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Article

On Characterizing a Three-Dimensional Sphere

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Abstract: In this paper, we find a characterization of the 3-sphere using 3-dimensional compact and simply connected trans-Sasakian manifolds of type (α, β).

Keywords: trans-Sasakian manifold; scalar curvature; sphere; Fischer–Marsden differential equation

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1. Introduction

The geometry of 3-dimensional Riemannian spaces has special importance owing to Thurston's conjecture (see [1]). In particular, spherical geometry, one of the eight Thurston geometries, is of primary relevance (cf. [2]). Remarkable examples of manifolds corresponding to this geometry are provided by the Poincaré dodecahedral space, three-dimensional spheres, and also lens spaces. We will focus our study on three-dimensional spheres, these spaces being of special importance also from the perspective of their own elegant geometry.

On the other hand, on a three-dimensional almost contact metric manifold $(N, \Psi, \zeta, \eta, g)$, there is a special structure depending on two smooth functions α and β on N, known as a trans-Sasakian structure, which leads to the concept of a trans-Sasakian manifold $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ (cf. [3]). It is known that beyond dimension three, a trans-Sasakian manifold $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ is a cosymplectic manifold, an α-Sasakian manifold, or a β-Kenmotsu manifold (cf. [4]), and because of this, almost all recent studies have been performed to elucidate the geometry of trans-Sasakian 3-manifolds (see, e.g., [5–8]). In this article, we shall use the abbreviation TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ for a 3-dimensional trans-Sasakian manifold $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$. A basic problem in studying the geometry of trans-Sasakian spaces consists in finding conditions under which such a space is homothetic to a Sasakian manifold (cf. [6,9–17]). Naturally, a 3-dimensional sphere $\mathbf{S}^3(c)$ of constant curvature c is a TRSM $(\mathbf{S}^3(c), \Psi, \zeta, \eta, g, \alpha, \beta)$ with $\alpha = \sqrt{c}$ and $\beta = 0$ (see next section). This leads to the following question: under what conditions is a compact and simply connected TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ isometric to $\mathbf{S}^3(c)$? Our goal is to get an answer to this question, providing a characterization of three-dimensional spheres using compact and simply connected trans-Sasakian 3-manifolds of type (α, β). A key role in this will be played by the famous Fischer–Marsden equation (see [18]). Recall that the Fischer–Marsden differential equation on a 3-dimensional Riemannian manifold (N, g) is as follows:

$$(\Delta f)g + fRic = H_f, \tag{1}$$

where H_f is the Hessian of function f, Δ is the Laplace operator, and Ric is the Ricci tensor of (N, g).

We are going to specifically answer the question raised above by showing that a compact and simply connected TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ with Ricci operator satisfying a Codazzi-type condition and β satisfying the differential Equation (1), and also with scalar curvature bounded above by a certain bound involving functions α and β, is isometric to a 3-sphere (see Theorem 1).

2. Preliminaries

Consider a 3-dimensional almost contact metric manifold $(N, \Psi, \zeta, \eta, g)$, where Ψ is a $(1,1)$-tensor field, ζ is a unit vector field, and η is the smooth 1-form dual to ζ with respect to the Riemannian metric g satisfying (cf. [19–21])

$$\Psi^2 = -I + \eta \otimes \zeta, \Psi\zeta = 0, \eta \circ \Psi = 0, g(\Psi E_1, \Psi E_2) = g(E_1, E_2) - \eta(E_1)\eta(E_2), \tag{2}$$

for all $E_1, E_2 \in \mathfrak{X}(N)$, where $\mathfrak{X}(N)$ is the Lie algebra of smooth vector fields on N. If there are smooth functions α and β defined on an almost contact metric manifold $(N, \Psi, \zeta, \eta, g)$ satisfying

$$(D\Psi)(E_1, E_2) = \alpha(g(E_1, E_2)\zeta - \eta(E_2)E_1) + \beta(g(\Psi E_1, E_2)\zeta - \eta(E_2)\Psi E_1), \tag{3}$$

then we get a TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$, where

$$(D\Psi)(E_1, E_2) = D_{E_1}\Psi(E_2) - \Psi(D_{E_1}E_2),\ E_1, E_2 \in \mathfrak{X}(N)$$

and D is the Levi–Civita connection with respect to the metric g (cf. [10–13,15–17,22]). We see that Equations (2) and (3) imply

$$D_E\zeta = -\alpha\Psi(E) + \beta(E - \eta(E)\zeta), \quad E \in \mathfrak{X}(N). \tag{4}$$

