha}\nu = {}_0D_t^{\alpha+1}(\rho u) + {}_0D_t^{\alpha}[(\sigma - D_t\rho)u] + {}_0D_t^{\alpha}\lambda \\
&= \rho\, {}_0D_t^{\alpha+1}u + \alpha D_t\rho\, {}_0D_t^{\alpha}u + \sigma\, {}_0D_t^{\alpha}u + {}_0D_t^{\alpha}\lambda + \sum_{k=1}^{\infty}\tbinom{\alpha}{k+1}{}_0D_t^{\alpha-k}u\, D_t^{k+1}\rho + \sum_{k=1}^{\infty}\tbinom{\alpha}{k}{}_0D_t^{\alpha-k}u\, D_t^{k}\sigma.
\end{aligned}$$

After substituting this representation into the determining equation, we can isolate the terms containing ${}_0D_t^{\alpha-k}u$ and set them equal to zero. As a result, we obtain an infinite chain of equations

$$\binom{\alpha}{k}D_t^k\sigma + \binom{\alpha}{k+1}D_t^{k+1}\rho + \binom{\alpha}{k+2}D_t^{k+2}\varphi = 0, \quad k = 1, 2, \ldots$$

which has the following solution:

$$\begin{aligned}
&\varphi \equiv \varphi(t,x) = \varphi_0(x) + \varphi_1(x)t + \varphi_2(x)t^2 + \varphi_3(x)t^3 + \varphi_4(x)t^4, \\
&\rho \equiv \rho(t,x) = \rho_0(x) + \rho_1(x)t + \rho_2(x)t^2 + 2(2-\alpha)\varphi_4(x)t^3, \\
&\sigma = \sigma(t,x) = \sigma_0(x) + (1-\alpha)[\rho_2(x) + (\alpha-2)\varphi_3(x)]t + (1-\alpha)(2-\alpha)\varphi_4(x)t^2,
\end{aligned} \tag{17}$$

where φ_i, ρ_i, σ_i are arbitrary functions of x. The initial conditions (11) and (16) yield

$$\varphi_0(x) = 0, \quad \varphi_1(x) = 0, \quad \rho_0(x) = 0.$$

Finally, we can isolate the terms containing ${}_0D_t^{\alpha-k}({}_0D_t^{\alpha}u)$. By setting all of them equal to zero, we get the system of equations

$$D_t^k\theta = 0, \quad k = 1, 2, \ldots.$$

It follows from this system that $\theta = \theta(x)$.

So, we prove that the function f is a linear function with respect to u and all their derivatives. It has the form

$$f = \varphi(t,x)u_{tt} + \psi(t,x)u_{tx} + \theta(x){}_0D_t^{\alpha}u + \rho(t,x)u_t + \mu(t,x)u_x + \sigma(t,x)u + \lambda(t,x).$$

Thus, all second-order symmetries of Equation (1) are linear with respect to u.

The determining Equation (8) reduces to

$$\begin{aligned}
&\left(\frac{\alpha(\alpha-1)}{2}\varphi_{tt} + \alpha\varphi_t + \alpha\rho_t\right){}_0D_t^{\alpha}u + \alpha\psi_t\,{}_0D_t^{\alpha}u_x + {}_0D_t^{\alpha}\lambda - (2\psi_x + \theta_{xx} + 2\rho_x + 2\mu_x){}_0D_t^{\alpha}\,u \\
&- 2\theta_x\,{}_0D_t^{\alpha}u_x - 2\varphi_x u_{ttx} - \varphi_{xx}u_{tt} - \psi_{xx}u_{tx} - \rho_{xx}u_t - \mu_{xx}u_x - \sigma_{xx}u - 2\sigma_x u_x - \lambda_{xx} = 0.
\end{aligned}$$

We split this equation with respect to terms containing u and their derivatives. As a result, we obtain the following system of equations:

$$
\begin{array}{ll}
{}_0D_t^\alpha u: & \dfrac{\alpha(\alpha-1)}{2}\varphi_{tt}+\alpha\varphi_t+\alpha\rho_t-2\psi_x-\theta_{xx}-2\rho_x-2\mu_x=0,\\
{}_0D_t^\alpha u_x: & \alpha\psi_t-2\theta_x=0,\\
u_{ttx}: & \varphi_x=0,\\
u_{tt}: & \varphi_{xx}=0,\\
u_{tx}: & \psi_{xx}=0,\\
u_t: & \rho_{xx}=0,\\
u_x: & \mu_{xx}+2\sigma_x=0,\\
u: & \sigma_{xx}=0,
\end{array}
$$

