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On Bonds for Generalized One-Sided Concept Lattices

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Abstract: The generalized one-sided concept lattices represent a generalization of the classical FCA method convenient for a hierarchical analysis of object-attribute models with different types of attributes. The mentioned types of object-attribute models are formalized within the theory as formal contexts of a certain type. The aim of this paper is to investigate some intercontextual relationships represented by the notion of bond. A composition of bonds is defined in order to introduce the category of formal contexts with bonds as morphisms. It is shown that there is a one-to-one correspondence between bonds and supremum preserving mappings between the corresponding generalized one-sided concept lattices. As the main theoretical result it is shown that the introduced category of formal contexts with bonds is equivalent to the category of complete lattices with supremum preserving mappings as morphisms.

Keywords: FCA generalization; intercontextual relations; supremum preserving mappings; category of contexts

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1. Introduction

The theory of concept lattices or formal concept analysis (FCA for short) represents a method of data analysis for identifying conceptual structures among data sets. Concept lattices are constructed from formal contexts, which represent a mathematical formalization of object-attribute models. The formal contexts are usually determined by a relation between a set of objects and a set of attributes, serving for characterization of the objects. In the classical setting [1] the relation is binary, i.e., particular objects are characterized only by the presence or the absence of considered attributes. However, in practice there are natural examples of object-attribute models for which relationship between objects and attributes are represented by many-valued (fuzzy) relations. Therefore, handling uncertainty, imprecise data or some kind of incomplete information has become an important research topic in the recent years within the field of FCA. From the approaches involving fuzzy logic framework the paper [2] can be mentioned, where complete commutative residuated lattices as structures for the truth degrees were used. The ideas described in the mentioned paper were further generalized, e.g., in [3], to the so-called multi-adjoint framework, where several non-commutative bi-residuated structures are used to mediate interaction between objects and attributes. The first mentioned approach involves the same underlying structure for objects and attributes evaluation, while the second one involves two different structures for evaluation of objects and attributes respectively. In [4] and independently in [5], two methods with different structures for evaluation of particular objects and attributes were described. In these cases each object as well as each attribute has assigned some truth value structure represented by a complete lattice. Concept lattices defined within this framework enable applying FCA-based methods to heterogeneous data tables, hence providing some kind of the most general types of concept lattices.

This paper is focused on the so-called generalized one-sided concept lattices introduced in [6]. This kind of concept lattices is convenient for analysis of object-attribute

models with fuzzy values from different types of truth value structures and they can be seen as specific subclass of fuzzy concept lattices introduced in [5]. The name “one-sided” refers to the fact that the output concepts are formed by crisp subsets of objects and vectors of fuzzy values, characterizing the objects in concepts. This type of concept lattices was studied by several authors, e.g., in [7] an extension concerning preference relation on attributes was described, papers [8,9] deal with problems of attribute reductions, while [10,11] deal with alternative definitions of concept forming operators.

Our aim is to study bonds between formal contexts corresponding to the mentioned generalized one-sided concept lattices. In the classical case, given two formal contexts, bonds are represented by certain binary relations between the objects of the first context and the set of attributes of the second context.

Bonds represent some intercontextual relationship between contexts. Perhaps the most versatile mathematical theory for study various relations among objects is category theory. Recall that formal contexts and concept lattices, for the classical as well as for the fuzzy case, were studied from a categorical point of view by several authors. For the classical case [12] is referred, where properties of several morphisms between formal contexts are studied or [13] where the notions of (dual) bonds, scale measures and informorphisms are studied. Concerning the fuzzy concept lattices, a categorical aspects of fuzzy concept lattices via generalized Chu spaces is investigated in the papers [14,15]. The L -fuzzy version of bonds was studied in [16] with respect to several concept forming operators or in [17,18], where its tight connection with generalized chu correspondence is discussed. In this case, an L -bond is defined as an L -valued multifunction, or it can be equivalently seen as an L -relation, such that the corresponding rows and columns form closed L -sets. Similarly as in the classical case, such definition of the notion of a bond is possible, since objects and attributes are evaluated in a single structure (residuated lattice L) and consequently concepts consist of pairs of L -sets. However, such definition is no longer possible in the case of generalized one-sided concept lattices, due to different structures for attribute evaluation. Hence, in order to obtain a reasonable definition of the notion of bond another approach is proposed in this paper.

In practice one can find two approaches to the categorical study of FCA. In the first case, morphisms between formal contexts are defined by some “natural way”, and consequently such category is investigated. On the other side, one can prescribe properties for a category of context to be satisfied, and then find the appropriate morphisms between contexts. In this paper the second approach to the study of bonds between the generalized one-sided formal contexts is adopted. To be more precise, our requirement is that bonds, similarly as in the classical FCA, were in one-to-one correspondence with $\vee$ -preserving mappings. Moreover, a composition of bonds is defined, and a categorical equivalence between the category of bonds and the category of complete lattices with $\vee$ -preserving mapping is presented.

The presented results are of the theoretical nature and can be seen as an extension of the similar results known for the classical concept lattices. From this point of view the introduced category of generalized one-sided contexts with bonds represents an enlargement of the category of binary contexts with classical bonds.

The structure of the paper is organized in the following way. In the next chapter the basic notions concerning generalized one-sided concept lattices is recalled. The main results are presented in the third section. The category of bonds for generalized one-sided formal contexts is introduced and its equivalence with the category of all complete lattices with $\vee$ -preserving mappings as morphisms is showed. In the last, conclusion section, some potential theoretical application of the introduced notion is discussed, e.g., in a reduction process of the mentioned types of concept lattices.

2. Generalized One-Sided Concept Lattices

In this section, the basic framework of generalized one-sided concept lattices is recalled, cf. [6] for more details. The starting point is the notion of a generalized one-sided formal context, which formalizes the notion of an object-attribute model with different types of attributes.

Formally, a 4-tuple $\mathbf{c} = (B, A, L, R)$ is said to be a generalized one-sided formal context if the following conditions are fulfilled:

- (a) $B \neq \emptyset$ is a set of objects and $A \neq \emptyset$ is a set of attributes,
- (b) $L: A \rightarrow \text{CL}$, where CL denotes the class of all complete lattices,
- (c) $R: B \times A \rightarrow \bigcup_{a \in A} L(a)$ is a mapping satisfying $R(b, a) \in L(a)$ for all $b \in B$ and $a \in A$.

