mathematics

Article

The Integral Mittag-Leffler, Whittaker and Wright Functions

Alexander Apelblat [1] and Juan Luis González-Santander [2,*]

[1] Department of Chemical Engineering, Ben Gurion University of the Negev, Beer Sheva 84105, Israel; apelblat@bgu.ac.il
[2] Department of Mathematics, Universidad de Oviedo, 33007 Oviedo, Spain
* Correspondence: gonzalezmarjuan@uniovi.es

Abstract: Integral Mittag-Leffler, Whittaker and Wright functions with integrands similar to those which already exist in mathematical literature are introduced for the first time. For particular values of parameters, they can be presented in closed-form. In most reported cases, these new integral functions are expressed as generalized hypergeometric functions but also in terms of elementary and special functions. The behavior of some of the new integral functions is presented in graphical form. By using the MATHEMATICA program to obtain infinite sums that define the Mittag-Leffler, Whittaker, and Wright functions and also their corresponding integral functions, these functions and many new Laplace transforms of them are also reported in the Appendices for integral and fractional values of parameters.

Keywords: integral Mittag-Leffler functions; integral Whittaker functions; integral Wright functions; Laplace transforms

Citation: Apelblat, A.; González-Santander, J.L. The Integral Mittag-Leffler, Whittaker and Wright Functions. *Mathematics* **2021**, *9*, 3255. https://doi.org/10.3390/math9243255

Academic Editors: Francesco Mainardi and Jaan Janno

Received: 24 November 2021
Accepted: 12 December 2021
Published: 15 December 2021

1. Introduction

The appearance of special functions of mathematical physics was associated with solutions of particular ordinary differential equations, while the integral special functions arrived much later in mathematical literature after properties of these functions were investigated. Integral special functions were introduced as new special functions, which can be applied in many circumstances, especially in operational calculus, where they are frequently serving as direct and inverse integral transforms. The form of an integrand is identical for all integral functions, but limits of integration are different in order to assure the convergence of defined integrals. There are two types of integral special functions: those with elementary functions in their integrands and those with special functions. To the first group belong the exponential integral $-\mathrm{Ei}(-x)$, the sine and cosine integrals, $\mathrm{si}(x)$, $\mathrm{Si}(x)$, $\mathrm{ci}(x)$ and $\mathrm{Ci}(x)$, and the corresponding integrals of hyperbolic trigonometric functions, $\mathrm{Shi}(x)$ and $\mathrm{Chi}(x)$. These functions are defined in the following way [1–5]

$$\begin{aligned}
&\mathrm{E}_1(x) = -\mathrm{Ei}(-x) = \int_x^\infty \frac{e^{-t}}{t}dt, \quad x>0, \\
&\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt, \\
&\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}dt = \mathrm{Si}(x) - \frac{\pi}{2}, \\
&\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt = \gamma + \ln x - \int_0^x \frac{1-\cos t}{t}dt = -\mathrm{ci}(x), \\
&\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}dt, \\
&\mathrm{Chi}(x) = \gamma + \ln x - \int_0^x \frac{1-\cosh t}{t}dt,
\end{aligned} \tag{1}$$

where γ is the Euler–Mascheroni constant. As can be observed in (1), the integral special functions have integrands in the form, $f(t)/t$, and the intervals of integrations are $0 < t < x$

or $x < t < \infty$. Few direct and inverse integral transforms are presented below to illustrate their applications, for example, in the Laplace transformation [6–8],

