

Article

On Some Generalized Simpson's and Newton's Inequalities for (α, m) -Convex Functions in q -Calculus

Ifra Bashir Sial ¹, Sun Mei ¹, Muhammad Aamir Ali ^{2,*}  and Kamsing Nonlaopon ³ 

¹ School of Mathematics Science, Jiangsu University, Zhenjiang 212114, China; ifrabashir92@gmail.com (I.B.S.); sunm@ujs.edu.cn (S.M.)

² Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

³ Department of Mathematics, Faculty of Science and Arts, Khon Kaen University, Khon Kaen 40000, Thailand; nkamsi@kku.ac.th

* Correspondence: mahr.muhammad.aamir@gmail.com

Abstract: In this paper, we first establish two right-quantum integral equalities involving a right-quantum derivative and a parameter $m \in [0, 1]$. Then, we prove modified versions of Simpson's and Newton's type inequalities using established equalities for right-quantum differentiable (α, m) -convex functions. The newly developed inequalities are also proven to be expansions of comparable inequalities found in the literature.

Keywords: Simpson's inequalities; Newton's inequalities; quantum calculus; (α, m) -convex functions



Citation: Sial, I.B.; Mei, S.; Ali, M.A.; Nonlaopon, K. On Some Generalized Simpson's and Newton's Inequalities for (α, m) -Convex Functions in q -Calculus. *Mathematics* **2021**, *9*, 3266. <https://doi.org/10.3390/math9243266>

Academic Editors: Juan Eduardo Nápoles Valdés and Miguel Vivas-Cortez

Received: 25 November 2021

Accepted: 10 December 2021

Published: 16 December 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In [1], Hudzik and Maligranda introduced the concept of generalized convexity, called s -convexity and stated as follows: A function $f : [0, \infty) \rightarrow \mathbb{R}$ is called s -convex or $f \in K_s^i$ if the inequality

$$f(tx + uy) \leq t^s f(x) + u^s f(y)$$

holds for all $x, y \in [0, \infty)$, $s \in (0, 1)$ and $t, u \in [0, 1]$.

Note that if $u^s + t^s = 1$, the above class of convex functions is referred to as s -convex in the first sense and denoted by K_s^1 , while if $u + t = 1$, the above class is referred to as s -convex in the second sense and denoted by K_s^2 .

Since much attention has been paid to investigating the idea of convexity and its variant forms in recent years, convexity concerning integral inequalities is an interesting research subject. Hermite inequality, Hadamard's Jensen's inequality, and Hardy's inequality are three of the most relevant inequalities linked to the integral mean of a convex function; see [2,3].

For a function to be convex, Hermite–Hadamard inequality is a necessary and sufficient condition. The Hermite–Hadamard inequality is given by:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

This double inequality is a development of the concept of convexity, and it readily follows from Jensen's inequality. Many researchers have investigated related Hermite–Hadamard type integral inequalities, and a remarkable diversity of generalizations and extensions for the concept of convexity have recently been considered.

In [4], Dragomir and Fitzpatrick used the s -convexity in the second sense for f and proved the following Hermite–Hadamard type inequality:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{1+s}.$$

A class of (α, m) -convex functions was introduced by Mishen and stated as:

Definition 1. Ref. [5] A function $f : [0, b] \rightarrow \mathbb{R}$ is called (α, m) -convex, if the inequality

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y)$$

holds for all $x, y \in [0, b]$, $t \in [0, 1]$, $(\alpha, m) \in [0, 1]^2$ and $m \in [0, 1]$.

In recent years, many authors have focused on Simpson's type inequality in different categories of mappings. Since convexity theory is an easy and efficient way to solve many problems from various branches of pure and applied mathematics, some mathematicians have used the results of Simpson's and Newton's types in obtaining a convex map. Dragomir et al. [6], for example, introduced the recent Simpson's inequalities and their applications in numerical integration formulas. Moreover, Alomari et al. in [7] determined several inequalities of Simpson's type for s -convex functions. The difference in Simpson's type inequality dependent on convexity was then noted by Sarikaya et al. in [8]. Newton's inequality for harmonic convex and p -harmonic convex mappings is presented in [9,10]. New Newton-type inequality for functions with the local fractional derivative is generalized convex as described by Iftikhar et al. in [11].

On the other hand, several works in the field of q -analysis are being carried out, beginning with Euler, in order to achieve mastery in the mathematics that drives quantum computing. Q -calculus is the link between physics and mathematics. It has a wide range of applications in many fields, e.g., mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, theory of relativity, and quantum theory [12,13]. q -calculus also has many applications in quantum information theory, which is an interdisciplinary area that encompasses computer science, information theory, philosophy, and cryptography, among other areas [14,15]. Euler is the inventor of this significant branch of mathematics. Newton used the q -parameter in his work on infinite series. The q -calculus that is known as without limits calculus was presented by Jackson [16] in a systematic manner. In 1966, Al-Salam [17] introduced a q -analogue of the Riemann–Liouville fractional integral operator. Since then, the related research has been increasing steadily. In particular, in 2013, Tariboon and Ntouyas introduced the left quantum difference operator and left quantum integral in [18]. In 2020, Bermudo et al. introduced the notion of the right quantum derivative and right quantum integral in [19].

Many integral inequalities have also been studied using quantum and post-quantum integrals for various types of functions. For example, in [19–28], the authors proved Hermite–Hadamard integral inequalities and their left–right estimates for convex and coordinated convex functions by using the quantum derivatives and integrals. In [29], the generalized version of q -integral inequalities was presented by Noor et al. In [30], Nwaeze et al. proved certain parameterized quantum integral inequalities for generalized quasi-convex functions. Khan et al. proved quantum Hermite–Hadamard inequality using the green function in [31]. For convex and coordinated convex functions, Budak et al. [32], Ali et al. [33,34], and Vivas-Cortez et al. [35] developed new quantum Simpson's and quantum Newton's type inequalities. For quantum Ostrowski's inequalities for convex and coordinated convex functions, one can consult [36–38].

Inspired by the ongoing studies, we give some new inequalities of Simpson's and Newton's formula type for (α, m) -convex functions using quantum calculus. The fundamental benefit of these inequalities is that these can be turned into quantum Simpson's and quantum Newton's inequalities for convex functions [32], classical Simpson's type inequalities for (α, m) -convex functions [39], and classical Simpson's type inequalities for convex functions [7] without having to prove each one separately.