The first condition is obvious, i.e., it is assumed that input data are in the form of an object-attribute model where particular objects are characterized by considered attributes as it is usual within various FCA-based methods. Condition (b) represents the main difference compared to the classical FCA as well as to other one-sided concept lattices approaches. In this case, each particular attribute can be evaluated (attains values) from arbitrary (possibly different) complete lattices. It allows using the FCA-based approach for analysis of data tables with attributes of different nature. In such non-homogeneous data model groups of attributes can be evaluated by various structures, e.g., one group of attributes can be evaluated by two values (binary attributes), other group using some ordinal scale etc. The appropriate choice of a complete lattice $L(a)$ corresponding to an attribute $a \in A$ is on a person performing the analysis of the data. For example, considering the attribute “color” with n values $\{\text{white, red, blue, } \dots\}$ one can use a nominal scale, consisting of n incomparable elements together with the top and bottom element respectively. However, for other application it would be more convenient to use ordinal scale, where colors are ordered with respect to some parameter, e.g., with respect to brightness. In this case the corresponding complete lattice $L(\text{color})$ will be represented by an n -element chain.

Let us note that the condition $L(a) \in \text{CL}$ for all attributes $a \in A$ is of pure mathematical nature and it is imposed by the requirement that hierarchical structure of concepts should form a complete lattice.

The third condition says that input object-attribute model is in the form of a data table where values in columns are from the corresponding complete lattice. Let $\mathbf{c}$ be a generalized one-sided formal context. From the information contained in $\mathbf{c}$ one can obtain the following concept forming operators between the power set $\mathbf{P}(B)$ and the direct product of lattices $\prod_{a \in A} L(a)$: for all $X \subseteq B$

$$\uparrow_{\mathbf{c}}(X)(a) = \bigwedge_{b \in X} R(b, a) \quad (1)$$

and for all $g \in \prod_{a \in A} L(a)$

$$\downarrow_{\mathbf{c}}(g) = \{b \in B : \forall a \in A, g(a) \leq R(b, a)\}. \quad (2)$$

The pair of these operators forms a Galois connection (see [6]), and fulfills the following properties for all $X, X_1, X_2 \subseteq B$ and $g, g_1, g_2 \in \prod_{a \in A} L(a)$

- (i) $X \subseteq \downarrow_{\mathbf{c}}(g)$ if and only if $g \leq \uparrow_{\mathbf{c}}(X)$,
- (ii) $X_1 \subseteq X_2$ implies $\uparrow_{\mathbf{c}}(X_2) \leq \uparrow_{\mathbf{c}}(X_1)$,
- (iii) $g_1 \leq g_2$ implies $\downarrow_{\mathbf{c}}(g_2) \subseteq \downarrow_{\mathbf{c}}(g_1)$,
- (iv) $X \subseteq \downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}}(X)$ and $g \leq \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}}(g)$,
- (v) $\uparrow_{\mathbf{c}}(X) = \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}}(X)$ and $\downarrow_{\mathbf{c}}(g) = \downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}}(g)$,
- (vi) $\uparrow_{\mathbf{c}}(\bigcup_{i \in I} X_i) = \bigwedge_{i \in I} \uparrow_{\mathbf{c}}(X_i)$ and $\downarrow_{\mathbf{c}}(\bigvee_{i \in I} g_i) = \bigcap_{i \in I} \downarrow_{\mathbf{c}}(g_i)$.

Let us note that (i) represents a defining property of a Galois connection in the particular case, when $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ are considered to be the underlying posets. Furthermore, condition (i) holds if and only if (ii), (iii) and (iv) are valid. Moreover, the mentioned three conditions implies that the compositions $\uparrow_{\mathbf{c}} \circ \downarrow_{\mathbf{c}}: \mathbf{P}(B) \rightarrow \mathbf{P}(B)$ and $\downarrow_{\mathbf{c}} \circ \uparrow_{\mathbf{c}}: \prod_{a \in A} L(a) \rightarrow \prod_{a \in A} L(a)$ induce closure operators on $\mathbf{P}(B)$ and $\prod_{a \in A} L(a)$ respectively.

It is a well known fact that the set of all fixed points

$$\mathfrak{C}(\mathbf{c}) = \{(X, g) : \uparrow_{\mathbf{c}}(X) = g \text{ and } \downarrow_{\mathbf{c}}(g) = X\}$$

forms a complete lattice, referred to as the generalized one-sided concept lattice corresponding to the formal context $\mathbf{c}$. The elements of $\mathfrak{C}(\mathbf{c})$ are called formal concepts and if $(X, g) \in \mathfrak{C}(\mathbf{c})$ then X is called an extent and g is called an intent of the concept. The functions assigning to each concept its extent and intent, respectively are denoted by $\text{Ext}[\cdot]$ and $\text{Int}[\cdot]$, i.e., $\text{Ext}[(X, g)] = X$ and $\text{Int}[(X, g)] = g$. The sets of all extents and intents corresponding to a context $\mathbf{c}$ are denoted by $\text{Ext}(\mathbf{c})$ and $\text{Int}(\mathbf{c})$ respectively.

Furthermore, the basic lattice operations on $\mathfrak{C}(\mathbf{c})$ are recalled. Note, that due to properties of Galois connections, there are several expressions for these operations. If $(X_i, g_i)_{i \in I} \subseteq \mathfrak{C}(\mathbf{c})$ is a family of concepts, then

$$\bigvee_{i \in I} (X_i, g_i) = (\downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}} (\bigcup_{i \in I} X_i), \bigwedge_{i \in I} g_i) = (\downarrow_{\mathbf{c}} (\bigwedge_{i \in I} g_i), \bigwedge_{i \in I} g_i) = (\downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}} (\bigcup_{i \in I} X_i), \uparrow_{\mathbf{c}} (\bigcup_{i \in I} X_i)), \quad (3)$$

$$\bigwedge_{i \in I} (X_i, g_i) = (\bigcap_{i \in I} X_i, \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}} (\bigvee_{i \in I} g_i)) = (\downarrow_{\mathbf{c}} (\bigvee_{i \in I} g_i), \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}} (\bigvee_{i \in I} g_i)) = (\bigcap_{i \in I} X_i, \uparrow_{\mathbf{c}} (\bigcap_{i \in I} X_i)). \quad (4)$$

Note that the lattice operations on the left side are computed in the lattice $\mathfrak{C}(\mathbf{c})$, while other ones appearing in the above formulas are computed in the direct product $\prod_{a \in A} L(a)$.