$$F(s) := \mathcal{L}[f(t)] := \int_0^\infty e^{-st} f(t)dt, \tag{2}$$

we have

$$\begin{aligned}
&\mathcal{L}\left[\frac{1}{\sqrt{t}}\mathrm{Ei}(-t)\right] = -2\sqrt{\frac{\pi}{s}}\ln\left(\sqrt{s}+\sqrt{s+1}\right), \quad \mathrm{Re}\, s > 0,\\
&\mathcal{L}[\mathrm{Si}(t)] = \frac{\cot^{-1}s}{s}, \quad \mathrm{Re}\, s > 0,\\
&\mathcal{L}[\mathrm{si}(t)] = \frac{\tan^{-1}s}{s},\\
&\mathcal{L}[\mathrm{Ci}(t)] = -\frac{\ln(1+s^2)}{2s},\\
&\mathcal{L}^{-1}\left[\frac{\ln(s+b)}{s+a}\right] = e^{-at}[\ln(b-a) - \mathrm{Ei}((a-b)t)], \quad \mathrm{Re}\,(s-a) > 0,\\
&\mathcal{L}^{-1}\left[\frac{\ln s}{s^2+1}\right] = \cos t\,\mathrm{Si}(t) - \sin t\,\mathrm{Ci}(t), \quad \mathrm{Re}\, s > 0,\\
&\mathcal{L}^{-1}\left[\frac{s\ln s}{s^2+1}\right] = -\sin t\,\mathrm{Si}(t) - \cos t\,\mathrm{Ci}(t).
\end{aligned} \tag{3}$$

Integrands in the second group of integral special functions include special functions, the most well-known and applied of which are the integral Bessel functions (see, e.g., [3,7,9–13])

$$\begin{aligned}
\mathrm{Ji}_\nu(x) &= -\int_x^\infty \frac{J_\nu(t)}{t}dt,\\
\mathrm{Yi}_\nu(x) &= -\int_x^\infty \frac{Y_\nu(t)}{t}dt,\\
\mathrm{Ii}_\nu(x) &= -\int_0^x \frac{I_\nu(t)}{t}dt,\\
\mathrm{Ki}_\nu(x) &= -\int_x^\infty \frac{K_\nu(t)}{t}dt.
\end{aligned} \tag{4}$$

Already in 1929, van der Pol [9] showed that it is possible to express the differentiation with respect to the order of the Bessel function of the first kind as a convolution integral, which includes the integral Bessel function of the zero-order:

$$\frac{\partial J_\nu(t)}{\partial \nu} = \frac{1}{2}\int_0^t \mathrm{Ji}_0(t-x)[J_{\nu-1}(x) - J_{\nu+1}(x)]dx. \tag{5}$$

The integral Bessel functions of the zero-order are inverse transforms of the following Laplace transforms [7]

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{\sinh^{-1}s}{s}\right] = \mathrm{Ji}_0(t),\\
&\mathcal{L}^{-1}\left[\frac{\left(\sinh^{-1}s\right)^2}{s}\right] = \mathrm{Yi}_0(t),\\
&\mathcal{L}^{-1}\left[\ln\left(s+\sqrt{s^2+1}\right) - \frac{\pi i}{2}\right] = \mathrm{Ii}_0(t),\\
&\mathcal{L}^{-1}\left[\frac{\left(\cosh^{-1}s\right)^2}{2s} + \frac{\pi^2}{8s}\right] = \mathrm{Ki}_0(t).
\end{aligned} \tag{6}$$

In analogy to the integral Bessel functions and with the possibility of extension to other special functions, this work introduces three new integral functions. Furthermore, these integral functions guide us toward the establishment of integrals and series. Section 2 explores the integral Mittag-Leffler functions. Sections 3 and 4 discuss the integral Whittaker and Wright functions, respectively. Section 5 contains concluding remarks.

In order to preserve the applied form of notation, the following two integral functions are introduced:

$$\mathrm{Fi}(x) = \int_0^x \frac{f(t) - f(0)}{t} dt, \tag{7}$$

and

$$\mathrm{fi}(x) = \int_x^\infty \frac{f(t)}{t} dt. \tag{8}$$

To ensure convergence of integrals in (7) or in (8), which depends on the behavior of $f(t)/t$ integrands at the origin and at infinity, the forms of integral functions $\mathrm{Fi}(x)$ or $\mathrm{fi}(x)$ are chosen. Since the explicit expressions for $f(t)$ functions are sometimes given in the form of $f(t^\alpha)$ where $\alpha = \pm\frac{1}{2}, \pm 1, 2, 3, \ldots$ the corresponding change of integration variables for these equations is desired.