The structure of this paper is as follows. The fundamentals of q -calculus, as well as other relevant topics in this subject, are briefly discussed in Section 2. In Section 3, we establish two fundamental identities that are crucial for the development of the paper's

primary findings. Sections 4 and 5 describe the Simpson's and Newton's type inequalities for q -differentiable functions via q -integrals. The findings provided here are compared to similar findings in the literature. Section 6 finishes with some research suggestions for the future.

2. Preliminaries of q -Calculus and Some Inequalities

The definitions and properties of quantum derivatives and quantum integrals are recalled first in this section. We also recall some well-known quantum integral inequalities. Throughout this paper, let $0 < q < 1$ be a constant.

The q -number or q -analogue of $n \in \mathbb{N}$ is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}. \quad (1)$$

Definition 2. Ref. [18] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the left q -derivative of function f at $x \in [a, b]$ is defined by

$${}_aD_q f(x) = \begin{cases} \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, & \text{if } x \neq a; \\ \lim_{x \rightarrow a} {}_aD_q f(x), & \text{if } x = a. \end{cases} \quad (2)$$

The function f is said to be q -differentiable on $[a, b]$ if ${}_aD_q f(x)$ exists for all $x \in [a, b]$.

Note that, if $a = 0$ and ${}_0D_q f(x) = D_q f(x)$, then (3) reduces to

$$D_q f(x) = \begin{cases} \frac{f(x) - f(qx)}{(1 - q)x}, & \text{if } x \neq 0; \\ \lim_{x \rightarrow 0} D_q f(x), & \text{if } x = 0. \end{cases}$$

which is the q -Jackson derivative; see [13,18] for more details.

Theorem 1. Ref. [18] If $f, g : J \rightarrow \mathbb{R}$ are q -differentiable functions, then the following identities hold:

(i) The product $fg : [a, b] \rightarrow \mathbb{R}$ is q -differentiable on $[a, b]$ with

$$\begin{aligned} {}_aD_q(fg)(x) &= f(x){}_aD_q g(x) + g(qx + (1 - q)x){}_aD_q(x) \\ &= g(x){}_aD_q f(x) + f(qx + (1 - q)x){}_aD_q g(x) \end{aligned}$$

(ii) If $g(x)g(qx + (1 - q)x) \neq 0$, then f/g is q -differentiable on $[a, b]$ with

$${}_aD_q\left(\frac{f}{g}\right)(x) = \frac{g(x){}_aD_q f(x) - f(x){}_aD_q g(x)}{g(x)g(qx + (1 - q)x)}$$

Definition 3. Ref. [18] Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, the left q -integral of function f at $z \in [a, b]$ is defined by

$$\int_a^z f(x)_a d_q x = (1 - q)(z - a) \sum_{n=0}^{\infty} q^n f(q^n z + (1 - q^n)a). \quad (3)$$

The function f is said to be q -integrable on $[a, b]$ if $\int_a^z f(x)_a d_q x$ exists for all $z \in [a, b]$.

Note that, if $a = 0$, then (4) reduces to

$$\int_0^z f(x)_0 d_q x = \int_0^z f(x) d_q x = (1 - q)z \sum_{n=0}^{\infty} q^n f(q^n z),$$

which is the q -Jackson integral; see [13,18] for more details.

Theorem 2. Ref. [18] If $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function and $z \in [a, b]$, then the following identities hold:

- (i) ${}_a D_q \int_a^z f(x) {}_a d_q x = f(z);$
- (ii) $\int_c^z {}_a D_q f(x) {}_a d_q x = f(z) - f(c)$ for $c \in (a, z).$

Alp et al. proved quantum Hermite–Hadamard inequalities for left q -integrals by utilizing the convex functions, as follows:

Theorem 3. Ref. [25] For a convex mapping $f : [a, b] \rightarrow \mathbb{R}$ that is differentiable on $[a, b]$, the following inequality holds:

$$f\left(\frac{qa+b}{[2]_q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}_a d_q x \leq \frac{qf(a)+f(b)}{[2]_q}. \quad (4)$$

Definition 4. Ref. [19] The right q -derivative of mapping $f : [a, b] \rightarrow \mathbb{R}$ is defined as:

$${}^b D_q f(x) = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \quad x \neq b.$$

If $x = b$, we define ${}^b D_q f(b) = \lim_{x \rightarrow b} {}^b D_q f(x)$ if it exists and it is finite.

Definition 5. Ref. [19] The right q -integral of mapping $f : [a, b] \rightarrow \mathbb{R}$ on $[a, b]$ is defined as:

$$\int_a^b f(x) {}^b d_q x = (1-q)(b-a) \sum_{k=0}^{\infty} q^k f\left(q^k a + (1-q^k)b\right).$$

Bermudo et al. also proved the corresponding quantum Hermite–Hadamard inequalities for the right q -integral:

Theorem 4. Ref. [19] For a convex mapping $f : [a, b] \rightarrow \mathbb{R}$ that is differentiable on $[a, b]$, the following inequality holds:

$$f\left(\frac{a+qb}{[2]_q}\right) \leq \frac{1}{b-a} \int_a^b f(x) {}^b d_q x \leq \frac{f(a) + qf(b)}{[2]_q}. \quad (5)$$

Now, we give another new lemma that helps us to prove the identities in the next section.

Lemma 1. For continuous functions $f, g : [a, b] \rightarrow \mathbb{R}$, the following equality holds:

$$\begin{aligned} \int_0^c g(t) {}^b D_q f(ta + (1-t)b) d_q t &= \frac{1}{b-a} \int_0^c D_q g(t) f(qta + (1-qt)b) d_q t \\ &\quad - \frac{g(t)f(ta + (1-t)b)}{b-a} \Big|_0^c. \end{aligned}$$

Proof. The lemma can be shown by straightforward calculations, so it is omitted. \square

3. Identities

In this section, we establish two quantum integral equalities using the integration by parts method for quantum integrals to obtain the main results.