At the end of this section a small illustrative example of a formal context and the corresponding generalized one-sided concept lattice will be provided.

Example 1. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$ be a formal context where $B_1 = \{b_1, \dots, b_5\}$ is a set of objects and $A_1 = \{a_1, \dots, a_4\}$ is a set of attributes. Furthermore, the following complete lattices for evaluation of particular attributes are considered: $L_1(a_1) = \{0, 1, 2, 3\}$ with $0 < 1 < 2 < 3$, $L_1(a_2) = [0, 1]$ (the real unit interval) and $L_1(a_3) = L_1(a_4) = \{0, 1\}$ with $0 < 1$. The input data table, represented by the generalized incidence relation R_1 is given in Table 1.

Table 1. Input data table corresponding to the relation R_1 .

R_1	a_1	a_2	a_3	a_4
b_1	2	0.25	0	1
b_2	3	0.50	1	0
b_3	1	0.35	0	0
b_4	1	0.25	0	1
b_5	2	0.70	1	0

Now applying the concept forming operators given by (1) and (2) respectively, the generalized one-sided concept lattice $\mathfrak{C}(\mathbf{c}_1)$ is obtained, where its hierarchical structure can be seen at Figure 1. Note that in this case, the elements of the direct product $\prod_{a \in A} L_1(a)$ are indicated as 4-tuples with elements from the corresponding complete lattices.

Figure 1. Generalized one-sided concept lattice $\mathfrak{C}(\mathbf{c}_1)$ corresponding to the context $\mathbf{c}_1$.

3. Bonds between Generalized One-Sided Formal Contexts

The notion of a bond between generalized one-sided formal contexts is defined in this section and its basic properties are investigated.

Definition 1. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$ and $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$ be two generalized one-sided formal contexts. A bond between the contexts $\mathbf{c}_1$ and $\mathbf{c}_2$ is a 4-tuple $\gamma = (B_1, A_2, L_2, R_{12})$ such that γ is a generalized one-sided formal context satisfying $\text{Ext}(\gamma) \subseteq \text{Ext}(\mathbf{c}_1)$ and $\text{Int}(\gamma) \subseteq \text{Int}(\mathbf{c}_2)$. The notation $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$ is used, provided γ is a bond between $\mathbf{c}_1$ and $\mathbf{c}_2$.

Note that although a bond between two contexts is itself a context, it is used a Greek small letter for denoting it, in order to emphasize that it represents some kind of relationship between the respective contexts.

Using such definition of bonds one can see an analogy with the category of sets and mappings, where each mapping is formally a set of certain ordered pairs, i.e., every morphism represents also some object in this category. In what follows an example of a bond will be presented.

Example 2. Consider a generalized one-sided formal context $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$ where $B_2 = \{b_1, b_2, b_3, b_4\}$, $A_2 = \{a'_1, a'_2, a'_3\}$. The truth value structures for particular attributes are defined as $L_2(a'_1) = L_2(a'_2) = \mathbf{3}$, where $\mathbf{3} = \{0, 1, 2\}$ is the three-element chain with $0 < 1 < 2$, and $L_2(a'_3) = M_3$, where $M_3 = \{a, b, c, O, I\}$ is the lattice consisting of three mutually incomparable elements a, b, c together with I the top and O the bottom element respectively.

Incidence relation R_2 is given in Table 2 and the resulting generalized one-sided concept lattice $\mathfrak{C}(\mathbf{c}_2)$ is depicted in Figure 2.

Table 2. Incidence relation R_2 .

R_2	a'_1	a'_2	a'_3
b_1	0	1	b
b_2	1	1	c
b_3	2	2	d
b_4	0	2	c

Figure 2. Generalized one-sided concept lattice $\mathfrak{C}(\mathbf{c}_2)$ corresponding to the context $\mathbf{c}_2$.

We give an example of a bond $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$, where $\mathbf{c}_1$ is the context from Example 1. According to Definition 1, the bond γ is represented as a generalized one-sided formal concept $\gamma = (B_1, A_2, L_2, R_{12})$. Hence $B = \{b_1, \dots, b_5\}$, $A_2 = \{a'_1, a'_2, a'_2\}$ and the attributes from A_2 are evaluated in the same truth value structures, i.e., $L_2(a'_1) = L_2(a'_2) = \mathbf{3}$ and $L_2(a'_3) = M_3$. Furthermore, R_{12} is given in Table 3.

Table 3. Incidence relation R_{12} corresponding to the bond γ .

R_{12}	a'_1	a'_2	a'_3
b_1	0	1	b
b_2	1	1	c
b_3	0	1	O
b_4	0	1	b
b_5	0	2	c

The following lemma is an easy consequence of the definition of bonds.

Lemma 1. Let $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$ be a bond. Then

$$\uparrow_{\gamma}(X) = \uparrow_{\gamma}(\downarrow_{\mathbf{c}_1}\uparrow_{\mathbf{c}_1}(X)) = \uparrow_{\mathbf{c}_2}\downarrow_{\mathbf{c}_2}(\uparrow_{\gamma}(X))$$

for all $X \subseteq B_1$.

Proof. Let $X \subseteq B_1$ be a subset. It follows that $\downarrow_{\gamma}\uparrow_{\gamma}(X) \in \text{Ext}(\gamma) \subseteq \text{Ext}(\mathbf{c}_1)$, i.e., $\downarrow_{\gamma}\uparrow_{\gamma}(X) = \downarrow_{\mathbf{c}_1}\uparrow_{\mathbf{c}_1}(\downarrow_{\gamma}\uparrow_{\gamma}(X))$ is a closed element in $\text{Ext}(\mathbf{c}_1)$. Consequently

$$X \subseteq \downarrow_{\mathbf{c}_1}\uparrow_{\mathbf{c}_1}(X) \subseteq \downarrow_{\gamma}\uparrow_{\gamma}(X),$$

where the first inclusion is due to the basic property of closure operators, while the second inclusion follows from $X \subseteq \downarrow_{\gamma}\uparrow_{\gamma}(X) \in \text{Ext}(\mathbf{c}_1)$ and the fact that $\downarrow_{\mathbf{c}_1}\uparrow_{\mathbf{c}_1}(X)$ is the smallest closed set in $\text{Ext}(\mathbf{c}_1)$ containing X . Furthermore, from the properties of Galois connections it follows that $\uparrow_{\gamma}\downarrow_{\gamma}\uparrow_{\gamma}(X) = \uparrow_{\gamma}(X)$.

