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# Approximation Properties of Chebyshev Polynomials in the Legendre Norm

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**Abstract:** In this paper, we present some important approximation properties of Chebyshev polynomials in the Legendre norm. We mainly discuss the Chebyshev interpolation operator at the Chebyshev–Gauss–Lobatto points. The cases of single domain and multidomain for both one dimension and multi-dimensions are considered, respectively. The approximation results in Legendre norm rather than in the Chebyshev weighted norm are given, which play a fundamental role in numerical analysis of the Legendre–Chebyshev spectral method. These results are also useful in Clenshaw–Curtis quadrature which is based on sampling the integrand at Chebyshev points.

**Keywords:** Chebyshev polynomials; Chebyshev interpolation operator; the Legendre norm; Legendre–Chebyshev spectral method; Clenshaw–Curtis quadrature; multidomain; multi-dimensions



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## 1. Introduction

Orthogonal polynomials are useful in many areas of numerical analysis and are powerful for function approximation, numerical integration and numerical solution of differential and integral equations [1,2]. The core idea of spectral methods is that any nice enough function can be expanded in a series of orthogonal polynomials so that orthogonal polynomials play a fundamental role in spectral methods [3–6]. Particularly, Chebyshev polynomials and Legendre polynomials are frequently used in spectral methods and are two important sequences in numerical analysis.

The related approximation results of typical Chebyshev and Legendre spectral approximation are discussed in many literatures [3,4,7–10]. These results of Chebyshev spectral approximation are usually in the weighted norm forms. The Legendre–Chebyshev spectral method is a popular numerical method, which enjoys advantages of better stability of the Legendre method and easy implementation of the Chebyshev method. Therefore, it is necessary to develop the approximation properties of Chebyshev polynomials in the Legendre norm. In [11,12], the approximation result of the Chebyshev interpolation operator without the Chebyshev weighted norm was first given. Some other valuable results related to Chebyshev polynomials can be referred to [2,13–19] and references therein.

In addition, Chebyshev polynomials have a special connection with Clenshaw–Curtis quadrature, which uses Chebyshev points instead of optimal nodes. Clenshaw–Curtis quadrature can be implemented in  $O(N \log N)$  operations using the fast Fourier transform (FFT) and is used in numerical integration and numerical analysis [20–25]. As we known, Gauss quadrature is a beautiful and powerful idea. Zeros of orthogonal polynomials are chosen as the nodes of Gauss-type quadratures and used to generate computational grids for spectral methods. Yet, the Clenshaw–Curtis formula has essentially the same performance for most integrands and can be implemented effortlessly by the FFT [26]. Thus, the Clenshaw–Curtis and Gauss formulas are employed in the numerical solution of Ordinary differential equations and Partial differential equations by spectral methods [5,26–28].

And, Chebyshev polynomials also have an important connection with the mock–Chebyshev subset interpolation exploited to cutdown the Runge phenomenon [29,30], which takes advantages of the optimality of the interpolation processes on Chebyshev–Lobatto nodes.

The purpose of this paper is to present some essential approximation results related to Chebyshev polynomials in the Legendre norm. The first fundamental result of orthogonal polynomials is the Weierstrass Theorem, which is an important element of the classical polynomial approximation theory [31,32]. In numerical analysis of the Legendre–Chebyshev spectral method, we need to consider the stability and approximation properties of the Chebyshev interpolation operator in the  $L^2$ -norm rather than in the Chebyshev weighted norm [13]. In the paper, we consider the Chebyshev interpolation operator at the Chebyshev–Gauss–Lobatto (CGL) points. The cases of single domain and multidomain for both one dimension and multi-dimensions are discussed. Some approximation results in the Legendre norm rather than in the Chebyshev weighted norm are given. These results serve as preparations for polynomial-based spectral methods.

The rest of the paper is organized as follows. In Section 2, Chebyshev polynomials are described, and some related notations are introduced. In Section 3, some approximation properties of Chebyshev interpolation operators in one dimension are given. The cases of single domain and multidomain are discussed, respectively. In Section 4, some approximation properties in multi-dimensions are given. The conclusion is given in Section 5.