Lemma 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a q -differentiable function. If $D_q f$ is continuous and integrable on $[a, b]$, then one has the following identity for $m \in [0, 1]$:

$$\begin{aligned} & \frac{1}{(mb - a)} \int_a^{mb} f(x) d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \\ &= (mb - a) \left[\int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) D_q f(ta + m(1-t)b) d_q t \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \right]. \end{aligned} \quad (6)$$

Proof. From the fundamental properties of quantum integrals, we have

$$\begin{aligned} & \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) D_q f(ta + m(1-t)b) d_q t + \int_{\frac{1}{2}}^1 \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &= \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) D_q f(ta + m(1-t)b) d_q t + \int_0^1 \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &\quad - \int_0^{\frac{1}{2}} \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &= I_1 + I_2 - I_3. \end{aligned} \quad (7)$$

Using Lemma 1, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left(qt - \frac{1}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &= \frac{q}{(mb - a)} \int_0^{\frac{1}{2}} f(qta + m(1 - qt)b) d_q t - \left(qt - \frac{1}{6} \right) \frac{f(ta + m(1-t)b)}{mb - a} \Big|_0^{\frac{1}{2}} \\ &= \frac{q}{(mb - a)} \int_0^{\frac{1}{2}} f(qta + m(1 - qt)b) d_q t - \frac{3q - 1}{6(mb - a)} f\left(\frac{a+mb}{2}\right) - \frac{1}{6(mb - a)} f(mb). \end{aligned} \quad (8)$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_0^1 \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &= \frac{q}{mb - a} \int_0^1 f(qta + m(1 - qt)b) d_q t - \frac{6q - 5}{6(mb - a)} f(a) - \frac{5}{6(mb - a)} f(mb) \\ &= \frac{1}{(mb - a)^2} \int_a^{mb} f(x) d_q x - \frac{1 - q}{(mb - a)} f(a) - \frac{6q - 5}{6(mb - a)} f(a) - \frac{5}{6(mb - a)} f(mb) \end{aligned} \quad (9)$$

and

$$\begin{aligned} I_3 &= \int_0^{\frac{1}{2}} \left(qt - \frac{5}{6} \right) D_q f(ta + m(1-t)b) d_q t \\ &= \frac{q}{b - ma} \int_0^{\frac{1}{2}} f(qta + m(1 - qt)b) d_q t - \frac{3q - 5}{6(mb - a)} f\left(\frac{a+mb}{2}\right) - \frac{5}{6(mb - a)} f(mb). \end{aligned} \quad (10)$$

Thus, from (8)–(10), we have

$$I_1 + I_2 - I_3 = \frac{1}{(mb - a)^2} \int_a^{mb} f(x) d_q x - \frac{1}{6(mb - a)} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \quad (11)$$

and we obtain the required equality (6) by multiplying $(mb - a)$ on both sides of (11). The proof is completed. \square

Remark 1. In Lemma 2, we have:

- (i) If we set $\alpha = m = 1$, then we find Lemma 2 in [32].
- (ii) If we set $q \rightarrow 1^-$, then we find Lemma 2.1 in [39].
- (iii) If we set $\alpha = m = 1$ and later taking limit as $q \rightarrow 1^-$, then we find Lemma 1 in [7].

We now observe how an equality emerges when we use the kernel mapping with three sections.

Lemma 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be a q -differentiable function on (a, b) . If $D_q f$ is continuous and integrable on $[a, b]$, then one has the following identity for $m \in [0, 1]$:

$$\begin{aligned} & \frac{1}{(mb - a)} \int_a^{mb} f(x) d_q x - \frac{3}{8} \left[\frac{f(mb)}{3} + f\left(\frac{a+2mb}{3}\right) + f\left(\frac{2a+mb}{3}\right) + \frac{f(b)}{3} \right] \\ &= (mb - a) \left[\int_0^{\frac{1}{3}} \left(qt - \frac{1}{8} \right) D_q f(ta + m(1-t)b) d_q t \right. \\ &\quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(qt - \frac{1}{2} \right) D_q f(ta + m(1-t)b) d_q t \\ &\quad \left. + \int_{\frac{2}{3}}^1 \left(qt - \frac{7}{8} \right) D_q f(ta + m(1-t)b) d_q t \right]. \end{aligned} \quad (12)$$

Proof. The desired result can be attained if the same steps used in the proof of Lemma 2 are used in this proof. \square

Remark 2. In Lemma 3, if we set $\alpha = m = 1$, then we find Lemma 3 in [32].

4. Simpson's 1/3 Formula Type Inequalities

In this section, we prove Simpson's 1/3 formula type inequalities for the differentiable (α, m) -convex function.

Theorem 5. Under the assumption of Lemma 2, if $|{}^b D_q f|$ is (α, m) -convex mapping over $[a, b]$, then we have the following Simpson-type inequality:

$$\begin{aligned} & \left| \frac{1}{(mb - a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ & \leq (mb - a) \left[(\Omega_1(\alpha; q) + \Omega_3(\alpha; q)) |{}^b D_q f(a)| + (\Omega_2(\alpha; q) + \Omega_4(\alpha; q)) m |{}^b D_q f(b)| \right], \end{aligned}$$

where

$$\begin{aligned} \Omega_1(\alpha; q) &= \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| t^\alpha d_q t \\ &= \begin{cases} \frac{1}{6 \cdot 2^{\alpha+1} [\alpha+1]_q} - \frac{q}{2^{\alpha+2} [\alpha+2]_q}, & 0 < q < \frac{1}{3}; \\ \frac{2 - (3q)^{\alpha+1}}{6 \cdot (6q)^{\alpha+1} [\alpha+1]_q} - \frac{q((3q)^{\alpha+2} - 2)}{(6q)^{\alpha+2} [\alpha+2]_q}, & \frac{1}{3} \leq q < 1, \end{cases} \end{aligned}$$

$$\Omega_2(\alpha; q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| (1 - t^\alpha) d_q t$$