Hence, applying the antitone operator $\uparrow_{\gamma}$ on the aforementioned two inclusions

$$\uparrow_{\gamma}(X) \supseteq \uparrow_{\gamma}(\downarrow_{\mathbf{c}_1}\uparrow_{\mathbf{c}_1}(X)) \supseteq \uparrow_{\gamma}\downarrow_{\gamma}\uparrow_{\gamma}(X) = \uparrow_{\gamma}(X)$$

is obtained. Since $\uparrow_{\gamma}(X) \in \text{Int}(\gamma) \subseteq \text{Int}(\mathbf{c}_2)$, $\uparrow_{\mathbf{c}_2}\downarrow_{\mathbf{c}_2}(\uparrow_{\gamma}(X)) = \uparrow_{\gamma}(X)$ holds as well. $\square$

For a system $(L(a') : a' \in A)$ of complete lattices and $x \in L(a)$ for some $a \in A$, $\mathbf{0}_x^a \in \prod_{a' \in A} L(a')$ denotes the element of the direct product of lattices $L(a')$, $a' \in A$ given by $\mathbf{0}_x^a(a') = x$ if $a' = a$ and $\mathbf{0}_x^a(a') = 0_{L(a')}$ otherwise. Note that $0_{L(a')}$ represents the bottom element of the complete lattice $L(a')$.

Lemma 2. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$ and $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$ be two generalized one-sided formal contexts. A 4-tuple $\gamma = (B_1, A_2, L_2, R_{12})$ is a bond between $\mathbf{c}_1$ and $\mathbf{c}_2$ if and only if γ satisfies the following two conditions:

- (1) $\uparrow_{\gamma}(b) \in \text{Int}(\mathbf{c}_2)$ for all $b \in B_1$,
- (2) $\downarrow_{\gamma}(\mathbf{0}_x^a) \in \text{Ext}(\mathbf{c}_1)$ for any $a \in A_2$ and $x \in L_2(a)$.

Proof. Obviously, if γ is a bond, then $\uparrow_{\gamma}(b) \in \text{Int}(\gamma) \subseteq \text{Int}(\mathbf{c}_2)$ for all $b \in B_1$ and $\downarrow_{\gamma}(\mathbf{0}_x^a) \in \text{Ext}(\gamma) \subseteq \text{Ext}(\mathbf{c}_1)$ for any $a \in A_2$ and $x \in L_2(a)$. Hence the conditions (1) and (2) are necessary for the bond γ .

Conversely let $g = \uparrow_{\gamma}(X) \in \text{Int}(\gamma)$ for some $X \subseteq B_1$ be an intent. From the properties of the Galois connections we obtain

$$g = \uparrow_{\gamma}(X) = \uparrow_{\gamma}\left(\bigcup_{b \in X} \{b\}\right) = \bigwedge_{b \in X} \uparrow_{\gamma}(b).$$

Since the set of intents is $\wedge$ -closed, the condition (1) yields $g \in \text{Int}(\mathbf{c}_2)$. This shows $\text{Int}(\gamma) \subseteq \text{Int}(\mathbf{c}_2)$.

Similarly, if $X = \downarrow_\gamma(g) \in \text{Ext}(\gamma)$ for some $g \in \prod_{a \in A_2} L_2(a)$, then the condition (2) yields

$$X = \downarrow_\gamma(g) = \downarrow_\gamma\left(\bigvee_{a \in A_2} \mathbf{0}_{g(a)}^a\right) = \bigcap_{a \in A_2} \downarrow_\gamma(\mathbf{0}_{g(a)}^a) \in \text{Ext}(\mathbf{c}_1).$$

Consequently, $\text{Ext}(\gamma) \subseteq \text{Ext}(\mathbf{c}_1)$ holds. $\square$

Remark 1. Condition (1) of the previous Lemma, can be used to obtain an upper bound for the number of bonds $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$.

It suffices to count the number of all contexts $\gamma = (B_1, A_2, L_2, R_{12})$ such that $\uparrow_\gamma(b) \in \text{Int}(\mathbf{c}_2)$ for all $b \in B_1$. However, from Formula (1), the definition of the concept forming operator $\uparrow$, it follows that the number of such contexts is equal to $|\text{Int}(\mathbf{c}_2)|^{|B_1|} = |\mathfrak{C}(\mathbf{c}_2)|^{|B_1|}$. As not all such contexts also satisfy (2), for the number of bonds the following is obtained $|\{\gamma : \mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2\}| \leq |\mathfrak{C}(\mathbf{c}_2)|^{|B_1|}$.

Applying this to Example 2 there is at most 9^5 bonds between the contexts $\mathbf{c}_1$ and $\mathbf{c}_2$.

Theorem 1. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$ and $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$ be two generalized one-sided formal contexts. Then any bond $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$ induces a $\vee$ -preserving mapping $f_\gamma : \mathfrak{C}(\mathbf{c}_1) \rightarrow \mathfrak{C}(\mathbf{c}_2)$ given by

$$f_\gamma((X, g)) = (\downarrow_{\mathbf{c}_2} \uparrow_\gamma(X), \uparrow_\gamma(X)) \quad (5)$$

for all concepts $(X, g) \in \mathfrak{C}(\mathbf{c}_1)$.

Conversely, if $f : \mathfrak{C}(\mathbf{c}_1) \rightarrow \mathfrak{C}(\mathbf{c}_2)$ is a $\vee$ -preserving mapping, the 4-tuple $\gamma_f = (B_1, A_2, L_2, R_f)$ where

$$R_f(b, a) = \text{Int}[f(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}(b), \uparrow_{\mathbf{c}_1}(b))](a), \quad (6)$$

for all $b \in B_1$ and $a \in A_2$, is a bond between the contexts $\mathbf{c}_1$ and $\mathbf{c}_2$.