### 2. Preliminaries and Notations

In this section, we give a brief description of Chebyshev polynomials and define the Chebyshev interpolation operators. Some notations are also given, which will be used in the following sections.

We consider orthogonal polynomials—Chebyshev polynomials, which are proportional to Jacobi polynomials  $\{J_n^{-\frac{1}{2},-\frac{1}{2}}\}$  and are orthogonal with respect to the weight function

$$\omega(x) = (1 - x^2)^{-\frac{1}{2}}, \quad x \in [-1, 1].$$

The three-term recurrence relation for the Chebyshev polynomials is as follows [6]:

$$\begin{cases} T_0(x) = 1, \\ T_1(x) = x, \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1, \end{cases}$$

which satisfies

$$\int_{-1}^1 T_i(x)T_j(x)(1 - x^2)^{-\frac{1}{2}} dx = \frac{c_i}{2} \delta_{ij},$$

where  $c_0 = 2, c_i = 1(i \geq 1)$ . As we known, there have been many useful properties of Chebyshev polynomials [4,6,28].

Denote  $(\cdot, \cdot)_Q$  and  $\|\cdot\|_Q$  be the inner product and the norm of the space  $L^2(Q)$ , respectively. We will drop the subscript  $Q$  whenever  $Q = I = (-1, 1)$ . Let  $\mathbb{P}_N(I)$  be the space of polynomials with the degree at most  $N$  on an interval  $I$ . And let  $H^\sigma(Q)(\sigma > 0)$  be the classical Sobolev space with norm  $\|\cdot\|_{H^\sigma(Q)}$ .

Define the Chebyshev interpolation operator at the CGL points by  $I_N^C : C(\bar{I}) \rightarrow \mathbb{P}_N(I)$  satisfying

$$I_N^C \varphi(x_j) = \varphi(x_j), \quad 0 \leq j \leq N, \tag{1}$$

where  $x_j = \cos \frac{\pi j}{N}$ .

### 3. Approximation Properties of Chebyshev Interpolation Operator in One Dimension

In this section, some approximation properties of the Chebyshev interpolation operator in one dimension are derived. The cases of both single domain and multidomain are considered respectively.

### 3.1. Case of Single Domain in One Dimension

Similar to the approximation results presented in [11] for the Chebyshev interpolation operator  $I_N^C$ , we give the following lemma.

**Lemma 1** ([11,15]). *If  $u \in H^1(I)$ , then*

$$N\|I_N^C u - u\| + \|\partial_x I_N^C u\| \leq C\|\partial_x u\|. \tag{2}$$

*In addition, if  $u \in H^\sigma(I)$  and  $\sigma > 1/2$ , then*

$$\|I_N^C u - u\|_{H^l(I)} \leq CN^{l-\sigma}\|u\|_{H^\sigma(I)}, \quad 0 \leq l \leq 1. \tag{3}$$

We note that the norm in the approximation results (2) and (3) is already without the Chebyshev weighted function and is in Legendre norm. The lemma is important in numerical analysis of Legendre–Chebyshev spectral method.

Next, the applications of the result of interpolation (3) to connect with the Clenshaw-Curtis quadrature are presented as follows. Given

$$I(u) = \int_{-1}^1 u(x)dx, \quad I_N(u) = \sum_{k=0}^N \omega_k u(x_k),$$

where the nodes  $x_k$  depend on  $N$ . Since the weights  $\omega_k$  are defined uniquely by the property that  $I_N$  is equal to the integral of the degree  $\leq N$  polynomial interpolation through the data points. Then we have

$$I_N(u) = \sum_{k=0}^N \omega_k u(x_k) = \int_{-1}^1 I_N^C u(x)dx.$$

For the Clenshaw-Curtis numerical integration in [26], the unique best approximation to  $u$  on  $[-1, 1]$  of degree  $\leq N$  with respect to the  $L^\infty$ -norm.

The following lemma shows that we simply use the  $L^2$ -norm estimation result (3) to get the desired error estimate.