$$\begin{aligned}
&= \begin{cases} \frac{1-2q}{12[2]_q} - \frac{1}{6 \cdot 2^{\alpha+1}[\alpha+1]_q} + \frac{q}{2^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{1}{3}; \\ \frac{6q-1}{36[2]_q} - \frac{2-(3q)^{\alpha+1}}{6 \cdot (6q)^{\alpha+1}[\alpha+1]_q} + \frac{q((3q)^{\alpha+2}-2)}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{1}{3} \leq q < 1, \end{cases} \\
\Omega_3(\alpha; q) &= \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| t^\alpha d_q t \\
&= \begin{cases} \frac{5(2^{\alpha+1}-1)}{6 \cdot 2^{\alpha+1}[\alpha+1]_q} + \frac{q(1-2^{\alpha+2})}{2^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{5}{6}; \\ \frac{5}{6[\alpha+1]_q} \left(\frac{2 \cdot 5^{\alpha+1} - 3^{\alpha+1}}{(6q)^{\alpha+1}} - 1 \right) + \frac{q}{[\alpha+2]_q} \left(\frac{3^{\alpha+2} - 2 \cdot 5^{\alpha+2}}{(6q)^{\alpha+2}} + 1 \right), & \frac{5}{6} \leq q < 1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_4(\alpha; q) &= \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| (1-t^\alpha) d_q t \\
&= \begin{cases} \frac{5-4q}{12[2]_q} - \frac{5(2^{\alpha+1}-1)}{6 \cdot 2^{\alpha+1}[\alpha+1]_q} - \frac{q(1-2^{\alpha+2})}{2^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{5}{6}; \\ \frac{5}{36[2]_q} - \frac{5}{6[\alpha+1]_q} \left(\frac{2 \cdot 5^{\alpha+1} - 3^{\alpha+1}}{(6q)^{\alpha+1}} - 1 \right) - \frac{q}{[\alpha+2]_q} \left(\frac{3^{\alpha+2} - 2 \cdot 5^{\alpha+2}}{(6q)^{\alpha+2}} + 1 \right), & \frac{5}{6} \leq q < 1. \end{cases}
\end{aligned}$$

Proof. By taking modulus in (6) and using the (α, m) -convexity of $|{}^b D_q f|$, we have

$$\begin{aligned}
&\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\
&\leq (mb-a) \left[\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| |{}^b D_q f(ta + m(1-t)b)| d_q t \right. \\
&\quad \left. + \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| |{}^b D_q f(ta + m(1-t)b)| d_q t \right] \\
&\leq (mb-a) \left[|{}^b D_q f(a)| \int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| t^\alpha d_q t + m |{}^b D_q f(b)| \int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| (1-t^\alpha) d_q t \right. \\
&\quad \left. + |{}^b D_q f(a)| \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| t^\alpha d_q t + m |{}^b D_q f(b)| \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{1}{6} \right) \right| (1-t^\alpha) d_q t \right] \\
&= (mb-a) \left[(\Omega_1(\alpha; q) + \Omega_3(\alpha; q)) |{}^b D_q f(a)| + (\Omega_2(\alpha; q) + \Omega_4(\alpha; q)) m |{}^b D_q f(b)| \right].
\end{aligned}$$

Thus, the proof is completed. \square

Remark 3. In Theorem 5, we have:

- (i) If we set $\alpha = m = 1$, then we find Theorem 4 in [32].
- (ii) If we set $q \rightarrow 1^-$, then we find Theorem 2.2 in [39].
- (iii) If we set $\alpha = m = 1$ and later taking limit as $q \rightarrow 1^-$, then we find Corollary 1 in [7].

Theorem 6. Under the assumption of Lemma 2, if $s \geq 1$ is a real number and $|{}^b D_q f|^s$ is (α, m) -convex mapping over $[a, b]$, then we have the following Simpson-type inequality:

$$\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right|$$

$$\leq (mb - a) \left[\Omega_5^{1-\frac{1}{s}}(q) \left(\Omega_1(\alpha; q) \left| {}^b D_q f(a) \right|^s + \Omega_2(\alpha; q) m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right. \\ \left. + \Omega_6^{1-\frac{1}{s}}(q) \left(\Omega_3(\alpha; q) \left| {}^b D_q f(a) \right|^s + \Omega_4(\alpha; q) m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right], \quad (13)$$

where $\Omega_i(\alpha; q), i = 1, 2, 3, 4$ are defined in Theorem 5,

$$\Omega_5(q) = \int_0^{\frac{1}{2}} \left| qt - \frac{1}{6} \right| d_q t = \begin{cases} \frac{1-2q}{12[2]_q}, & 0 < q < \frac{1}{3}; \\ \frac{6q-1}{36[2]_q}, & \frac{1}{3} \leq q < 1, \end{cases}$$

and

$$\Omega_6(q) = \int_{\frac{1}{2}}^1 \left| qt - \frac{5}{6} \right| d_q t = \begin{cases} \frac{5-4q}{12[2]_q}, & 0 < q < \frac{5}{6}; \\ \frac{5}{36[2]_q}, & \frac{5}{6} \leq q < 1. \end{cases}$$

Proof. By taking the modulus in (6) and using the power mean inequality, we have

$$\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ \leq (mb-a) \left[\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| {}^b D_q f(ta + m(1-t)b) |d_q t \right. \\ \left. + \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| {}^b D_q f(ta + m(1-t)b) |d_q t \right] \\ \leq (mb-a) \left[\left(\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| d_q t \right)^{1-\frac{1}{s}} \left(\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| {}^b D_q f(ta + m(1-t)b) |^s d_q t \right)^{\frac{1}{s}} \right. \\ \left. + \left(\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| d_q t \right)^{1-\frac{1}{s}} \left(\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| {}^b D_q f(ta + m(1-t)b) |^s d_q t \right)^{\frac{1}{s}} \right].$$

Now, applying (α, m) -convexity, we have

$$\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ \leq (mb-a) \left[\left(\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| d_q t \right)^{1-\frac{1}{s}} \right. \\ \times \left(\left| {}^b D_q f(a) \right|^s \int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| t^\alpha d_q t + m \left| {}^b D_q f(b) \right|^s \int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right| (1-t^\alpha) d_q t \right)^{\frac{1}{s}} \\ \left. + \left(\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| d_q t \right)^{1-\frac{1}{s}} \right. \\ \times \left(\left| {}^b D_q f(a) \right|^s \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| t^\alpha d_q t + m \left| {}^b D_q f(b) \right|^s \int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right| (1-t^\alpha) d_q t \right)^{\frac{1}{s}} \\ \left. = (mb-a) \left[\Omega_5^{1-\frac{1}{s}}(q) \left(\Omega_1(\alpha; q) \left| {}^b D_q f(a) \right|^s + \Omega_2(\alpha; q) m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right] \right]$$

$$+ \Omega_6^{1-\frac{1}{s}}(q) \left(\Omega_3(\alpha; q) \left| {}^b D_q f(a) \right|^s + \Omega_4(\alpha; q) m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \Big].$$

Thus, the proof is completed. \square

Remark 4. In Theorem 6, we have:

- (i) If we set $\alpha = m = 1$, then we find Theorem 6 in [32].
- (ii) If we set $q \rightarrow 1^-$, then we find Theorem 2.10 in [39].
- (iii) If we set $\alpha = m = 1$ and later taking limit as $q \rightarrow 1^-$, then we find Theorem 7 for $s = 1$ in [7].