Proof. Let γ be a bond between the contexts $\mathbf{c}_1$ and $\mathbf{c}_2$. Lemma 1 yields $\uparrow_{\mathbf{c}_2} \downarrow_{\mathbf{c}_2}(\uparrow_\gamma(X)) = \uparrow_\gamma(X)$ for any $X \in \text{Ext}(\mathbf{c}_1)$, thus $(\downarrow_{\mathbf{c}_2} \uparrow_\gamma(X), \uparrow_\gamma(X)) \in \mathfrak{C}(\mathbf{c}_2)$. Hence f_γ is a mapping with the domain $\mathfrak{C}(\mathbf{c}_1)$ and the range $\text{Rng}(f_\gamma) \subseteq \mathfrak{C}(\mathbf{c}_2)$. Furthermore, let $((X_i, g_i) : i \in I) \subseteq \mathfrak{C}(\mathbf{c}_1)$ be an indexed system of concepts from $\mathfrak{C}(\mathbf{c}_1)$. Then expressing the supremum in $\mathfrak{C}(\mathbf{c}_1)$, using Lemma 1 and the basic properties of Galois connections we obtain

$$\begin{aligned} f_\gamma\left(\bigvee_{i \in I} (X_i, g_i)\right) &= f_\gamma\left(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}\left(\bigcup_{i \in I} X_i\right), \bigwedge_{i \in I} g_i\right) = (\downarrow_{\mathbf{c}_2} \uparrow_\gamma(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}(\bigcup_{i \in I} X_i)), \uparrow_\gamma(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}(\bigcup_{i \in I} X_i))) = \\ &= (\downarrow_{\mathbf{c}_2} \uparrow_\gamma(\bigcup_{i \in I} X_i), \uparrow_\gamma(\bigcup_{i \in I} X_i)) = (\downarrow_{\mathbf{c}_2}(\bigwedge_{i \in I} \uparrow_\gamma(X_i)), \bigwedge_{i \in I} \uparrow_\gamma(X_i)). \end{aligned}$$

On the other side, with respect to (5) and involving the formula for computing the supremum in the lattice $\mathfrak{C}(\mathbf{c}_2)$, it follows that

$$\bigvee_{i \in I} f_\gamma((X_i, g_i)) = \bigvee_{i \in I} (\downarrow_{\mathbf{c}_2} \uparrow_\gamma(X_i), \uparrow_\gamma(X_i)) = (\downarrow_{\mathbf{c}_2}(\bigwedge_{i \in I} \uparrow_\gamma(X_i)), \bigwedge_{i \in I} \uparrow_\gamma(X_i)).$$

Hence, the mapping f_γ is $\vee$ -preserving, which completes the first part of the proof.

Now assume that a mapping $f : \mathfrak{C}(\mathbf{c}_1) \rightarrow \mathfrak{C}(\mathbf{c}_2)$ is $\vee$ -preserving. To obtain the assertion, Lemma 2 is applied. Let $b \in B_1$ be an element. It follows that $\uparrow_{\gamma_f}(b)(a) = R_f(b, a)$ for all $a \in A_2$. However, according to (6) for each $a \in A_2$, $\uparrow_{\gamma_f}(b)(a) = \text{Int}[f(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}(b), \uparrow_{\mathbf{c}_1}(b))](a)$ holds, thus $\uparrow_{\gamma_f}(b) = \text{Int}[f(\downarrow_{\mathbf{c}_1} \uparrow_{\mathbf{c}_1}(b), \uparrow_{\mathbf{c}_1}(b))] \in \text{Int}(\mathbf{c}_2)$.

Furthermore, let $a \in A_2$ and $x \in L_2(a)$ be arbitrary elements and

$$Z = \downarrow_{\gamma_f}(\mathbf{0}_x^a) = \{b \in B_1 : R_f(b, a) \geq x\}.$$

Hence according to the definition (6) of $R_f(b, a)$ it follows that $b \in Z$ if and only if $\text{Int}[f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))] \geq \mathbf{0}_x^a$, which is equivalent to

$$b \in Z \quad \text{iff} \quad f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b)) \leq (\downarrow_{c_2}(\mathbf{0}_x^a), \uparrow_{c_2} \downarrow_{c_2}(\mathbf{0}_x^a)).$$

We show that $Z \in \text{Ext}(\mathbf{c}_1)$, i.e., $Z = \downarrow_{c_1} \uparrow_{c_1}(Z)$. Obviously $Z \subseteq \downarrow_{c_1} \uparrow_{c_1}(Z)$. Since $\downarrow_{c_1}(\uparrow_{c_1}(Z)) = \bigvee_{b \in Z} \downarrow_{c_1} \uparrow_{c_1}(b)$ from the fact that f is $\vee$ -preserving

$$f(\downarrow_{c_1} \uparrow_{c_1}(Z), \uparrow_{c_1}(Z)) = f\left(\bigvee_{b \in Z} (\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))\right) =$$

$$\bigvee_{b \in Z} f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b)) \leq (\downarrow_{c_2}(\mathbf{0}_x^a), \uparrow_{c_2} \downarrow_{c_2}(\mathbf{0}_x^a))$$

is obtained. Since for any element $b' \in \downarrow_{c_1} \uparrow_{c_1}(Z)$ the inequality $(\downarrow_{c_1} \uparrow_{c_1}(b'), \uparrow_{c_1}(b')) \leq (\downarrow_{c_1} \uparrow_{c_1}(Z), \uparrow_{c_1}(Z))$ holds, the inequality $f(\downarrow_{c_1} \uparrow_{c_1}(b'), \uparrow_{c_1}(b')) \leq (\downarrow_{c_2}(\mathbf{0}_x^a), \uparrow_{c_2} \downarrow_{c_2}(\mathbf{0}_x^a))$ for every $b' \in \downarrow_{c_1} \uparrow_{c_1}(Z)$ is obtained as well, which yields $\downarrow_{c_1} \uparrow_{c_1}(Z) \subseteq Z$. $\square$

To illustrate the mentioned correspondence between bonds and $\vee$ -preserving mappings the following example is provided.

Example 3. Let $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$ be the bond presented in Example 2. The corresponding generalized one-sided concept lattices $\mathfrak{C}(\mathbf{c}_1)$ and $\mathfrak{C}(\mathbf{c}_2)$ are depicted in Figure 1 and Figure 2 respectively. The corresponding $\vee$ -preserving mapping f_γ derived by Formula (5) is depicted in Figure 3.

Figure 3. The $\vee$ -preserving mapping f_γ corresponding to the bond γ from Example 2.

Theorem 2. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$ and $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$ be two generalized one-sided formal contexts. The assignment $f \mapsto \gamma_f$ represents a one-to-one correspondence between the set of all $\vee$ -preserving mappings and the set of all $\mathbf{c}_1, \mathbf{c}_2$ bonds.