In the case of Mittag-Leffler, Whittaker and Wright functions, for some values of parameters, by using the MATHEMATICA program, it was possible to obtain these integral functions in a closed-form. Derived integral functions are tabulated and also in some cases graphically presented (see [3]).

2. The Integral Mittag-Leffler Functions

The classical one-parameter and the two-parameter Mittag-Leffler functions are defined by [14]:

$$\begin{aligned} \mathrm{E}_\alpha(x) &= \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \mathrm{Re}\,\alpha > 0, \\ \mathrm{E}_{\alpha,\beta}(x) &= \sum_{k=0}^\infty \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \mathrm{Re}\,\alpha > 0,\ \mathrm{Re}\,\beta > 0. \end{aligned} \tag{9}$$

In this investigation, they are only considered for positive real values of the argument, i.e., $x > 0$. In the particular case of positive rational α with $\alpha = p/q$ and p and q positive coprimes, Mittag-Leffler functions are given as a finite sum of generalized hypergeometric functions (see (A3) in Appendix A).

The Laplace transforms of the Mittag-Leffler functions are derived directly from (2) and (9), and we have:

$$\begin{aligned} &\mathcal{L}[\mathrm{E}_\alpha(t)] = \int_0^\infty e^{-st} \left[\sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k + 1)}\right] dt = \sum_{k=0}^\infty \frac{k!}{\Gamma(\alpha k + 1)} \left(\frac{1}{s}\right)^{k+1}, \\ &\mathcal{L}\left[\mathrm{E}_{\alpha,\beta}(t)\right] = \int_0^\infty e^{-st} \left[\sum_{k=0}^\infty \frac{t^k}{\Gamma(\alpha k + \beta)}\right] dt = \sum_{k=0}^\infty \frac{k!}{\Gamma(\alpha k + \beta)} \left(\frac{1}{s}\right)^{k+1}, \\ &\mathrm{Re}\,s > 1. \end{aligned} \tag{10}$$

For particular values of parameters α and β, the explicit form of the Mittag-Leffler functions can be obtained by applying the MATHEMATICA program to sums of infinite series in (9), and these results are presented in Appendix A. Using Equation (10), many new Laplace transforms of the Mittag-Leffler functions were evaluated, and they are also reported in Appendix A. Similarly as in the case when α is positive rational, the Laplace transforms of the Mittag-Leffler functions can be expressed by the finite sum of products of generalized hypergeometric functions (see (A4) in Appendix A).

The integral Mittag-Leffler functions are introduced by considering their exponential behavior as a function of real, positive variable x (see Appendix A).

$$\begin{aligned} \mathrm{Ei}_{\alpha}(x) &= \int_0^x \frac{\mathrm{E}_{\alpha}(t)-1}{t}dt, \\ \mathrm{Ei}_{\alpha,\beta}(x) &= \int_0^x \frac{\mathrm{E}_{\alpha,\beta}(t)-1/\Gamma(\beta)}{t}dt. \end{aligned} \tag{11}$$

Formally, by introducing (9) into (11) we have

$$\begin{aligned} \mathrm{Ei}_{\alpha}(x) &= \sum_{k=1}^{\infty} \frac{x^k}{k\,\Gamma(\alpha k+1)}, \\ \mathrm{Ei}_{\alpha,\beta}(x) &= \sum_{k=1}^{\infty} \frac{x^k}{k\,\Gamma(\alpha k+\beta)}. \end{aligned} \tag{12}$$

For several values of parameters α and β, it is possible to derive the integral Mittag-Leffler functions in a closed-form by applying the MATHEMATICA program to the sums of infinite series in (12). These functions are presented in Tables 1 and 2. As it is observable, most of these integral functions are expressed as generalized hypergeometric series. Typical behavior of one-parameter and two-parameter integral Mittag-Leffler functions is illustrated in Figures 1 and 2.