**Lemma 2.** *If  $u \in H^\sigma(I)$  and  $\sigma > 1/2$ , then*

$$|I_N(u) - I(u)| \leq CN^{-\sigma}\|u\|_{H^\sigma(I)}. \tag{4}$$

### 3.2. Case of Multidomain in One Dimension

For  $1 \leq k \leq K$ , we denote  $-1 = a_0 < a_1 < \dots < a_K = 1$ , and set

$$I_k = (a_{k-1}, a_k], \quad I = \cup_{k=0}^K I_k, \quad h_k = a_k - a_{k-1}, \quad v^k \equiv v|_{I_k}.$$

Let  $\mathbb{P}_{N_k}(I_k)$  be the space of polynomials with the degree at most  $N_k$  on the interval  $I_k$ . Denote  $\mathcal{N} = (N_1, \dots, N_K)$ .

Define the following space

$$\mathbb{P}_{\mathcal{N}}(I) = \{u : u|_{I_k} \in \mathbb{P}_{N_k}(I_k), \quad 1 \leq k \leq K\}.$$

Set the relationship between  $I_k$  and  $I$  as follows:

$$v(x) = \hat{v}(\hat{x}), \quad x = \frac{1}{2}(h_k \hat{x} + a_{k-1} + a_k), \quad x \in I_k, \hat{x} \in I.$$

Define the operator  $I_{\mathcal{N}}^C : C(\bar{I}) \rightarrow \mathbb{P}_{\mathcal{N}}$  such that

$$\widehat{(I_{\mathcal{N}}^C u)}^k = I_{N_k}^C \hat{u}^k(\hat{x}), \quad 1 \leq k \leq K, \tag{5}$$

where  $I_{N_k}^C : C(\bar{I}) \rightarrow \mathbb{P}_{N_k}(I)$  is the CGL interpolation operator defined as (1).

**Lemma 3.** *If  $v \in H^\sigma(I_k)$  ( $\sigma \geq 0$ ), then*

$$\|\hat{v}\|_{H^\sigma(I)} \leq Ch_k^{\sigma-\frac{1}{2}} \|v\|_{H^\sigma(I_k)}, \tag{6}$$

$$\|v\|_{H^\sigma(I_k)} \leq Ch_k^{\frac{1}{2}-\sigma} \|\hat{v}\|_{H^\sigma(I)}. \tag{7}$$

Denote  $\hat{h} = \max_{1 \leq k \leq K} \{ \frac{h_k}{N_k} \}$ . We arrive at the following approximation result.

**Theorem 1.** *If  $u \in H^\sigma(I)$  ( $\sigma \geq 1$ ), then*

$$\|I_{N'}^C u - u\|_{H^l(I)} \leq Ch^{\sigma-l} \|u\|_{H^\sigma(I)}, \quad 0 \leq l \leq 1. \tag{8}$$

**Proof.** Applying Lemma 1 and Lemma 3, we get

$$\begin{aligned} \|I_{N'}^C u - u\|_{H^l(I)}^2 &= \sum_{1 \leq k \leq K} \|(I_{N'}^C u)^k - u^k\|_{H^l(I_k)}^2 \\ &\leq C \sum_{1 \leq k \leq K} h_k^{1-2l} \|I_{N_k}^C \hat{u}^k - \hat{u}^k\|_{H^l(I)}^2 \\ &\leq C \sum_{1 \leq k \leq K} h_k^{1-2l} N_k^{2(l-\sigma)} \|\hat{u}^k\|_{H^\sigma(I)}^2 \\ &\leq C \sum_{1 \leq k \leq K} h_k^{1-2l} N_k^{2(l-\sigma)} h_k^{2\sigma-1} \|u^k\|_{H^\sigma(I_k)}^2 \\ &\leq C \hat{h}^{2(\sigma-l)} \|u\|_{H^\sigma(I)}^2. \end{aligned}$$

Thus, the theorem is proved.  $\square$

#### 4. Approximation Properties of Chebyshev Interpolation Operator in Multi-Dimensions

Set  $I^i = (-1, 1)$  ( $i = 1, \dots, d$ ),  $\Omega_d = I^1 \times I^2 \times \dots \times I^d$ . If  $d = 2$ , we use  $\Omega = I^x \times I^y$  instead of  $\Omega_2 = I^1 \times I^2$ .