Proof. We show that the mappings defined by $f \mapsto \gamma_f$ and $\gamma \mapsto f_\gamma$ are mutually inverse, i.e., the identities $f_{\gamma_f} = f$ and $\gamma_{f_\gamma} = \gamma$ are valid. Let $f: \mathfrak{C}(\mathbf{c}_1) \rightarrow \mathfrak{C}(\mathbf{c}_2)$ be a $\vee$ -preserving mapping. Since for each concept $(X, g) \in \mathfrak{C}(\mathbf{c}_1)$ holds

$$f((X, g)) = f\left(\bigvee_{b \in X} (\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))\right) = \bigvee_{b \in X} f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b)),$$

it suffices to show that f_{γ_f} and f agree on concepts $(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))$, $b \in B_1$. The intent of $f_{\gamma_f}(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))$ equals to $\uparrow_{\gamma_f}(\downarrow_{c_1} \uparrow_{c_1}(b)) = \uparrow_{\gamma_f}(b)$, where the last equality is due to Lemma 1. However, by (1) for each $a \in A_2$

$$\uparrow_{\gamma_f}(b)(a) = R_f(b, a) = \text{Int}[f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))](a)$$

holds, i.e., the intents of $f_{\gamma_f}(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))$ and $f(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))$ are the same. Consequently, $f_{\gamma_f} = f$.

Furthermore, let $\gamma = (B_1, A_2, L_2, R)$ be a bond. Obviously it suffices to show $R = R_{f_\gamma}$, where R_{f_γ} is defined by (6). In this case, using Formula (5) and Lemma 1

$$R_{f_\gamma}(b, a) = \text{Int}[f_\gamma(\downarrow_{c_1} \uparrow_{c_1}(b), \uparrow_{c_1}(b))](a) = \text{Int}[(\downarrow_{c_2} \uparrow_\gamma(\downarrow_{c_1} \uparrow_{c_1}(b)), \uparrow_\gamma(\downarrow_{c_1} \uparrow_{c_1}(b)))](a) = \\ \uparrow_\gamma(\downarrow_{c_1} \uparrow_{c_1}(b))(a) = \uparrow_\gamma(b)(a) = R(b, a)$$

for all $b \in B_1$ and $a \in A_2$ is obtained, which completes the proof. $\square$

Definition 2. Let $\mathbf{c}_1 = (B_1, A_1, L_1, R_1)$, $\mathbf{c}_2 = (B_2, A_2, L_2, R_2)$, $\mathbf{c}_3 = (B_3, A_3, L_3, R_3)$ be generalized one-sided contexts and $\mathbf{c}_1 \xrightarrow{\gamma_{12}} \mathbf{c}_2$, $\mathbf{c}_2 \xrightarrow{\gamma_{23}} \mathbf{c}_3$ be bonds. The composition $\gamma_{12} \diamond \gamma_{23}$ of bonds is defined as a 4-tuple $\gamma_{13} = (B_1, A_3, L_3, R_{13})$, where

$$R_{13}(b, a) = \bigwedge_{c \in \downarrow_{c_2} \uparrow_{\gamma_{12}}(b)} R_{23}(c, a), \quad (7)$$

for all $b \in B_1$ and for all $a \in A_3$.

Lemma 3. Let $\mathbf{c}_1 \xrightarrow{\gamma_{12}} \mathbf{c}_2$, $\mathbf{c}_2 \xrightarrow{\gamma_{23}} \mathbf{c}_3$ be bonds and $\gamma_{13} = \gamma_{12} \diamond \gamma_{23}$. Then γ_{13} is a bond between $\mathbf{c}_1$ and $\mathbf{c}_3$, and $\uparrow_{\gamma_{13}}(X) = \uparrow_{\gamma_{23}} \downarrow_{c_2} \uparrow_{\gamma_{12}}(X)$ for all $X \subseteq B_1$.

Proof. Let $X \subseteq B_1$ be a subset. First $\uparrow_{\gamma_{13}}(X) = \uparrow_{\gamma_{23}} \downarrow_{c_2} \uparrow_{\gamma_{12}}(X)$ is proved. From the properties of Galois connections and due to Lemma 1 it follows that

$$\downarrow_{c_2} \uparrow_{c_2} \left(\bigcup_{b \in X} \downarrow_{c_2} \uparrow_{\gamma_{12}}(b) \right) = \downarrow_{c_2} \left(\bigwedge_{b \in X} \uparrow_{c_2} \downarrow_{c_2} (\uparrow_{\gamma_{12}}(b)) \right) = \downarrow_{c_2} \left(\bigwedge_{b \in X} \uparrow_{\gamma_{12}}(b) \right) = \downarrow_{c_2} \uparrow_{\gamma_{12}}(X).$$

Using this and applying Lemma 1 again

$$\uparrow_{\gamma_{23}}(\downarrow_{c_2} \uparrow_{\gamma_{12}}(X)) = \uparrow_{\gamma_{23}}(\downarrow_{c_2} \uparrow_{c_2} \left(\bigcup_{b \in X} \downarrow_{c_2} \uparrow_{\gamma_{12}}(b) \right)) = \uparrow_{\gamma_{23}} \left(\bigcup_{b \in X} \downarrow_{c_2} \uparrow_{\gamma_{12}}(b) \right)$$

is obtained. According to (1), for any $a \in A_3$

$$\uparrow_{\gamma_{23}} \left(\bigcup_{b \in X} \downarrow_{c_2} \uparrow_{\gamma_{12}}(b) \right)(a) = \bigwedge_{c \in \bigcup_{b \in X} \downarrow_{c_2} \uparrow_{\gamma_{12}}(b)} R_{23}(c, a) = \bigwedge_{b \in X} \left(\bigwedge_{c \in \downarrow_{c_2} \uparrow_{\gamma_{12}}(b)} R_{23}(c, a) \right)$$

holds. On the other side

$$\uparrow_{\gamma_{13}}(X)(a) = \bigwedge_{b \in X} R_{13}(b, a) = \bigwedge_{b \in X} \left(\bigwedge_{c \in \downarrow_{c_2} \uparrow_{\gamma_{12}}(b)} R_{23}(c, a) \right),$$

which shows $\uparrow_{\gamma_{13}}(X) = \uparrow_{\gamma_{23}} \downarrow_{c_2} \uparrow_{\gamma_{12}}(X)$.