The Ricci tensor Ric of a Riemannian manifold (N, g) gives a symmetric $(1,1)$ tensor field S, called Ricci operator, defined by

$$Ric(E_1, E_2) = g(SE_1, E_2),\ E_1, E_2 \in \mathfrak{X}(N).$$

Furthermore, the scalar curvature $\mathbf{s}$ is defined by

$$\mathbf{s} = TrS.$$

We have the following relations on a trans-Sasakian manifold (cf. [15–17]):

$$\zeta(\alpha) = -2\alpha\beta, \tag{5}$$

$$S(\zeta) = \Psi(\nabla\alpha) - \nabla\beta + 2(\alpha^2 - \beta^2)\zeta - \zeta(\beta)\zeta, \tag{6}$$

where $\nabla\alpha$, $\nabla\beta$ are gradients of functions α, β respectively. Note that Equation (4) implies

$$\mathrm{div}\zeta = 2\beta. \tag{7}$$

Thus, on a compact TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$, using Equation (7), one finds

$$\int_N \beta = 0.$$

For a smooth function σ on a Riemannian manifold (N, g), the Hessian operator A_σ of σ is defined by

$$A_\sigma(E) = D_E\nabla\sigma, \quad E \in \mathfrak{X}(N), \tag{8}$$

while the Hessian H_σ is given by

$$H_\sigma(E_1, E_2) = g(A_\sigma(E_1), E_2), \quad E_1, E_2 \in \mathfrak{X}(N).$$

The Laplace operator Δ on a Riemannian manifold (N, g) is defined by

$$\Delta\sigma = \operatorname{div}(\nabla\sigma).$$

We also have

$$\Delta\sigma = TrA_\sigma.$$

Next, we shall show that the sphere $\mathbf{S}^3(c)$ of constant curvature c has a trans-Sasakian structure. It is clear that $\mathbf{S}^3(c)$ is an embedded surface in the Euclidean space $\mathbf{R}^4$ with unit normal ξ and shape operator B given by $B = -\sqrt{c}I$. Using complex structure $\mathbf{J}$ on $\mathbf{R}^4$ that is compatible with the Euclidean metric $\langle , \rangle$ and makes $\left(\mathbf{R}^4, \mathbf{J}, \langle , \rangle\right)$ a Kähler manifold, we define an operator Ψ on $\mathbf{S}^3(c)$ by

$$\mathbf{J}E = \Psi(E) + \eta(E)\zeta, \quad E \in \mathfrak{X}(\mathbf{S}^3(c)), \tag{9}$$

where $\zeta = -\mathbf{J}\xi$ is the unit vector field on $\mathbf{S}^3(c)$, η is the 1-form dual to ζ with respect to the induced metric g on $\mathbf{S}^3(c)$, and $\Psi(E)$ is the projection of $\mathbf{J}E$ on $\mathbf{S}^3(c)$. Then, it follows that the quadruplet (Ψ, ζ, η, g) satisfies (2) by virtue of the properties of the operator $\mathbf{J}$, that is, $\left(\mathbf{S}^3(c), \Psi, \zeta, \eta, g\right)$ is an almost contact metric manifold. Now, the fundamental equations of the hypersurface $\mathbf{S}^3(c) \subset \mathbf{R}^4$ are

$$\overline{D}_{E_1}E_2 = D_{E_1}E_2 - \sqrt{c}g(E_1, E_2)\xi, \quad \overline{D}_{E_1}\xi = \sqrt{c}E_1, \quad E_1, E_2 \in \mathfrak{X}(\mathbf{S}^3(c)), \tag{10}$$

where $\overline{D}$ is the Euclidean connection on $\mathbf{R}^4$, and D is the induced Riemannian connection on $\mathbf{S}^3(c)$. Taking the covariant derivative in Equation (9) and using $\zeta = -\mathbf{J}\xi$, as well as Equation (10) and the fact that $\mathbf{J}$ is parallel, we get

$$(D\Psi)(E_1, E_2) = \sqrt{c}(g(E_1, E_2)\zeta - \eta(E_2)E_1), \quad E_1, E_2 \in \mathfrak{X}(\mathbf{S}^3(c)).$$