Evidently, also direct integration, by using (11), leads to the integral Mittag-Leffler functions. For example, for $\mathrm{E}_2(x) = \cosh\sqrt{x}$, according to (1), we have

$$\mathrm{Ei}_2(x) = \int_0^x \frac{\cosh\sqrt{t}-1}{t}dt = -2\gamma - \ln x + 2\,\mathrm{Chi}\sqrt{x}, \tag{13}$$

and as expected, this result is identical to that derived from (12) (see Tables 1 and 2).

Applying the formulas (A1) and (A2) given in Appendix A, the integral Mittag-Leffler function for positive rational values of parameter α with $\alpha = p/q$ and p, q positive coprimes is

$$\begin{aligned} &\mathrm{Ei}_{p/q,\beta}(x) \\ &= \sum_{k=1}^{q} \frac{x^k}{k\,\Gamma(k/q+\beta)}\, {}_2F_q\left(\begin{matrix} 1, k/q \\ b_0, \dots, b_{p-1}, k/q+1 \end{matrix} \middle| \frac{x^q}{p^p} \right). \end{aligned} \tag{14}$$

where

$$b_j = \frac{k}{q} + \frac{\beta+j}{p}.$$

In addition, using the sums in (12), it is possible to derive the Laplace transforms of the integral Mittag-Leffler functions:

$$\begin{aligned} \mathcal{L}[\mathrm{Ei}_{\alpha}(t)] &= \int_0^{\infty} e^{-st}\left[\sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)\Gamma(\alpha(k+1)+1)}\right]dt \\ &= \sum_{k=0}^{\infty} \frac{(k+1)!}{(k+1)\Gamma(\alpha(k+1)+1)}\left(\frac{1}{s}\right)^{k+2}, \quad \mathrm{Re}\, s > 1. \\ \mathcal{L}\left[\mathrm{Ei}_{\alpha,\beta}(t)\right] &= \int_0^{\infty} e^{-st}\left[\sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)\Gamma(\alpha(k+\beta)+1)}\right]dt \\ &= \sum_{k=0}^{\infty} \frac{(k+1)!}{(k+1)\Gamma(\alpha(k+1)+\beta)}\left(\frac{1}{s}\right)^{k+2}, \quad \mathrm{Re}\, s > 1. \end{aligned} \tag{15}$$

The evaluated Laplace transforms of the integral Mittag-Leffler functions are presented in Tables 3 and 4.

Table 1. The integral Mittag-Leffler functions derived for some values of parameters α and β by using (12).