Define the following space

$$\mathbb{P}_N(\Omega_d) = \mathbb{P}_N(I^1) \otimes \mathbb{P}_N(I^2) \otimes \dots \otimes \mathbb{P}_N(I^d).$$

Denote  $x = (x_1, \dots, x_d) \in \Omega_d$ . With each function  $v$  in  $C(\bar{\Omega}_d)$ , we associate the  $d$ -function  $v_j$  defined by

$$v_j(x_j)(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) = v(x_1, \dots, x_d), \quad 1 \leq j \leq d.$$

Define the operator  $I_{N'}^C$  by

$$I_{N'}^C = I_{N,1}^C \circ \dots \circ I_{N,d}^C, \tag{9}$$

where  $I_{N,i}^C$  is the CGL interpolation operator  $I_{N,i}^C : C(\bar{I}^i) \rightarrow \mathbb{P}_N(I^i)$  defined as (1).

##### 4.1. Case of Single Domain in Multi-Dimensions

According to the one dimensional approximation results, we give some approximation results of the Chebyshev interpolation operators for the case of single domain in multi-dimensions ( $d \geq 2$ ).

**Theorem 2.** If  $u \in H^\sigma(\Omega_d)$  and  $\sigma > \frac{d}{2}$ , then

$$\|I_{\mathbb{N}}^C u - u\|_{\Omega_d} \leq CN^{-\sigma} \|u\|_{H^\sigma(\Omega_d)}. \tag{10}$$

**Proof.** Applying (3) in Lemma 1 and noting  $L^2(\Omega_d) = L^2(I^j; L^2(\Omega_{d-1}))$ , we have

$$\begin{aligned} \|u - I_{\mathbb{N}}^C u\|_{\Omega_d} &= \|u - I_{N,1}^C \circ \dots \circ I_{N,d}^C u\|_{\Omega_d} \\ &\leq \|u - I_{N,1}^C u\|_{L^2(I^1; L^2(\Omega_{d-1}))} + \|u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u\|_{L^2(I^1; L^2(\Omega_{d-1}))} \\ &\quad + \|(I - I_{N,1}^C)(u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u)\|_{L^2(I^1; L^2(\Omega_{d-1}))} \\ &\leq CN^{-\sigma} \|u\|_{H^\sigma(I^1; L^2(\Omega_{d-1}))} + \|u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u\|_{L^2(I^1; L^2(\Omega_{d-1}))} \\ &\quad + CN^{-\frac{\sigma}{d}} \|u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u\|_{H^{\frac{\sigma}{d}}(I^1; L^2(\Omega_{d-1}))}. \end{aligned}$$

Repeating the above discussion  $d$  times for  $\|u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u\|_{L^2(I^1; L^2(\Omega_{d-1}))}$  and  $\|u - I_{N,2}^C \circ \dots \circ I_{N,d}^C u\|_{H^{\frac{\sigma}{d}}(I^1; L^2(\Omega_{d-1}))}$ , and by the following imbedding relationship

$$H^s(\Omega_d) \subset H^k(I^j; H^{s-k}(\Omega_{d-1})), \quad s \geq k,$$

the desired result is obtained.  $\square$

**Theorem 3.** If  $u \in H^\sigma(\Omega_d)$  and  $\sigma > \frac{d+1}{2}$ , then

$$\|I_{\mathbb{N}}^C u - u\|_{H^1(\Omega_d)} \leq CN^{1-\sigma} \|u\|_{H^\sigma(\Omega_d)}. \tag{11}$$

**Proof.** For  $1 \leq j \leq d$ , we have

$$\|u - I_{\mathbb{N}}^C u\|_{H^1(\Omega_d)}^2 \leq \sum_{j=1}^d \|u - I_{\mathbb{N}}^C u\|_{H^1(I^j; L^2(\Omega_{d-1}))}^2.$$