Theorem 7. Under the assumption of Lemma 2, if $s > 1$ is a real number and $\left| {}^b D_q f \right|^s$ is (α, m) -convex mapping over $[a, b]$, then we have the following Simpson-type inequality:

$$\begin{aligned} & \left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ & \leq \frac{(mb-a)}{6} \left[2^{1-\frac{1}{r}} \left(\frac{1}{2^{\alpha+1}[\alpha+1]_q} \left| {}^b D_q f(a) \right|^s + \frac{2^\alpha [\alpha+1]_q - 1}{2^{\alpha+1}[\alpha+1]_q} m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right. \\ & \quad \left. + \left(5^r - 2^{r-1} \right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1} - 1}{2^{\alpha+1}[\alpha+1]_q} \left| {}^b D_q f(a) \right|^s + \frac{2^\alpha ([\alpha+1]_q - 2) + 1}{2^{\alpha+1}[\alpha+1]_q} m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right], \end{aligned} \quad (14)$$

where $s^{-1} + r^{-1} = 1$.

Proof. Taking the modulus in Lemma 2 and applying Hölder's inequality, we have

$$\begin{aligned} & \left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ & \leq (mb-a) \left[\left(\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right|^r d_q t \right)^{\frac{1}{r}} \left(\int_0^{\frac{1}{2}} \left| {}^b D_q f(ta + m(1-t)b) \right|^s d_q t \right)^{\frac{1}{s}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right|^r d_q t \right)^{\frac{1}{r}} \left(\int_{\frac{1}{2}}^1 \left| {}^b D_q f(ta + m(1-t)b) \right|^s d_q t \right)^{\frac{1}{s}} \right]. \end{aligned}$$

Now, applying (α, m) -convexity of $\left| {}^b D_q f \right|^s$, we have

$$\begin{aligned} & \left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{1}{6} \left[f(mb) + 4f\left(\frac{a+mb}{2}\right) + f(a) \right] \right| \\ & \leq (mb-a) \left[\left(\int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right|^r d_q t \right)^{\frac{1}{r}} \left(\left| {}^b D_q f(a) \right|^s \int_0^{\frac{1}{2}} t^\alpha d_q t + m \left| {}^b D_q f(b) \right|^s \int_0^{\frac{1}{2}} (1-t^\alpha) d_q t \right)^{\frac{1}{s}} \right. \\ & \quad \left. + \left(\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right|^r d_q t \right)^{\frac{1}{r}} \left(\left| {}^b D_q f(a) \right|^s \int_{\frac{1}{2}}^1 t^\alpha d_q t + m \left| {}^b D_q f(b) \right|^s \int_{\frac{1}{2}}^1 (1-t^\alpha) d_q t \right)^{\frac{1}{s}} \right] \\ & \leq \frac{(mb-a)}{6} \left[2^{1-\frac{1}{r}} \left(\frac{1}{2^{\alpha+1}[\alpha+1]_q} \left| {}^b D_q f(a) \right|^s + \frac{2^\alpha [\alpha+1]_q - 1}{2^{\alpha+1}[\alpha+1]_q} m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}} \right] \end{aligned}$$

$$+ \left(5^r - 2^{r-1} \right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1}-1}{2^{\alpha+1}[\alpha+1]_q} \left| {}^b D_q f(a) \right|^s + \frac{2^\alpha ([\alpha+1]_q - 2) + 1}{2^{\alpha+1}[\alpha+1]_q} m \left| {}^b D_q f(b) \right|^s \right)^{\frac{1}{s}},$$

where one can easily observe that

$$\begin{aligned} \int_0^{\frac{1}{2}} \left| \left(qt - \frac{1}{6} \right) \right|^r d_q t &= \frac{1-q}{2} \sum_{n=0}^{\infty} q^n \left| \frac{q^{n+1}}{2} - \frac{1}{6} \right|^r \\ &\leq \frac{1-q}{2} \sum_{n=0}^{\infty} q^n \left| \frac{1}{2} - \frac{1}{6} \right|^r \\ &= \frac{1}{2 \cdot 3^r} \end{aligned}$$

and, similarly,

$$\int_{\frac{1}{2}}^1 \left| \left(qt - \frac{5}{6} \right) \right|^r d_q t \leq \frac{5^r - 2^{r-1}}{6}.$$

Thus, the proof is completed. \square

Remark 5. In Theorem 7, if we set $\alpha = m = 1$, then we find Theorem 5 in [32].

5. Simpson's 3/8 Formula Type Inequalities

In this section, we prove Simpson's 3/8 formula type inequalities for the differentiable (α, m) -convex function.