This proves that $\mathbf{S}^3(c)$ has a trans-Sasakian structure $(\Psi, \zeta, \eta, g, \alpha, \beta)$, where $\alpha = \sqrt{c}$ and $\beta = 0$.

3. A Characterization of 3-Spheres

We are interested in characterizing 3-spheres using a TRSM $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$. In the following result, we see that the combination of the Fischer–Marsden differential equation and an upper bound on the scalar curvature involving the functions α, β helps us in reaching the goal.

Theorem 1. *Let $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ be a 3-dimensional compact and simply connected trans-Sasakian manifold. Then, the following three conditions are satisfied if and only if α is a nonzero constant and (N, g) is isometric to the sphere $\mathbf{S}^3(\alpha^2)$:*

(1) β is a solution of Fischer–Marsden differential Equation (1);
(2) The scalar curvature $\mathbf{s}$ satisfies the inequality: $\mathbf{s}\beta^2 \leq 4(\alpha^2 - \beta^2)\beta^2$;
(3) The Ricci operator S satisfies the Codazzi condition:

$$(DS)(E, \zeta) = (DS)(\zeta, E), \quad E \in \mathfrak{X}(N). \tag{11}$$

Proof. Suppose $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$ satisfies the hypothesis of the Theorem. Then, we have

$$(\Delta\beta)g + \beta Ric = H_\beta \tag{12}$$

and

$$\beta^2\left(\alpha^2 - \beta^2 - \frac{\mathbf{s}}{4}\right) \geq 0, \tag{13}$$

as well as Equation (11).

Using $D_\zeta \zeta = 0$ (an outcome of Equation (4)), we get

$$H_\beta(\zeta, \zeta) = \zeta\zeta(\beta).$$

Moreover, on using Equation (6), we obtain

$$Ric(\zeta, \zeta) = 2\left(\alpha^2 - \beta^2 - \zeta(\beta)\right).$$

Combining the last two equations with (12), we conclude

$$\Delta\beta + 2\beta\left(\alpha^2 - \beta^2 - \zeta(\beta)\right) = \zeta\zeta(\beta),$$

or

$$\Delta\beta + 2\beta\left(\alpha^2 - \beta^2\right) = \zeta\zeta(\beta) + 2\beta\zeta(\beta). \tag{14}$$

Using Equation (7), we derive

$$\mathrm{div}(\zeta(\beta)\zeta) = \zeta\zeta(\beta) + 2\beta\zeta(\beta),$$

and inserting this in Equation (14), we obtain

$$\beta\Delta\beta + 2\beta^2\left(\alpha^2 - \beta^2\right) = \beta\mathrm{div}(\zeta(\beta)\zeta).$$

But the above equation, in view of

$$\mathrm{div}(\beta\zeta(\beta)\zeta) = (\zeta(\beta))^2 + \beta\mathrm{div}(\zeta(\beta)\zeta),$$

takes the form

$$\beta\Delta\beta + 2\beta^2\left(\alpha^2 - \beta^2\right) = \mathrm{div}(\beta\zeta(\beta)\zeta) - (\zeta(\beta))^2. \tag{15}$$

Taking the trace in Equation (12), we get

$$3\Delta\beta + \beta\mathbf{s} = \Delta\beta,$$

that is

$$\beta\Delta\beta = -\frac{1}{2}\beta^2\mathbf{s},$$

and inserting it in Equation (15), we get

$$-\frac{1}{2}\beta^2\mathbf{s} + 2\beta^2\left(\alpha^2 - \beta^2\right) = \mathrm{div}(\beta\zeta(\beta)\zeta) - (\zeta(\beta))^2.$$