By (3) in Lemma 1 and Theorem 2, we get

$$\begin{aligned} &\|u - I_{\mathbb{N}}^C u\|_{H^1(I^j; L^2(\Omega_{d-1}))} \\ &\leq \|u - I_{N,j}^C u\|_{H^1(I^j; L^2(\Omega_{d-1}))} \\ &\quad + \|I_{N,j}^C(u - I_{N,1}^C \circ \dots \circ I_{N,j-1}^C \circ I_{N,j+1}^C \circ \dots \circ I_{N,d}^C u)\|_{H^1(I^j; L^2(\Omega_{d-1}))} \\ &\leq CN^{1-\sigma} \|u\|_{H^\sigma(I^j; L^2(\Omega_{d-1}))} \\ &\quad + C \|u - I_{N,1}^C \circ \dots \circ I_{N,j-1}^C \circ I_{N,j+1}^C \circ \dots \circ I_{N,d}^C u\|_{H^1(I^j; L^2(\Omega_{d-1}))} \\ &\leq CN^{1-\sigma} \|u\|_{H^\sigma(I^j; L^2(\Omega_{d-1}))} + CN^{1-\sigma} \|u\|_{H^1(I^j; H^{\sigma-1}(\Omega_{d-1}))} \\ &\leq CN^{1-\sigma} \|u\|_{H^\sigma(\Omega_d)}. \end{aligned}$$

Thus, the theorem is proved.  $\square$

#### 4.2. Case of Multidomain in Multi-Dimensions

In this subsection, we give some approximation properties of the CGL interpolation operator for the case of multidomain in multi-dimensions ( $d \geq 2$ ).

For simplicity, we make the same subdivision in each direction of space. Similar to the case of multidomain in one dimension, for  $1 \leq k \leq K$ , denote  $-1 = a_0 < a_1 < \dots < a_K = 1$ , and set

$$I_k^i = (a_{k-1}, a_k], \quad I^i = \cup_{k=0}^K I_k^i, \quad i = 1, \dots, d.$$

Let  $\mathbb{P}_{N_k}(I_k^i)$  be the space of polynomials with the degree at most  $N_k$  on the interval  $I_k^i$ .

We introduce the space  $\mathbb{P}_{\mathbb{N}}(\Omega_d) = \mathbb{P}_{\mathcal{N}}(I^1) \otimes \dots \otimes \mathbb{P}_{\mathcal{N}}(I^d)$ . Define the Chebyshev–Gauss–Lobatto interpolation operator  $\mathbf{I}_{\mathcal{N}}^{\mathbb{C}}$  by

$$\mathbf{I}_{\mathcal{N}}^{\mathbb{C}} = I_{\mathcal{N},1}^{\mathbb{C}} \circ I_{\mathcal{N},2}^{\mathbb{C}} \circ \dots \circ I_{\mathcal{N},d}^{\mathbb{C}}, \tag{12}$$

where  $I_{\mathcal{N},i}^{\mathbb{C}} : C(\bar{I}^i) \rightarrow \mathbb{P}_{\mathcal{N}}(I^i)$  is the CGL interpolation operator defined as (5).

By the assumption of the same subdivision in each direction of space, we set  $\hbar = \max_{1 \leq k \leq K} \{ \frac{h_k}{N_k} \}$  and give the following approximation results.

**Theorem 4.** Assume that  $d = 2$ . If  $u \in H^\sigma(\Omega)$  and  $\sigma > 1$ , then

$$\|\mathbf{I}_{\mathcal{N}}^{\mathbb{C}}u - u\|_{\Omega} \leq C\hbar^\sigma \|u\|_{H^\sigma(\Omega)}. \tag{13}$$

**Proof.** By Theorem 1 and Lemma 1, we get

$$\begin{aligned} \|u - \mathbf{I}_{\mathcal{N}}^{\mathbb{C}}u\| &= \|u - I_{\mathcal{N},1}^{\mathbb{C}} \circ I_{\mathcal{N},2}^{\mathbb{C}}u\| \\ &\leq \|u - I_{\mathcal{N},1}^{\mathbb{C}}u\|_{L^2(I^x;L^2(I^y))} + \|u - I_{\mathcal{N},2}^{\mathbb{C}}u\|_{L^2(I^x;L^2(I^y))} \\ &\quad + \|(I - I_{\mathcal{N},1}^{\mathbb{C}})(u - I_{\mathcal{N},2}^{\mathbb{C}}u)\|_{L^2(I^x;L^2(I^y))} \\ &\leq C\hbar^\sigma \|u\|_{H^\sigma(I^x;L^2(I^y))} + \|u - I_{\mathcal{N},2}^{\mathbb{C}}u\|_{L^2(I^x;L^2(I^y))} \\ &\quad + C\hbar \|u - I_{\mathcal{N},2}^{\mathbb{C}}u\|_{H^1(I^x;L^2(I^y))} \\ &\leq C\hbar^\sigma \|u\|_{H^\sigma(I^x;L^2(I^y))} + C\hbar^\sigma \|u\|_{L^2(I^x;H^\sigma(I^y))} + C\hbar^\sigma \|u\|_{H^1(I^x;H^{\sigma-1}(I^y))} \\ &\leq C\hbar^\sigma \|u\|_{H^\sigma(\Omega)}. \end{aligned}$$