α	β	$\mathbf{Ei}_{\alpha,\beta}(x)$
$\frac{1}{3}$	$\frac{1}{5}$	$\frac{x}{\Gamma\left(\frac{8}{15}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{1}{3}\\ \frac{8}{15},\frac{4}{3}\end{matrix}\middle\vert\, x^3\right)+\frac{x^2}{2\,\Gamma\left(\frac{13}{15}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{2}{3}\\ \frac{13}{15},\frac{5}{3}\end{matrix}\middle\vert\, x^3\right)+\frac{5\,x^3}{3\,\Gamma\left(\frac{1}{5}\right)}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{6}{5},2\end{matrix}\middle\vert\, x^3\right)$
$\frac{1}{3}$	$\frac{1}{4}$	$\frac{x}{\Gamma\left(\frac{7}{12}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{1}{3}\\ \frac{7}{12},\frac{4}{3}\end{matrix}\middle\vert\, x^3\right)+\frac{x^2}{2\,\Gamma\left(\frac{11}{12}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{2}{3}\\ \frac{11}{12},\frac{5}{3}\end{matrix}\middle\vert\, x^3\right)+\frac{4x^3}{3\,\Gamma\left(\frac{1}{4}\right)}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{5}{4},2\end{matrix}\middle\vert\, x^3\right)$
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2\,x^3}{3\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{3}{2},2\end{matrix}\middle\vert\, x^3\right)+\frac{x}{\Gamma\left(\frac{5}{6}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{1}{3}\\ \frac{5}{6},\frac{4}{3}\end{matrix}\middle\vert\, x^3\right)+\frac{3\,x^2}{\Gamma\left(\frac{1}{6}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{2}{3}\\ \frac{7}{6},\frac{5}{3}\end{matrix}\middle\vert\, x^3\right)$
$\frac{1}{3}$	$\frac{3}{2}$	$\frac{x}{\Gamma\left(\frac{11}{6}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{1}{3}\\ \frac{4}{3},\frac{11}{6}\end{matrix}\middle\vert\, x^3\right)+\frac{18\,x^2}{7\,\Gamma\left(\frac{1}{6}\right)}\,{}_2F_2\left(\begin{matrix}1,\frac{2}{3}\\ \frac{5}{3},\frac{13}{6}\end{matrix}\middle\vert\, x^3\right)+\frac{4\,x^3}{9\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}1,1\\ 2,\frac{5}{2}\end{matrix}\middle\vert\, x^3\right)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{x^2}{\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{3}{2},2\end{matrix}\middle\vert\, x^2\right)+e^{x^2}F(x),\quad F(x)=e^{-x^2}\int_0^x e^{t^2}dt$
$\frac{1}{2}$	1	$-\frac{\gamma}{2}-\ln x+\frac{\mathrm{Ei}\left(x^2\right)}{2}+\frac{2\,x}{\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert\, x^2\right)$
$\frac{1}{2}$	2	$\frac{1}{2}\left(1-\gamma+\mathrm{Ei}\left(x^2\right)+\frac{1-e^{x^2}}{x^2}\right)-\ln x+\frac{4\,x}{3\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1\\ \frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert\, x^2\right)$
$\frac{1}{2}$	3	$\frac{2+4x^2+(3-2\gamma)x^4-2e^{x^2}\left(1+x^2\right)+2x^4\left(\mathrm{Ei}\left(x^2\right)-2\ln x\right)}{8x^4}+\frac{8\,x}{15\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1\\ \frac{3}{2},\frac{7}{2}\end{matrix}\middle\vert\, x^2\right)$
$\frac{1}{2}$	4	$\frac{12+18x^2\left(1+x^2\right)+(11-6\gamma)x^6-6e^{x^2}\left(2+x^2+x^4\right)+3x^6\left(2\mathrm{Ei}\left(x^2\right)-4\ln x\right)}{72x^6}+\frac{16\,x}{105\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1\\ \frac{3}{2},\frac{9}{2}\end{matrix}\middle\vert\, x^2\right)$
$\frac{1}{2}$	β	$\frac{x^2}{2\,\Gamma(\beta+1)}\,{}_2F_2\left(\begin{matrix}1,1\\ 2,\beta+1\end{matrix}\middle\vert\, x^2\right)+\frac{x}{\Gamma\left(\beta+\frac{1}{2}\right)}\,{}_2F_2\left(\begin{matrix}\frac{1}{2},1\\ \frac{3}{2},\beta+\frac{1}{2}\end{matrix}\middle\vert\, x^2\right)$
1	$\frac{1}{4}$	$\frac{x}{\Gamma\left(\frac{5}{4}\right)}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{5}{4},2\end{matrix}\middle\vert\, x\right)$
1	$\frac{1}{3}$	$\frac{x}{\Gamma\left(\frac{4}{3}\right)}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{4}{3},2\end{matrix}\middle\vert\, x\right)$
1	$\frac{1}{2}$	$\frac{2x}{\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{3}{2},2\end{matrix}\middle\vert\, x\right)$
1	1	$-\gamma-\ln x+\mathrm{Chi}(x)+\mathrm{Shi}(x)$
1	$\frac{3}{2}$	$\frac{4x}{3\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{5}{2},2\end{matrix}\middle\vert\, x\right)$
1	β	$\frac{x}{\Gamma(\beta+1)}\,{}_2F_2\left(\begin{matrix}1,1\\ 2,1+\beta\end{matrix}\middle\vert\, x\right)$