Theorem 8. Under the assumption of Lemma 3, if $\left| {}^b D_q f \right|$ is (α, m) -convex mapping over $[a, b]$, then we have the following Newton's type inequality:

$$\begin{aligned} &\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{3}{8} \left[f\left(\frac{mb}{3}\right) + f\left(\frac{a+2mb}{3}\right) + f\left(\frac{2a+mb}{3}\right) + f(b) \right] \right| \\ &\leq (mb-a) \left[\begin{array}{l} (\Omega_7(\alpha; q) + \Omega_9(\alpha; q) + \Omega_{11}(\alpha; q)) \left| {}^b D_q f(a) \right| \\ + (\Omega_8(\alpha; q) + \Omega_{10}(\alpha; q) + \Omega_{12}(\alpha; q)) m \left| {}^b D_q f(b) \right| \end{array} \right], \end{aligned} \quad (15)$$

where

$$\begin{aligned} \Omega_7(\alpha; q) &= \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| t^\alpha d_q t \\ &= \begin{cases} \frac{1}{8 \cdot 3^{\alpha+1} [\alpha+1]_q} - \frac{q}{3^{\alpha+2} [\alpha+2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{3^{\alpha+1} - 4(8q)^{\alpha+1}}{4 \cdot (24q)^{\alpha+1} [\alpha+1]_q} - \frac{q((8q)^{\alpha+2} - 2 \cdot 3^{\alpha+2})}{(24q)^{\alpha+2} [\alpha+2]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\ \Omega_8(\alpha; q) &= \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| (1-t^\alpha) d_q t \\ &= \begin{cases} \frac{3-5q}{72[2]_q} - \frac{1}{8 \cdot 3^{\alpha+1} [\alpha+1]_q} + \frac{q}{3^{\alpha+2} [\alpha+2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{20q-3}{288[2]_q} - \frac{3^{\alpha+1} - 4(8q)^{\alpha+1}}{4 \cdot (24q)^{\alpha+1} [\alpha+1]_q} + \frac{q((8q)^{\alpha+2} - 2 \cdot 3^{\alpha+2})}{(24q)^{\alpha+2} [\alpha+2]_q}, & \frac{3}{8} \leq q < 1, \end{cases} \\ \Omega_9(\alpha; q) &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| t^\alpha d_q t \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \frac{2^{\alpha+1}-1}{2 \cdot 3^{\alpha+1}[\alpha+1]_q} + \frac{q(1-2^{\alpha+1})}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{2 \cdot 3^{\alpha+1} - (2q)^{\alpha+1}(1+2^{\alpha+1})}{2 \cdot (6q)^{\alpha+1}[\alpha+1]_q} + \frac{q(-2 \cdot 3^{\alpha+2} - (2q)^{\alpha+2}(1+2^{\alpha+2}))}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{4} \leq q < 1, \end{cases} \\
\Omega_{10}(\alpha; q) &= \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| (1-t^\alpha) d_q t \\
&= \begin{cases} \frac{3-3q}{18[2]_q} - \frac{2^{\alpha+1}-1}{2 \cdot 3^{\alpha+1}[\alpha+1]_q} - \frac{q(1-2^{\alpha+1})}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{q}{18[2]_q} - \frac{2 \cdot 3^{\alpha+1} - (2q)^{\alpha+1}(1+2^{\alpha+1})}{2 \cdot (6q)^{\alpha+1}[\alpha+1]_q} - \frac{q(-2 \cdot 3^{\alpha+2} - (2q)^{\alpha+2}(1+2^{\alpha+2}))}{(6q)^{\alpha+2}[\alpha+2]_q}, & \frac{3}{4} \leq q < 1, \end{cases} \\
\Omega_{11}(\alpha; q) &= \int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right| t^\alpha d_q t \\
&= \begin{cases} \frac{7(3^{\alpha+1}-2^{\alpha+1})}{8 \cdot 3^{\alpha+1}[\alpha+1]_q} - \frac{q(2^{\alpha+2}-32)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{7}{8[\alpha+1]_q} \left(2 \left(\frac{7}{8q} \right)^{\alpha+1} - \left(\frac{2}{3} \right)^{\alpha+1} - 1 \right) + \frac{q}{[\alpha+2]_q} \left(\left(\frac{2}{3} \right)^{\alpha+2} - 2 \left(\frac{7}{8q} \right)^{\alpha+2} + 1 \right), & \frac{7}{8} \leq q < 1, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{12}(\alpha; q) &= \int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right| (1-t^\alpha) d_q t \\
&= \begin{cases} \frac{21-19q}{72[2]_q} - \frac{7(3^{\alpha+1}-2^{\alpha+1})}{8 \cdot 3^{\alpha+1}[\alpha+1]_q} + \frac{q(2^{\alpha+2}-32)}{3^{\alpha+2}[\alpha+2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{21-4q}{288[2]_q} - \frac{7}{8[\alpha+1]_q} \left(2 \left(\frac{7}{8q} \right)^{\alpha+1} - \left(\frac{2}{3} \right)^{\alpha+1} - 1 \right) - \frac{q}{[\alpha+2]_q} \left(\left(\frac{2}{3} \right)^{\alpha+2} - 2 \left(\frac{7}{8q} \right)^{\alpha+2} + 1 \right), & \frac{7}{8} \leq q < 1. \end{cases}
\end{aligned}$$

Proof. By addressing the equality (12), the proof of this theorem follows the same lines as the proof of Theorem 5. \square

Remark 6. In Theorem 8, if we set $\alpha = m = 1$, then we find Theorem 7 in [32].

Theorem 9. Under the assumption of Lemma 3, if $s \geq 1$ is a real number and $|{}^b D_q f|^s$ is (α, m) -convex mapping over $[a, b]$, then we have the following Newton's type inequality:

$$\begin{aligned}
&\left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{3}{8} \left[\frac{f(mb)}{3} + f\left(\frac{a+2mb}{3}\right) + f\left(\frac{2a+mb}{3}\right) + \frac{f(b)}{3} \right] \right| \\
&\leq (mb-a) \left[\Omega_{13}^{1-\frac{1}{s}}(q) \left(\Omega_7(\alpha; q) |{}^b D_q f(a)|^s + \Omega_8(\alpha; q) m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \right. \\
&\quad + \Omega_{14}^{1-\frac{1}{s}}(q) \left(\Omega_9(\alpha; q) |{}^b D_q f(a)|^s + \Omega_{10}(\alpha; q) m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \\
&\quad \left. + \Omega_{15}^{1-\frac{1}{s}}(q) \left(\Omega_{11}(\alpha; q) |{}^b D_q f(a)|^s + \Omega_{12}(\alpha; q) m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \right], \tag{16}
\end{aligned}$$

where $\Omega_j(\alpha; q), j = 7, 8, 9, \dots, 12$ are defined in Theorem 8,

$$\Omega_{13}(q) = \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| t^\alpha d_q t = \begin{cases} \frac{3-5q}{72[2]_q}, & 0 < q < \frac{3}{8}; \\ \frac{20q-3}{288[2]_q}, & \frac{3}{8} \leq q < 1, \end{cases}$$

$$\Omega_{14}(q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| d_q t = \begin{cases} \frac{3-3q}{18[2]_q}, & 0 < q < \frac{3}{4}; \\ \frac{q}{18[2]_q}, & \frac{3}{4} \leq q < 1, \end{cases}$$

and

$$\Omega_{15}(q) = \int_{\frac{2}{3}}^1 \left| qt - \frac{7}{8} \right| d_q t = \begin{cases} \frac{21-19q}{72[2]_q}, & 0 < q < \frac{7}{8}; \\ \frac{21-4q}{288[2]_q}, & \frac{7}{8} \leq q < 1. \end{cases}$$

Proof. By addressing the equality (12), the proof of this theorem follows the same lines as the proof of Theorem 6. \square

Remark 7. In Theorem 9, if we set $\alpha = m = 1$, then we find Theorem 9 in [32].