Thus, the proof is completed.  $\square$

**Theorem 5.** Assume that  $d = 2$ . If  $u \in H^\sigma(\Omega)$  and  $\sigma > \frac{d+1}{2} = \frac{3}{2}$ , then

$$\|u - \mathbf{I}_{\mathcal{N}}^{\mathbb{C}}u\|_{H^1(\Omega)} \leq C\hbar^{\sigma-1} \|u\|_{H^\sigma(\Omega)}. \tag{14}$$

**Proof.** By Theorem 1 and Lemma 1, we have

$$\begin{aligned} \|u - \mathbf{I}_{\mathcal{N}}^{\mathbb{C}}u\|_{H^1(I^x;L^2(I^y))} &\leq \|u - I_{\mathcal{N},1}^{\mathbb{C}}u\|_{H^1(I^x;L^2(I^y))} + \|I_{\mathcal{N},1}^{\mathbb{C}}(u - I_{\mathcal{N},2}^{\mathbb{C}}u)\|_{H^1(I^x;L^2(I^y))} \\ &\leq C\hbar^{\sigma-1} \|u\|_{H^\sigma(I^x;L^2(I^y))} + \|(I - I_{\mathcal{N},1}^{\mathbb{C}})(u - I_{\mathcal{N},2}^{\mathbb{C}}u)\|_{H^1(I^x;L^2(I^y))} \\ &\quad + \|u - I_{\mathcal{N},2}^{\mathbb{C}}u\|_{H^1(I^x;L^2(I^y))} \\ &\leq C\hbar^{\sigma-1} \|u\|_{H^\sigma(I^x;L^2(I^y))} + C\|u - I_{\mathcal{N},2}^{\mathbb{C}}u\|_{H^1(I^x;L^2(I^y))} \\ &\leq C\hbar^{\sigma-1} \|u\|_{H^\sigma(I^x;L^2(I^y))} + C\hbar^{\sigma-1} \|u\|_{H^1(I^x;H^{\sigma-1}(I^y))} \\ &\leq C\hbar^{\sigma-1} \|u\|_{H^\sigma(\Omega)}. \end{aligned}$$

Therefore, the desired result is obtained.  $\square$

### 5. Numerical Experiments

In this section, we give some numerical experiments to confirm the theoretical results. The cases of continuous and discontinuous functions are considered, respectively.

The discrete  $L^2$ -error used in the following experiments is defined as

$$\text{Err}(u) = \left( \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |u(x_i, y_j) - \mathbf{I}_{\mathcal{N}}^{\mathbb{C}}u(x_i, y_j)|^2 \Delta x \Delta y \right)^{\frac{1}{2}}, \tag{15}$$

where  $x_i = i\Delta x, y_j = j\Delta y, \Delta x = \Delta y = \frac{1}{n}$ , and  $n = 100$ .