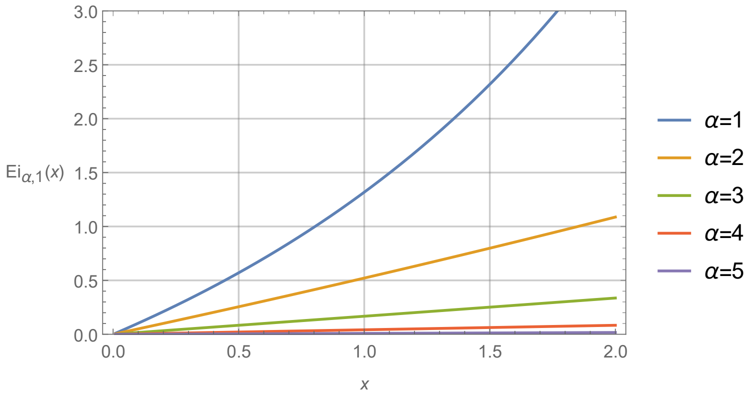

Figure 1. The integral one-parameter Mittag-Leffler function $\mathrm{Ei}_{\alpha,1}(x)$ as a function of variable x and parameters α.

Table 2. The integral Mittag-Leffler functions derived for some values of parameters α and β by using (12).

α	β	$\mathbf{Ei}_{\alpha,\beta}(x)$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{4x^2}{15\sqrt{\pi}}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{7}{6},\frac{3}{2},\frac{11}{6},2\end{matrix}\middle\vert\frac{x^2}{27}\right)+\frac{x}{48}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{2}{3},\frac{4}{3},\frac{3}{2}\end{matrix}\middle\vert\frac{x^2}{27}\right)$
$\frac{3}{2}$	1	$\frac{4x}{3\sqrt{\pi}}\,{}_2F_4\left(\begin{matrix}\frac{1}{2},1\\ \frac{5}{6},\frac{7}{6},\frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert\frac{x^2}{27}\right)+\frac{x^2}{12}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{4}{3},\frac{5}{3},2,2\end{matrix}\middle\vert\frac{x^2}{27}\right)$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{x}{2}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{4}{3},\frac{3}{2},\frac{5}{3}\end{matrix}\middle\vert\frac{x^2}{27}\right)+\frac{8x^2}{105\sqrt{\pi}}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},\frac{11}{6},2,\frac{13}{6}\end{matrix}\middle\vert\frac{x^2}{27}\right)$
$\frac{3}{2}$	2	$\frac{8x}{15\sqrt{\pi}}\,{}_2F_4\left(\begin{matrix}\frac{1}{2},1\\ \frac{7}{6},\frac{3}{2},\frac{3}{2},\frac{11}{6}\end{matrix}\middle\vert\frac{x^2}{27}\right)+\frac{x^2}{48}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{5}{3},2,2,\frac{7}{3}\end{matrix}\middle\vert\frac{x^2}{27}\right)$
2	$\frac{1}{4}$	$\frac{16x}{5\,\Gamma\left(\frac{1}{4}\right)}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{9}{8},\frac{11}{8},2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	$\frac{1}{3}$	$\frac{9x}{4\,\Gamma\left(\frac{1}{3}\right)}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{7}{6},\frac{5}{3},2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	$\frac{1}{2}$	$\frac{9x}{3\sqrt{\pi}}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{5}{4},\frac{7}{4},2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	1	$-2\gamma-\ln x+2\mathrm{Chi}\left(\sqrt{x}\right)$
2	2	$2-2\gamma-\ln x-\frac{2\sinh\sqrt{x}}{\sqrt{x}}+2\mathrm{Chi}\left(\sqrt{x}\right)$
2	3	$\frac{x}{24}\,{}_2F_3\left(\begin{matrix}1,1\\ 2,\frac{5}{2},3\end{matrix}\middle\vert\frac{x}{4}\right)$
2	4	$\frac{x}{120}\,{}_2F_3\left(\begin{matrix}1,1\\ 2,\frac{7}{2},3\end{matrix}\middle\vert\frac{x}{4}\right)$
2	β	$\frac{x}{\Gamma(\beta+2)}\,{}_2F_3\left(\begin{matrix}1,1\\ 2,\frac{\beta}{2}+1,\frac{\beta+3}{2}\end{matrix}\middle\vert\frac{x}{4}\right)$
3	1	$\frac{x}{6}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{4}{3},\frac{5}{3},2,2\end{matrix}\middle\vert\frac{x}{27}\right)$
3	β	$\frac{x}{\Gamma(\beta+3)}\,{}_2F_4\left(\begin{matrix}1,1\\ 2,\frac{\beta}{3}+1,\frac{\beta+4}{3},\frac{\beta+5}{3}\end{matrix}\middle\vert\frac{x}{27}\right)$
4	1	$\frac{x}{24}\,{}_2F_5\left(\begin{matrix}1,1\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},2,2\end{matrix}\middle\vert\frac{x}{256}\right)$
4	β	$\frac{x}{\Gamma(\beta+4)}\,{}_2F_5\left(\begin{matrix}1,1\\ 2,\frac{\beta}{4}+1,\frac{\beta+5}{4},\frac{3}{2}+\frac{\beta}{4},\frac{\beta+7}{4}\end{matrix}\middle\vert\frac{x}{256}\right)$
5	1	$\frac{x}{120}\,{}_2F_6\left(\begin{matrix}1,1\\ \frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5},2,2\end{matrix}\middle\vert\frac{x}{3125}\right)$
5	β	$\frac{x}{\Gamma(\beta+5)}\,{}_2F_6\left(\begin{matrix}1,1\\ 2,\frac{\beta}{5}+1,\frac{\beta+6}{5},\frac{\beta+7}{5},\frac{\beta+8}{5},\frac{\beta+9}{5}\end{matrix}\middle\vert\frac{x}{3125}\right)$