In view of representations (15) and (17), one can obtain the solution of this system in the form

$$
\sigma=C_0,\quad \mu=C_1+C_2\frac{\alpha}{2}x,\quad \theta=C_3+C_4\frac{\alpha}{2}x+C_5\frac{\alpha^2}{4}x^2,
$$
$$
\rho=C_3t+C_5\Big(1-\frac{\alpha}{2}\Big)t,\quad \psi=C_4t+C_5\alpha tx,\quad \varphi=C_5t^2,
$$

where C_i $(i=1,\dots,5)$ are arbitrary constants. The corresponding symmetries are

$$
f_0=u,\quad f_1=u_x,\quad f_2=tu_t+\frac{\alpha}{2}xu_x,\quad f_3={}_0D_t^\alpha u,\quad f_4=tu_{tx}+\frac{\alpha}{2}x\,{}_0D_t^\alpha u,
$$
$$
f_5=\Big(1-\frac{\alpha}{2}\Big)tu_t+t^2u_{tt}+\alpha txu_{tx}+\frac{\alpha^2}{4}x^2\,{}_0D_t^\alpha u.
$$

The remaining part of the determining equation is

$$
{}_0D_t^\alpha\lambda-\lambda_{xx}=0. \tag{18}
$$

This equation is identical to the initial linear fractional diffusion Equation (1). It gives an infinity number of symmetries of the form $f_\infty=\lambda(t,x)$, where $\lambda(t,x)$ is an arbitrary solution to Equation (18).

Note that symmetries f_3, f_4 and f_5 are nonlocal ones because they contain the fractional derivative ${}_0D_t^\alpha u$. Nevertheless, these symmetries are local on solutions of Equation (1). Indeed, in view of Equation (1), they can be rewritten in the form of local symmetries as

$$
\tilde f_3=u_{xx},\quad \tilde f_4=tu_{tx}+\frac{\alpha}{2}xu_{xx},\quad \tilde f_5=\Big(1-\frac{\alpha}{2}\Big)tu_t+t^2u_{tt}+\alpha txu_{tx}+\frac{\alpha^2}{4}x^2u_{xx}.
$$

These symmetries are the second-order symmetries of the fractional anomalous diffusion Equation (1).

Note that the symmetries f_0, f_1, f_2 and f_∞ correspond to the classical Lie point symmetries of Equation (1) (see [20])

$$
X_0=u\frac{\partial}{\partial u},\quad X_1=\frac{\partial}{\partial x},\quad X_2=t\frac{\partial}{\partial t}+\frac{\alpha}{2}x\frac{\partial}{\partial x},\quad X_\infty=\lambda(t,x)\frac{\partial}{\partial u}.
$$

4. Recursion Operators

It is easy to see that

$$
f_1=D_xf_0,\quad \tilde f_3=D_xf_1,\quad \tilde f_4=D_xf_2-\frac{\alpha}{2}f_1=D_x\Big(f_2-\frac{\alpha}{2}f_0\Big)
$$

and

$$
f_2=\Big(tD_t+\frac{\alpha}{2}xD_x\Big)f_0,
$$
$$
\tilde f_5=\Big(tD_t+\frac{\alpha}{2}xD_x\Big)f_2-\frac{\alpha}{2}f_2=\Big(tD_t+\frac{\alpha}{2}xD_x\Big)\Big(f_2-\frac{\alpha}{2}f_0\Big).
$$

Thus, we can introduce two differential operators

$$R_1 = D_x, \quad R_2 = tD_t + \frac{\alpha}{2} x D_x \tag{19}$$

that permit to obtain symmetries f_1, f_2, $\tilde{f}_3$, $\tilde{f}_4$ and $\tilde{f}_5$ from the single symmetry $f_0 = u$. In particular, for the Lie point symmetries we have

$$f_1 = R_1 f_0, \quad f_2 = R_2 f_0.$$

Similarly to the case of integer-order differential equations [1,8], we will call operators R_1 and R_2 the *recursion operators*.