Integrating the above equation, we have

$$\int_M 2\beta^2\left[\left(\alpha^2 - \beta^2\right) - \frac{\mathbf{s}}{4}\right] + \int_M (\zeta(\beta))^2 = 0.$$

Using inequality (13) in the above equation, we conclude

$$\beta^2\left[\left(\alpha^2 - \beta^2\right) - \frac{\mathbf{s}}{4}\right] = 0 \text{ and } \zeta(\beta) = 0. \tag{16}$$

Hence, we derive

$$\beta = 0 \text{ or } \alpha^2 - \beta^2 = \frac{\mathbf{s}}{4}. \tag{17}$$

Now, summing Equation (11) over an orthonormal frame $\{w_1, w_2, w_3\}$ and using the well-known identity

$$\frac{1}{2}\nabla\mathbf{s} = \sum_{i=1}^{3}(DS)(w_i, w_i),$$

we get

$$\frac{1}{2}\zeta(\mathbf{s}) = \zeta(\mathbf{s}).$$

Thus, we conclude

$$\zeta(\mathbf{s}) = 0.$$

We claim that $\alpha \neq 0$. If we suppose $\alpha = 0$, then (4) assumes the form

$$D_E\zeta = \beta[E - \eta(E)\zeta], \quad E \in \mathfrak{X}(N),$$

and by virtue of the above equation, we derive immediately that η is closed. However, N being simply connected, we have $\eta = dh$ for some smooth function h on N. Thus, with the assumption made, we get $\zeta = \nabla h$, and on compact N, there is a point $p \in N$ such that $\zeta(p) = 0$, which is contrary to the fact that ζ is a unit vector field. Hence, we have $\alpha \neq 0$.

We claim now that (17) always implies $\beta = 0$. Suppose that $\alpha^2 - \beta^2 = \frac{\mathbf{s}}{4}$ holds in Equation (17). Then, using Equations (5) and (16), we get $2\alpha(-2\alpha\beta) = 0$, that is $\alpha^2\beta = 0$, and as $\alpha \neq 0$ on connected M (being simply connected), we get $\beta = 0$. Thus, in view of (17), we derive that, indeed, we always have $\beta = 0$, and consequently, Equation (4) assumes the form

$$D_E\zeta = -\alpha\Psi(E), \quad E \in \mathfrak{X}(N), \tag{18}$$

which proves that ζ is a Killing vector field, and therefore the flow of ζ consists of isometries of N. We get

$$\pounds_\zeta S = 0,$$

which in view of Equation (18) gives

$$(DS)(\zeta, E) = \alpha S(\Psi(E)) - \alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N).$$

Using Equation (11) in the above equation, we derive

$$(DS)(E, \zeta) = \alpha S(\Psi(E)) - \alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N),$$

which in view of Equation (18) implies

$$D_E S(\zeta) = -\alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N). \tag{19}$$

Next, using $\beta = 0$ in Equation (6), we get

$$S(\zeta) = \Psi(\nabla\alpha) + 2\alpha^2\zeta,$$

and inserting this equation in Equation (19) we obtain

$$D_E\left(\Psi(\nabla\alpha) + 2\alpha^2\zeta\right) = -\alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N),$$

that is,

$$(D\Psi)(E, \nabla\alpha) + \Psi A_\alpha(E) + 4\alpha E(\alpha)\zeta - 2\alpha^3\Psi(E) = -\alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N),$$

where we have used Equations (8) and (18). Using now (3), (5), and $\beta = 0$ in the above equation, we arrive at

$$5\alpha E(\alpha)\zeta + \Psi A_\alpha(E) - 2\alpha^3\Psi(E) = -\alpha\Psi(S(E)), \quad E \in \mathfrak{X}(N), \tag{20}$$

which on taking the inner product with ζ gives

$$5\alpha E(\alpha) = 0,$$

that is, $E(\alpha^2) = 0$, $E \in \mathfrak{X}(N)$. This proves that α is a constant. Hence, because we have already shown that $\alpha \neq 0$, we conclude that α is a nonzero constant. Now, the Equations (6) and (20) become