The Laplace transforms of the integral Mittag-Leffler functions with positive rational parameter α with $\alpha=p/q$ and p,q positive coprimes can be evaluated from:

$$
\begin{aligned}
&\mathcal{L}\left[\mathrm{Ei}_{p/q,\beta}(t)\right] \\
=\;& \frac{1}{s^2}\sum_{k=0}^{q-1}\frac{k!\,s^{-k}}{\Gamma\left(\frac{p}{q}(k+1)+\beta\right)}\,{}_{q+1}F_p\left(\begin{matrix}1,a_0,\ldots,a_{q-1}\\ b_0,\ldots,b_{p-1}\end{matrix}\middle\vert\frac{(q/s)^q}{p^p}\right).
\end{aligned}
\tag{16}
$$

where

$$\begin{aligned} a_j &= \frac{k+1+j}{q}, \\ b_j &= \frac{k}{q} + \frac{\beta+j}{p}. \end{aligned}$$

Furthermore, the following relation is satisfied:

$$\mathcal{L}\left[\mathrm{Ei}_{p/q,\beta}(t)\right] = \frac{1}{p^{p/q}s}\mathcal{L}\left[\mathrm{E}_{p/q,\beta}(t)\right]. \tag{17}$$

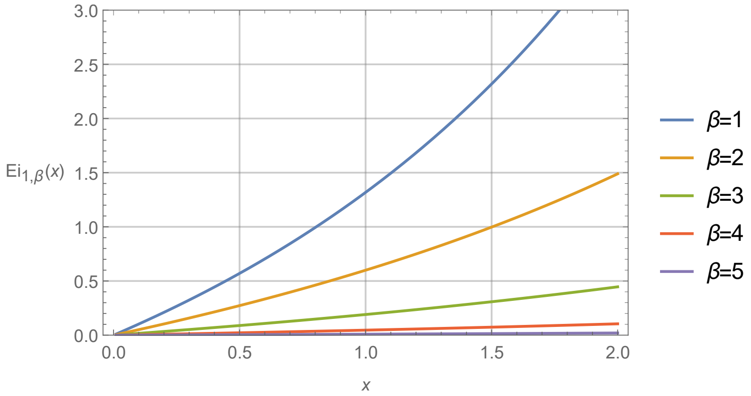

Figure 2. The integral two-parameter Mittag-Leffler function $\mathrm{Ei}_{1,\beta}(x)$ as a function of variable x and parameters β.