Note that the operator R_1 coincides with the first recursion operator obtained in [19] (see also [1]) for the linear heat (diffusion) equation. The second recursion operator $2tD_x + x$ for this equation can be also obtained from the operator R_2 in view of the fact that for $\alpha = 1$, we have $D_t u = D_x^2 u$.

Let us introduce the linear fractional differential operator

$$L = {}_0D_t^\alpha - D_x^2.$$

Then Equation (1) can be written as $Lu = 0$, and the determining Equation (8) is $Lf|_{\{Lu=0\}} = 0$. Since $D_x({}_0D_t^\alpha) = {}_0D_t^\alpha(D_x)$, it is obvious that the commutator

$$[L, R_1] \equiv LR_1 - R_1 L = 0.$$

Additionally, it is easy to prove that

$$[L, R_2] \equiv LR_2 - R_2 L = 0.$$

So, when $f = R_i u$ $(i = 1, 2)$, the determining equation is fulfilled identically because

$$L(R_i u)|_{\{Lu=0\}} = R_i(Lu)|_{\{Lu=0\}} \equiv 0. \tag{20}$$

Thus, the recursion operators R_1 and R_2 convert any solution of the determining equation into another solution of this equation. Note that these operators satisfy the commutation relation

$$[R_1, R_2] = \frac{\alpha}{2} R_1,$$

i.e., the linear span of these operators is the two-dimensional Lie algebra.

We can also introduce fractional recursion operators for the Equation (1). Indeed, since $D_x^2 u = {}_0D_t^\alpha u$, the operator

$$R_3 = {}_0D_t^\alpha$$

is also the recursion operator for Equation (1), and we have $f_3 = R_3 f_0$. It is easy to prove that

$$[L, R_3] = 0, \quad [R_1, R_3] = 0, \quad [R_2, R_3] = -\alpha R_3.$$

The composition of recursion operators R_1 and R_2 gives the fractional recursion operator

$$R_4 = tD_t D_x + \frac{\alpha}{2} x {}_0D_t^\alpha$$

such that $f_4 = R_4 f_0$, and

$$[L, R_4] = 0, \quad [R_1, R_4] = \frac{\alpha}{2} R_3, \quad [R_2, R_4] = -\frac{\alpha}{2} R_4, \quad [R_3, R_4] = \alpha R_1 R_3.$$

Similarly, one can find other fractional recursion operators. Such operators can be also used for constructing higher-order symmetries and can be useful for investigating qualitative properties of Equation (1). Nevertheless, it is necessary to note that all these operators can be obtained in view of Equation (1) from the operators R_1 and R_2, and, therefore, such fractional operators should not be considered as primary recursion operators.

5. Conclusions

The obtained results indicate that higher-order symmetries of linear fractional partial differential equations can be found explicitly in the same manner as Lie point symmetries of such equations. The local higher-order symmetries for linear FDEs can be obtained from the Lie point symmetries by local recursion operators that can be obtained from the analysis of the first-order and second-order symmetries. It is natural to expect that such equations have infinite sequences of higher-order symmetries. For example, any linear FDE of the form

$$ {}_0D_t^{\alpha} u = b(t) + \sum_{k=0}^{n} a_k(t) \frac{\partial^k u}{\partial x^k} $$

has the symmetries $f_0 = u$ and $f_1 = u_x$. As a result, there is a recursion operator $R = D_x$, and the equation given above has an infinite sequence of higher-order symmetries $f_k = \dfrac{\partial^k u}{\partial x^k}$ $(k = 0, 1, 2, \ldots)$. Additionally, some of such symmetries can be rewritten as fractional symmetries by initial FDE.

Nevertheless, the algorithm of finding higher-order symmetries described in this paper is not applicable for nonlinear FDEs because in this algorithm, all fractional integral variables are assumed to be independent and several linearization procedures are used. Moreover, it is likely that the fractional-order form of higher symmetries is more suitable for nonlinear FDEs. Thus, the development of methods of finding higher-order symmetries for nonlinear FDEs remains a challenging problem of modern group analysis of fractional differential equations.