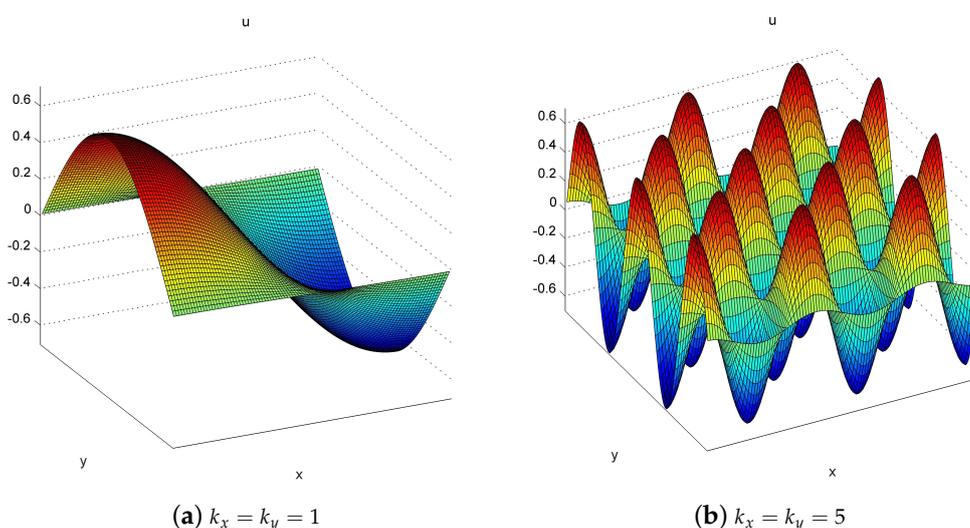
**Example 1.** We consider the following continuous function in  $\bar{\Omega} = [0, 1]^2$ :

$$u(x, y) = \frac{k_y}{w} \cos(k_x \pi x) \sin(k_y \pi y), \quad w = (k_x^2 + k_y^2)^{\frac{1}{2}}, \tag{16}$$

which is approximated by the multidomain Chebyshev-Gauss-Lobatto interpolation  $\mathbf{I}_{\mathcal{N}}^C u$ . We make the same subdivision in  $x$  and  $y$  directions as follows:

$$\bar{\Omega} = \{[0, 0.5] \cup [0.5, 1]\} \times \{[0, 0.5] \cup [0.5, 1]\}.$$

Figure 1 displays the shape of  $u(x, y)$  with low frequency  $k_x = k_y = 1$  and high frequency  $k_x = k_y = 5$ , respectively. Table 1 gives the discrete  $L^2$ -errors of the Chebyshev-Gauss-Lobatto interpolation for function  $u$ . The results show the spectral accuracy of the multidomain interpolation.



**Figure 1.** The shapes of  $u(x, y)$  with low frequency  $k_x = k_y = 1$  and high frequency  $k_x = k_y = 5$ .

**Table 1.**  $L^2$ -errors of Chebyshev-Gauss-Lobatto interpolation for  $u(x, y)$ .

$k_x = k_y = 1$			$k_x = k_y = 5$		
$\mathcal{N}_x = \mathcal{N}_y$	$\text{Err}(u)$	$\text{Order}$	$\mathcal{N}_x = \mathcal{N}_y$	$\text{Err}(u)$	$\text{Order}$
(7,7)	$2.15 \times 10^{-8}$	-	(14,14)	$1.76 \times 10^{-8}$	-
(10,10)	$1.28 \times 10^{-12}$	$\tilde{h}^{27.28}$	(18,18)	$2.93 \times 10^{-12}$	$\tilde{h}^{34.62}$
(13,13)	$1.61 \times 10^{-16}$	$\tilde{h}^{34.23}$	(22,22)	$6.31 \times 10^{-16}$	$\tilde{h}^{42.07}$

**Example 2.** We consider the following discontinuous functions in  $\bar{\Omega} = [0, 1]^2$ :

$$u_1(x, y) = \frac{k_y}{\epsilon w} \cos(k_x \pi x) \sin(k_y \pi y), \tag{17}$$

$$u_2(x, y) = -\frac{k_y}{\epsilon w} \sin(k_x \pi x) \cos(k_y \pi y), \tag{18}$$

where  $w = \left(\frac{k_x^2 + k_y^2}{\epsilon}\right)^{\frac{1}{2}}$ . The functions are approximated by the multidomain Chebyshev-Gauss-Lobatto interpolation  $\mathbf{I}_{\mathcal{N}}^C u$ .