Theorem 10. Under the assumption of Lemma 3, if $s > 1$ is a real number and $|{}^b D_q f|^s$ is (α, m) -convex mapping over $[a, b]$, then we have the following Newton's type inequality:

$$\begin{aligned} & \left| \frac{1}{(mb-a)} \int_a^{mb} f(x) {}^{mb} d_q x - \frac{3}{8} \left[\frac{f(mb)}{3} + f\left(\frac{a+2mb}{3}\right) + f\left(\frac{2a+mb}{3}\right) + \frac{f(b)}{3} \right] \right| \\ & \leq (mb-a) \left[\left(\frac{5^r}{3.8^r} \right)^{\frac{1}{r}} \left(\frac{1}{3^{\alpha+1}[\alpha+1]_q} |{}^b D_q f(a)|^s + \frac{3^\alpha[\alpha+1]_q - 1}{3^{\alpha+1}[\alpha+1]_q} m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \right. \\ & \quad + \left(\frac{2.3^r - 1}{3.6^r} \right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1} - 1}{3^{\alpha+1}[\alpha+1]_q} |{}^b D_q f(a)|^s + \frac{3^\alpha[\alpha+1]_q - 2^{\alpha+1} + 1}{3^{\alpha+1}[\alpha+1]_q} m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \\ & \quad \left. + \left(\frac{3.7^r - 2}{3.8^r} \right)^{\frac{1}{r}} \left(\frac{2^{\alpha+1} - 2^{\alpha+1}}{3^{\alpha+1}[\alpha+1]_q} |{}^b D_q f(a)|^s + \frac{3^\alpha([\alpha+1]_q - 3) + 2^{\alpha+1}}{3^{\alpha+1}[\alpha+1]_q} m |{}^b D_q f(b)|^s \right)^{\frac{1}{s}} \right], \quad (17) \end{aligned}$$

where $s^{-1} + r^{-1} = 1$.

Proof. By addressing the equality (12), the proof of this theorem follows the same lines as the proof of Theorem 7. \square

Remark 8. In Theorem 10, if we set $\alpha = m = 1$, then we find Theorem 8 in [32].

6. Conclusions

In this work, we proved two quantum integral identities to establish some new quantum Simpson's and quantum Newton's formula type inequalities for differentiable (α, m) -convex functions, which was the main motivation of this paper. We also proved that the newly established inequalities could be turned into quantum Simpson's and quantum Newton's inequalities for convex functions [32], classical Simpson's type inequalities for (α, m) -convex functions [39], and classical Simpson's type inequalities for convex functions [7] without having to prove each one separately. This is a new and interesting

problem, and researchers can obtain similar inequalities for other kinds of convexity and coordinated (α, m) -convexity in future works.

Author Contributions: conceptualization, I.B.S., S.M, M.A.A. and K.N.; Methodology, I.B.S., S.M, M.A.A. and K.N.; software, I.B.S., S.M, M.A.A. and K.N.; validation, I.B.S., S.M, M.A.A. and K.N.; formal analysis, I.B.S., S.M, M.A.A. and K.N.; investigation, I.B.S., S.M, M.A.A. and K.N.; resources, I.B.S., S.M, M.A.A. and K.N.; writing original draft preparation, I.B.S., S.M, M.A.A. and K.N.; writing review and editing, I.B.S., S.M, M.A.A. and K.N.; visualization, I.B.S., S.M, M.A.A. and K.N.; supervision, S.M and M.A.A; project administration, S.M. and K.N.; funding acquisition, K.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-63-KNOW-21.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We are grateful to the referees for their valuable comments. This research received funding support from the National Science, Research and Innovation Fund (NSRF), Thailand.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Hudzik, H.; Maligrada, L. Some remarks on s -convex functions. *Aequ. Math.* **1994**, *48*, 100–111. [[CrossRef](#)]
2. Dragomir, S.S.; Pearce, C.E.M. *Selected Topics on Hermite-Hadamard Inequalities and Applications*; RGMIA Monographs, Victoria University: Melbourne, Australia, 2000.
3. Pečarić, J.E.; Proschan, F.; Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications*; Academic Press: Boston, MA, USA, 1992.
4. Dragomir, S.S.; Fitzpatrick, S. The Hadamard's inequality for s -convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696.
5. Miheşan, V.G. A generalization of the Convexity. In *Seminar on Functional Equations, Approx. and Convex*; Convex: Cluj-Napoca, Romania, 1993.
6. Dragomir, S.S.; Agarwal, R.P.; Cerone, P. On Simpson's inequality and applications. *J. Inequal. Appl.* **2000**, *5*, 533–579. [[CrossRef](#)]
7. Alomari, M.; Darus, M.; Dragomir, S.S. New inequalities of Simpson's type for s -convex functions with applications. *RGMIA Res. Rep. Coll.* **2009**, *12*. Available online: <https://rgmia.org/papers/v12n4/Simpson.pdf> (accessed on 10 December 2021).
8. Sarikaya, M.Z.; Set, E.; Özdemir, M.E. On new inequalities of Simpson's type for s -convex functions. *Comput. Math. Appl.* **2010**, *60*, 2191–2199. [[CrossRef](#)]
9. Noor, M.A.; Noor, K.I.; Iftikhar, S. Some Newton's type inequalities for harmonic convex functions. *J. Adv. Math. Stud.* **2016**, *9*, 7–16.
10. Noor, M.A.; Noor, K.I.; Iftikhar, S. Newton inequalities for p -harmonic convex functions. *Honam Math. J.* **2018**, *40*, 239–250.
11. Iftikhar, S.; Kumam, P.; Erden, S. Newton's type integral inequalities via local fractional integrals. *Fractals* **2020**, *28*, 2050037. [[CrossRef](#)]
12. Ernst, T. *A Comprehensive Treatment of q -Calculus*; Springer: Basel, Switzerland, 2012.
13. Kac, V.; Cheung, P. *Quantum Calculus*; Springer: Berlin/Heidelberg, Germany, 2001.
14. Benatti, F.; Fannes, M.; Floreanini, R.; Petritis, D. *Quantum Information, Computation and Cryptography: An Introductory Survey of Theory, Technology and Experiments*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2010.
15. Bokulich, A.; Jaeger, G. *Philosophy of Quantum Information Theory and Entanglement*; Cambridge University Press: Cambridge, UK, 2010.
16. Jackson, F.H. On a q -definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
17. Al-Salam, W. Some fractional q -integrals and q -derivatives. *Proc. Edinburgh Math. Soc.* **1966**, *15*, 135–140. [[CrossRef](#)]
18. Tariboon, J.; Ntouyas, S.K. Quantum calculus on finite intervals and applications to impulsive difference equations. *Adv. Differ. Equ.* **2013**, *2013*, 282. [[CrossRef](#)]
19. Bermudo, S.; Kórus, P.; Valdés, J.N. On q -Hermite-Hadamard inequalities for general convex functions. *Acta Math. Hung.* **2020**, *162*, 364–374. [[CrossRef](#)]
20. Ali, M.A.; Alp, N.; Budak, H.; Chu, Y.M.; Zhang, Z. On some new quantum midpoint type inequalities for twice quantum differentiable convex functions. *Open Math.* **2021**, *19*, 427–439. [[CrossRef](#)]
21. Ali, M.A.; Budak, H.; Abbas, M.; Chu, Y.M. QuantumHermite-Hadamard-type inequalities for functions with convex absolute values of second q^b -derivatives. *Adv. Differ. Equ.* **2021**, *2021*, 1–12. [[CrossRef](#)]