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References

1. Ibragimov, N.H. *Transformation Groups Applied to Mathematical Physics*; Reidel: Dordrecht, The Netherlands, 1985.
2. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA, 1986.
3. Bluman, G.; Kumei, S. *Symmetries and Differential Equations*; Springer: Berlin, Germany, 1989.
4. Grigoriev, Y.N.; Ibragimov, N.K.; Kovalev, V.F.; Meleshko S.V. *Symmetries of Integro-Differential Equations: With Applications in Mechanics and Plasma Physics*; Springer: Dordrecht, The Netherlands, 2010.
5. Ibragimov, N.H.; Anderson, R.L. Groups of Lie–Bäcklund contact transformations. *Sov. Math. Dokl.* **1976**, *17*, 437–441.
6. Anderson, R.L.; Ibragimov, N.H. Lie-Bäcklund Transformations in Applications. In *Studies in Applied and Numerical Mathematics*; Ovsyannikov, L.V., Ed.; Lavrentyev Institute of Hydrodynamics of the Siberian Branch of the Russian Academy of Sciences: Novosibirsk, Russia, 1979; pp. 34–46.
7. Anderson, R.L.; Ibragimov, N.H. Lie–Bäcklund tangent transformations. *J. Math. Anal. Appl.* **1977**, *59*, 145–162.
8. Ibragimov, N.H.; Shabat, A.B. Korteveg-de Vries equation from the group point of view. *Sov. Phys. Dokl.* **1979**, *24*, 15.
9. Anderson, R.L.; Ibragimov, N.H. *Lie–Backlund Transformations in Applications*; SIAM: Philadelphia, PA, USA, 1979.

10. Ibragimov, N.H. (Ed.) *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 1. Symmetries, Exact Solutions and Conservation Laws*; CRC Press Inc.: Boca Raton, FL, USA, 1994.
11. Ibragimov, N.H. (Ed.) *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3. New Trends in Theoretical Developments and Computational Methods*; CRC Press Inc.: Boca Raton, FL, USA, 1996.
12. Adler, V.E.; Shabat, A.B.; Yamilov, R.I. The symmetry approach to the problem of integrability. *Theor. Math. Phys.* **2000**, *125*, 355–424. [CrossRef]
13. Mikhailov, A.V.; Sokolov, V.V. Symmetries of differential equations and the problem of integrability. *Lect. Notes Phys.* **2009**, *767*, 19–88.
14. Bluman, G.W.; Cheviakov, A.F.; Anco, S.C. *Applications of Symmetry Methods to Partial Differential Equations*; Springer: New York, NY, USA, 2010.
15. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: New York, NY, USA, 2006.
16. Gazizov, R.K.; Kasatkin, A.A.; Lukashchuk, S.Y. Symmetries and group invariant solutions of fractional ordinary differential equations. In *Handbook of Fractional Calculus with Applications. Vol. 2 Fractional Differential Equations*; Kochubei, A., Luchko, Y., Eds.; De Gruyter: Berlin, Germany, 2019; pp. 65–90.
17. Gazizov, R.K.; Kasatkin, A.A.; Lukashchuk, S.Y. Symmetries, conservation laws and group invariant solutions of fractional PDEs. In *Handbook of Fractional Calculus with Applications. Vol. 2 Fractional Differential Equations*; Kochubei, A., Luchko, Y., Eds.; De Gruyter: Berlin, Germany, 2019; pp. 353–382.
18. Hashemi, M.S.; Baleanu, D. *Lie Symmetry Analysis of Fractional Differential Equations*; Chapman and Hall: New York, NY, USA; CRC Press: Boca Raton, FL, USA, 2020.
19. Ibragimov, N.H. Sur l'équivalence des équations d'evolution qui admettent une algèbre de Lie–Bäcklund infinie. *C. R. Acad. Sci. Paris Ser. 1* **1981**, *293*, 657–660.
20. Gazizov, R.K.; Kasatkin, A.A.; Lukashchuk, S.Y. Symmetry properties of fractional diffusion equations. *Phys. Scr.* **2009**, *T136*, 014016. [CrossRef]