Suppose that the parameters  $\epsilon$  and  $k_x$  are piecewise constants:

$$\epsilon = \begin{cases} 1, & 0 \leq x \leq 0.5, 0 \leq y \leq 1, \\ 4, & 0.5 \leq x \leq 1, 0 \leq y \leq 1, \end{cases} \quad k_x = \begin{cases} 4, & 0 \leq x \leq 0.5, 0 \leq y \leq 1, \\ 16, & 0.5 \leq x \leq 1, 0 \leq y \leq 1, \end{cases}$$

and  $k_y = 8$ . The functions are discontinuous at  $x = 0.5$ . The domain is decomposed as follows:

$$\bar{\Omega} = \{[0, 0.5] \cup [0.5, 1]\} \times \{[0, 1]\}.$$

Figure 2 displays the shape of  $u_1(x, y)$  and  $u_2(x, y)$ . It is clear that  $u_1(x, y)$  is discontinuous and  $u_2(x, y)$  is weak discontinuous at  $x = 0.5$ . Table 2 gives the discrete  $L^2$ -errors of the Chebyshev–Gauss–Lobatto interpolation for functions  $u_1, u_2$ . The results show the spectral accuracy of the multidomain interpolation for the discontinuous functions.

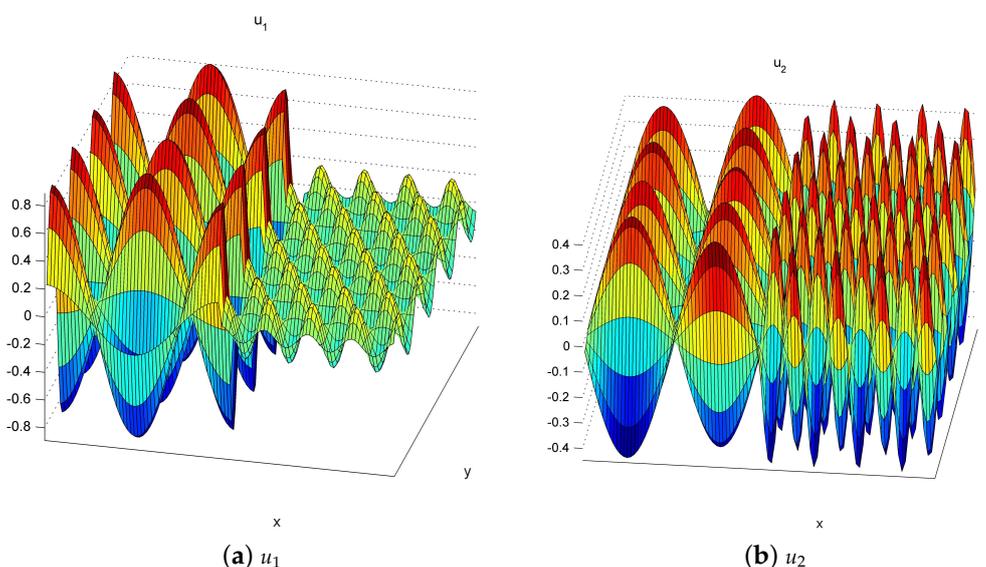


Figure 2. The shapes of  $u_1(x, y)$  and  $u_2(x, y)$ .

Table 2.  $L^2$ -errors of Chebyshev–Gauss–Lobatto interpolation for  $u_1(x, y)$  and  $u_2(x, y)$ .

$\mathcal{N}_x$	$\mathcal{N}_y$	$Err(u_1)$	Order	$Err(u_2)$	Order
(12,26)	26	$8.30 \times 10^{-8}$	-	$4.87 \times 10^{-8}$	-
(16,32)	32	$8.32 \times 10^{-12}$	$\tilde{h}^{32.00}$	$4.99 \times 10^{-12}$	$\tilde{h}^{31.93}$
(20,38)	38	$7.52 \times 10^{-16}$	$\tilde{h}^{54.18}$	$7.74 \times 10^{-16}$	$\tilde{h}^{51.04}$

### 6. Conclusions

In the paper, we have given some important approximation results of Chebyshev interpolation operators in Legendre norm. The Chebyshev interpolation operators at the Chebyshev–Gauss–Lobatto points is discussed mainly. Moreover, we considered the cases of single domain and multidomain for both one dimension and multi-dimensions, respectively. The approximation results in the Legendre norm are derived. These results play an important role in numerical integration and numerical analysis of the Legendre–Chebyshev spectral method and the Clenshaw–Curtis quadrature.

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