$$S(\zeta) = 2\alpha^2\zeta \text{ and } \Psi(S(E)) = 2\alpha^2\Psi(E),$$

and by operating Ψ on the second equation while using the first equation, we get

$$S(E) = 2\alpha^2 E, \quad E \in \mathfrak{X}(N). \tag{21}$$

However, the above equation implies

$$\mathbf{s} = 6\alpha^2. \tag{22}$$

Now, using the following expression for the Riemannian curvature tensor field Rm of a 3-dimensional manifold (N, g):

$$\begin{aligned} Rm(E_1, E_2)E_3 &= g(E_2, E_3)S(E_1) - g(E_1, E_3)S(E_2) + Ric(E_2, E_3)E_1 \\ &\quad - Ric(E_1, E_3)E_2 - \frac{\mathbf{s}}{2}\{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\}, \end{aligned}$$

as well as Equations (21) and (22), we arrive at

$$Rm(E_1, E_2)E_3 = \alpha^2\{g(E_2, E_3)E_1 - g(E_1, E_3)E_2\}, \quad E_1, E_2, E_3 \in \mathfrak{X}(N).$$

This proves that (N, g) is a space of constant curvature α^2. As (N, g) is compact, it is complete, and as it is also simply connected, it is isometric to $\mathbf{S}^3(\alpha^2)$.

Conversely, we have seen in the previous section that $\mathbf{S}^3(\alpha^2)$ admits a trans-Sasakian structure $(\Psi, \zeta, \eta, g, \alpha, \beta)$, where $\beta = 0$, and therefore β trivially satisfies Fischer–Marsden Equation (1). Moreover, it is clear that the scalar curvature $\mathbf{s}$ of $\mathbf{S}^3(\alpha^2)$ satisfies the equality case of the inequality

$$\mathbf{s}\beta^2 \leq 4\left(\alpha^2 - \beta^2\right)\beta^2.$$

Furthermore, $S = 2\alpha^2 I$, with α constant, and this implies that S is parallel. Therefore, S satisfies Equation (11). Hence, we conclude that all the requirements are attended by the TRSM $(\mathbf{S}^3(\alpha^2), \Psi, \zeta, \eta, g, \alpha, \beta)$. $\square$

Remark 1. *Observe that if a Riemannian manifold (M, g) admits a non-trivial solution of the Fischer–Marsden Equation (1), then the scalar curvature $\mathbf{s}$ is a constant (cf. [18]). However, we would like to point out that in the statement of Theorem 1, the solution β of Equation (1) is not supposed to be a non-trivial solution (actually, in the proof of the theorem, it turns out to be exactly zero), and therefore we could not use the above argument to conclude the constancy of the scalar curvature.*

Remark 2. *The assumption of Theorem 1 that β is a solution of Fischer–Marsden Equation (1) has the following justification. The two smooth functions involved in the definition of a trans-Sasakian manifold $(N, \Psi, \zeta, \eta, g, \alpha, \beta)$, namely α and β, could be natural candidates for solutions of Equation (1). Moreover, our aim is at getting a characterization of a 3-dimensional sphere $\mathbf{S}^3(c)$, knowing that $\mathbf{S}^3(c)$ admits a trans-Sasakian structure $(\Psi, \zeta, \eta, g, \alpha, \beta)$, where $\alpha = \sqrt{c}$ and $\beta = 0$. However, $\alpha = \sqrt{c}$ does not satisfy the Fischer–Marsden equation. Therefore, in view of our goal, the assumption that α satisfies Equation (1) is ruled out. This motivates the hypothesis of Theorem 1 that function β involved in the definition of a trans-Sasakian manifold satisfies the Fischer–Marsden equation. The proof of Theorem 1 shows that this assumption implies, in fact, that β is a trivial solution of (1), provided that the scalar curvature of N satisfies a certain inequality and the Ricci operator of N satisfies a Codazzi-type condition. Moreover, these assumptions also imply that α is a* *non-zero constant and N is a space of constant curvature, which ultimately leads to the conclusion that N is isometric to* $\mathbf{S}^3(\alpha^2)$.

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