Furthermore, for any $X \subseteq B_1$ and $g \in \prod_{a \in A_3} L_3(a)$ the following two equivalent assertions are obtained:

$$g \leq \uparrow_{\gamma_{23}}(\downarrow_{c_2} \uparrow_{\gamma_{12}}(X)) \quad \text{iff} \quad \downarrow_{c_2} \uparrow_{\gamma_{12}}(X) \subseteq \downarrow_{\gamma_{23}}(g)$$

$$\uparrow_{\gamma_{12}}(X) \supseteq \uparrow_{c_2} \downarrow_{\gamma_{23}}(g) \quad \text{iff} \quad X \subseteq \downarrow_{\gamma_{12}}(\uparrow_{c_2} \downarrow_{\gamma_{23}}(g)).$$

Since $\uparrow_{\gamma_{12}}(X) \in \text{Int}(\mathbf{c}_2)$ as well as $\downarrow_{\gamma_{23}}(g) \in \text{Ext}(\mathbf{c}_2)$, it follows $\downarrow_{c_2} \uparrow_{\gamma_{12}}(X) \subseteq \downarrow_{\gamma_{23}}(g)$ if and only if $X \subseteq \downarrow_{\gamma_{12}}(\uparrow_{c_2} \downarrow_{\gamma_{23}}(g))$. Hence the mapping $\downarrow_{\gamma_{12}} \uparrow_{c_2} \downarrow_{\gamma_{23}}(g)$ represents the dual adjoint to $\uparrow_{\gamma_{13}}$, i.e., $\downarrow_{\gamma_{13}}(g) = \downarrow_{\gamma_{12}} \uparrow_{c_2} \downarrow_{\gamma_{23}}(g)$ for all $g \in \prod_{a \in A_3} L_3(a)$. Consequently, $\text{Int}(\gamma_{13}) \subseteq \text{Int}(\gamma_{23}) \subseteq \text{Int}(\mathbf{c}_3)$ and $\text{Ext}(\gamma_{13}) \subseteq \text{Ext}(\gamma_{12}) \subseteq \text{Ext}(\mathbf{c}_1)$, proving that γ_{13} is a bond between $\mathbf{c}_1$ and $\mathbf{c}_3$. $\square$

Given a generalized one-sided formal context $\mathbf{c}$, it can be itself considered to be a bond between $\mathbf{c}$ and $\mathbf{c}$. This is indicated by $\mathbf{c} \xrightarrow{1_{\mathbf{c}}} \mathbf{c}$, i.e., formally $1_{\mathbf{c}} = \mathbf{c}$ holds.

Corollary 1. Let $\mathbf{c} = (B_{\mathbf{c}}, A_{\mathbf{c}}, L_{\mathbf{c}}, R_{\mathbf{c}})$ be a generalized one-sided formal context. Then $\mathbf{c} \xrightarrow{1_{\mathbf{c}}} \mathbf{c}$ represent the identity on $\mathbf{c}$ with respect to operation $\diamond$. If $\mathbf{c}_1 \xrightarrow{\gamma_{12}} \mathbf{c}_2$ and $\mathbf{c}_2 \xrightarrow{\gamma_{23}} \mathbf{c}_3$ are bonds, then $f_{\gamma_{12}} \circ f_{\gamma_{23}} = f_{\gamma_{12} \diamond \gamma_{23}}$

Proof. Let $\mathbf{b} \xrightarrow{\gamma} \mathbf{c}$, where $\mathbf{b} = (B_{\mathbf{b}}, A_{\mathbf{b}}, L_{\mathbf{b}}, R_{\mathbf{b}})$, be a bond. Then according to the previous Lemma and Lemma 1 $\uparrow_{\gamma \circ 1_{\mathbf{c}}}(X) = \uparrow_{\gamma}(\downarrow_{\mathbf{c}} \uparrow_{\mathbf{c}}(X)) = \uparrow_{\gamma}(X)$ is valid for all $X \subseteq B_{\mathbf{b}}$. Since the corresponding formal contexts are determined uniquely, it follows $\gamma \circ 1_{\mathbf{c}} = \gamma$. Similarly, for a bond $\mathbf{c} \xrightarrow{\gamma} \mathbf{b}$ the equality $\uparrow_{1_{\mathbf{c}} \diamond \gamma}(X) = \uparrow_{\mathbf{c}} \downarrow_{\mathbf{c}}(\uparrow_{\gamma}(X)) = \uparrow_{\gamma}(X)$ is obtained for all $X \subseteq B_{\mathbf{b}}$, which yields $1_{\mathbf{c}} \diamond \gamma = \gamma$.

Furthermore, put $\gamma_{12} \diamond \gamma_{23} = \gamma_{13}$. Then for any concept $(X, g) \in \mathfrak{C}(\mathbf{c}_1)$

$$f_{\gamma_{13}}(X, g) = (\downarrow_{\mathbf{c}_3}(\uparrow_{\gamma_{13}}(X)), \uparrow_{\gamma_{13}}(X)) = (\downarrow_{\mathbf{c}_3} \uparrow_{\gamma_{23}} \downarrow_{\mathbf{c}_2} \uparrow_{\gamma_{12}}(X), \uparrow_{\gamma_{23}} \downarrow_{\mathbf{c}_2} \uparrow_{\gamma_{12}}(X))$$

is valid, and

$$(f_{\gamma_{12}} \circ f_{\gamma_{23}})(X, g) = f_{\gamma_{23}}(\downarrow_{\mathbf{c}_2} \uparrow_{\gamma_{12}}(X), \uparrow_{\gamma_{12}}(X)) = (\downarrow_{\mathbf{c}_3} \uparrow_{\gamma_{23}} \downarrow_{\mathbf{c}_2} \uparrow_{\gamma_{12}}(X), \uparrow_{\gamma_{23}} \downarrow_{\mathbf{c}_2} \uparrow_{\gamma_{12}}(X)),$$

hence the identity $f_{\gamma_{12}} \circ f_{\gamma_{23}} = f_{\gamma_{12} \diamond \gamma_{23}}$ holds. $\square$

As the final step in order to define a category of contexts with bonds, it remains to show that the introduced composition of bonds is associative. Although it could be verified directly, it is used more transparent approach using the previous corollary and the well known fact that the composition of mappings is associative.