22. Alp, N.; Sarikaya, M.Z. Hermite Hadamard’s type inequalities for co-ordinated convex functions on quantum integral. *Appl. Math. E-Notes* **2020**, *20*, 341–356.
23. Budak, H. Some trapezoid and midpoint type inequalities for newly defined quantum integrals. *Proyecciones* **2021**, *40*, 199–215. [[CrossRef](#)]
24. Budak, H.; Ali, M.A.; Tarhanaci, M. Some new quantum Hermite-Hadamard-like inequalities for coordinated convex functions. *J. Optim. Theory Appl.* **2020**, *186*, 899–910. [[CrossRef](#)]
25. Alp, N.; Sarikaya, M.Z.; Kunt, M.; İşcan, İ. q -Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. *J. King Saud Univ. Sci.* **2018**, *30*, 193–203. [[CrossRef](#)]
26. Jhanthanam, S.; Tariboon, J.; Ntouyas, S.K.; Nonlaopon, K. On q -Hermite-Hadamard inequalities for differentiable convex functions. *Mathematics* **2019**, *7*, 632. [[CrossRef](#)]
27. Liu, W.; Hefeng, Z. Some quantum estimates of Hermite-Hadamard inequalities for convex functions. *J. Appl. Anal. Comput.* **2016**, *7*, 501–522.
28. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum estimates for Hermite-Hadamard inequalities. *Appl. Math. Comput.* **2015**, *251*, 675–679. [[CrossRef](#)]
29. Noor, M.A.; Noor, K.I.; Awan, M.U. Some quantum integral inequalities via preinvex functions. *Appl. Math. Comput.* **2015**, *269*, 242–251. [[CrossRef](#)]
30. Nwaeze, E.R.; Tameru, A.M. New parameterized quantum integral inequalities via η -quasiconvexity. *Adv. Differ. Equ.* **2019**, *2019*, 425. [[CrossRef](#)]
31. Khan, M.A.; Noor, M.; Nwaeze, E.R.; Chu, Y.-M. Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 1–20.
32. Budak, H.; Erden, S.; Ali, M.A. Simpson and Newton type inequalities for convex functions via newly defined quantum integrals. *Math. Meth. Appl. Sci.* **2020**, *44*, 378–390. [[CrossRef](#)]
33. Ali, M.A.; Budak, H.; Zhang, Z.; Yıldırım, H. Some new Simpson’s type inequalities for co-ordinated convex functions in quantum calculus. *Math. Meth. Appl. Sci.* **2021**, *44*, 4515–4540. [[CrossRef](#)]
34. Ali, M.A.; Abbas, M.; Budak, H.; Agarwal, P.; Murtaza, G.; Chu, Y.M. New quantum boundaries for quantum Simpson’s and quantum Newton’s type inequalities for preinvex functions. *Adv. Differ. Equ.* **2021**, *2021*, 64. [[CrossRef](#)]
35. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Sial, I.B.; Zhang, Z. Some new Newton’s type integral inequalities for coordinated convex functions in quantum calculus. *Symmetry* **2020**, *12*, 1476. [[CrossRef](#)]
36. Ali, M.A.; Chu, Y.M.; Budak, H.; Akkurt, A.; Yıldırım, H.; Zahid, M.A. Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables. *Adv. Differ. Equ.* **2021**, *2021*, 25. [[CrossRef](#)]
37. Ali, M.A.; Budak, H.; Akkurt, A.; Chu, Y.M. Quantum Ostrowski type inequalities for twice quantum differentiable functions in quantum calculus. *Open Math.* **2021**, *19*, 440–449. [[CrossRef](#)]
38. Budak, H.; Ali, M.A.; Alp, N.; Chu, Y.-M. Quantum Ostrowski type integral inequalities. *J. Math. Inequal.* **2021**, in press.
39. Qaisar, S.; He, C.; Hussain, S. A generalizations of Simpson’s type inequality for differentiable functions using (α, m) -convex functions and applications. *J. Inequal. Appl.* **2013**, *2013*, 1–13. [[CrossRef](#)]