Let $\mathbf{c}_1 \xrightarrow{\gamma_{12}} \mathbf{c}_2$, $\mathbf{c}_2 \xrightarrow{\gamma_{23}} \mathbf{c}_3$ and $\mathbf{c}_3 \xrightarrow{\gamma_{34}} \mathbf{c}_4$ be bonds between generalized one-sided contexts. Then

$$f_{(\gamma_{12} \diamond \gamma_{23}) \diamond \gamma_{34}} = (f_{\gamma_{12}} \circ f_{\gamma_{23}}) \circ f_{\gamma_{34}} = f_{\gamma_{12}} \circ (f_{\gamma_{23}} \circ f_{\gamma_{34}}) = f_{\gamma_{12} \diamond (\gamma_{23} \diamond \gamma_{34})}$$

and since the correspondence $\gamma \mapsto f_{\gamma}$ is injective, it follows that

$$(\gamma_{12} \diamond \gamma_{23}) \diamond \gamma_{34} = \gamma_{12} \diamond (\gamma_{23} \diamond \gamma_{34}),$$

i.e., the partial operation $\diamond$ on bonds is associative.

In what follows denote by $\mathcal{K}$ the category of generalized one-sided contexts with bonds and the operation $\diamond$ as the composition of bonds. Furthermore, let $\mathcal{C}$ denotes the category of complete lattices with $\vee$ -preserving mappings. Corollary 1 shows that the correspondence $F: \mathcal{K} \rightarrow \mathcal{C}$, given by $F(\mathbf{c}) = \mathfrak{C}(\mathbf{c})$ for all contexts $\mathbf{c} \in \mathcal{K}$ and $F(\gamma) = f_{\gamma}$ for all bonds $\mathbf{c}_1 \xrightarrow{\gamma} \mathbf{c}_2$, represents a functor. As the final result it is shown that the two respective categories are equivalent.

Theorem 3. The categories $\mathcal{K}$ of generalized one-sided formal contexts with bonds and $\mathcal{C}$ of complete lattices with $\vee$ -preserving mappings are equivalent.

Proof. Recall, see [19], that it is the well known fact that two categories are equivalent if and only if there is a functor between them which is faithful, full and essentially surjective on objects. We show that the functor F fulfills these properties.

According to Theorem 2, for arbitrary $\mathbf{c}_1, \mathbf{c}_2 \in \mathcal{K}$ the mapping $F': \text{Hom}_{\mathcal{K}}(\mathbf{c}_1, \mathbf{c}_2) \rightarrow \text{Hom}_{\mathcal{C}}(\mathfrak{C}(\mathbf{c}_1), \mathfrak{C}(\mathbf{c}_2))$ given by $F'(\gamma) = F(\gamma) = f_{\gamma}$ is bijective. Since F' is injective and surjective, the functor F is faithful and full respectively.

Furthermore, any complete lattice can be represented as a lattice of fixed points of a Galois connection, i.e., for any $L \in \mathcal{C}$ there is a context $\mathbf{c} \in \mathcal{K}$ such that $L \cong \mathfrak{C}(\mathbf{c})$, cf. [6] or [1]. Hence the functor F is also essentially surjective on objects. $\square$

Categorical methods can be successfully applied to the problematics of representation and reduction of various mathematical objects. Concerning fuzzy concept lattices, these

topics represents a very active research area, cf. [20] or [8,9] for the case of generalized one-sided concept lattices reduction, and [21], where a representation of generalized fuzzy concept lattices as the classical ones is described. A possible application of bonds to the mentioned problematic is briefly discussed in the next section.

4. Conclusions

The notion of bond within the theory of generalized one-sided concept lattices was introduced. For these types of concept lattices, a bond between two formal contexts is a formal context with the object set of the former one and with the attribute set from the second one. The main construction in various FCA-based methods of hierarchical analysis is a creation of the concept lattice corresponding to some object-attribute model (formal context). The notion of bond was defined in order to find suitable morphisms between generalized one-sided contexts so that the mentioned fundamental construction of concept lattices becomes functorial. A composition of bond was defined in an appropriate way, such that the class of all generalized one-sided formal contexts together with bonds as morphisms forms a category. It was shown that every bond uniquely induces a supremum preserving mapping between concept lattices and vice versa. As the main result it was proved that the introduced category of generalized one-sided contexts with bonds is equivalent to the category of complete lattices with supremum preserving mappings as the morphisms.

A possible application of bonds can be seen in the area concerning a reduction of generalized one-sided concept lattices. A reduction is basically performed on two levels. The first one is connected with attribute reduction, i.e., reducing the number of attributes while maintaining the entire structure of the resulting concept lattice. The second consists of reduction of overall number of concepts. As is well known, in the worst case the size of a concept lattice can be exponential with respect to number of objects, what can be problematic in some practical applications. Both types of reduction can be obtained using the fact, that a bond is also a generalized one-sided formal context. Attribute reduction of a given generalized one-sided formal context c can be obtained by finding a suitable bond γ between c and c_1 , where c_1 is a context with reduced set of attributes and γ is an isomorphism. As γ is “equivalent” to identity bond $1_c = c$, the definition of bond yields that the mappings $\uparrow_\gamma$ and its dual adjoint $\downarrow_\gamma$ form a concept forming operators corresponding to the reduced context γ with $\mathcal{C}(\gamma)$ isomorphic to $\mathcal{C}(c)$. Such method can be also used in the problem of representation of fuzzy concept lattices, where the aim is to find another type of context such that the corresponding concept lattices being isomorphic. In our opinion the method described above can incorporate representation of fuzzy concept lattices within the classical ones or other types of fuzzy concept lattices.

Reduction of concept lattice can be obtained by finding a suitable bond γ between the identical contexts c and c . In this case the corresponding concept lattice $\mathcal{C}(\gamma)$ will be a join subsemilattice of the former concept lattice $\mathcal{C}(c)$. Hence, bonds can be useful in reduction process of generalized one-sided concept lattices. In a future work, our goal is to find and describe some efficient ways how to define suitable bonds useful in reduction and representation of generalized one-sided concept lattices.

For the classical concept lattices the notion of bond was introduced for characterization complete sublattices of a direct product of concept lattices for which the projection mappings are surjective. Sublattices of a direct product with this property are known as the so-called subdirect product, representing one of the most important construction in universal algebra. Our further aim will be possible study of subdirect products of generalized one-sided concept lattices by means of bonds, and also to study the notions related to this construction e.g., subdirectly irreducible lattices.

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Abbreviation

The following abbreviation is used in this manuscript:

FCA	Formal Concept Analysis
MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	linear dichroism

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