Table 3. The Laplace transforms of the integral Mittag-Leffler functions $\mathrm{Ei}_{\alpha,\beta}$ derived for some values of parameters α and β by using (15).

$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\mathcal{L}\left[\mathbf{Ei}_{\alpha,\beta}(t)\right]$
1	$\frac{1}{5}$	$\frac{5}{\Gamma\left(\frac{1}{5}\right)s^2}\,{}_2F_1\left(\begin{array}{c}1,1\\ \frac{6}{5}\end{array}\middle\vert\,\frac{1}{s}\right)$
1	$\frac{1}{4}$	$\frac{4}{\Gamma\left(\frac{1}{4}\right)s^2}\,{}_2F_1\left(\begin{array}{c}1,1\\ \frac{5}{4}\end{array}\middle\vert\,\frac{1}{s}\right)$
1	$\frac{1}{3}$	$\frac{3}{\Gamma\left(\frac{1}{6}\right)s^2}\,{}_2F_1\left(\begin{array}{c}1,1\\ \frac{4}{3}\end{array}\middle\vert\,\frac{1}{s}\right)$
1	$\frac{1}{2}$	$\frac{2\csc^{-1}\left(\sqrt{s}\right)}{\sqrt{\pi}s\sqrt{s-1}}$
1	1	$-\frac{1}{s}\ln\left(1-\frac{1}{s}\right)$
1	$\frac{3}{2}$	$\frac{4}{\sqrt{\pi}s}\left[1-\sqrt{s-1}\csc^{-1}\left(\sqrt{s}\right)\right]$
1	β	$\frac{1}{s^2\Gamma(\beta+1)}\,{}_2F_1\left(\begin{array}{c}1,1\\ \beta+1\end{array}\middle\vert\,\frac{1}{s}\right)$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{8}{15\sqrt{\pi}s^3}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{7}{6},\frac{11}{6}\end{array}\middle\vert\,\frac{4}{27s^2}\right)+\frac{1}{s^2}\,{}_2F_2\left(\begin{array}{c}\frac{1}{2},1\\ \frac{2}{3},\frac{4}{3}\end{array}\middle\vert\,\frac{4}{27s^2}\right)$
$\frac{3}{2}$	1	$\frac{4}{3\sqrt{\pi}s^2}\,{}_3F_3\left(\begin{array}{c}\frac{1}{2},1,1\\ \frac{5}{6},\frac{7}{6},\frac{3}{2}\end{array}\middle\vert\,\frac{4}{27s^2}\right)+\frac{1}{6s^3}\,{}_3F_3\left(\begin{array}{c}1,1,\frac{3}{2}\\ \frac{4}{3},\frac{5}{6},2\end{array}\middle\vert\,\frac{4}{27s^2}\right)$

Table 3. *Cont.*

α	β	$\mathcal{L}[\mathbf{Ei}_{\alpha,\beta}(t)]$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2s^2}\,{}_2F_2\left(\begin{array}{c}\frac{1}{2},1\\ \frac{4}{3},\frac{5}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)+\frac{16}{105\sqrt{\pi}s^3}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{11}{6},\frac{13}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)$
$\frac{3}{2}$	2	$\frac{8}{15\sqrt{\pi}s^2}\,{}_3F_3\left(\begin{array}{c}\frac{1}{2},1,1\\ \frac{7}{6},\frac{3}{2},\frac{11}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)+\frac{1}{24s^3}\,{}_3F_3\left(\begin{array}{c}1,1,\frac{3}{2}\\ \frac{5}{3},2,\frac{7}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)$
2	$\frac{1}{5}$	$\frac{25}{6\,\Gamma\left(\frac{1}{5}\right)s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{11}{10},\frac{8}{5}\end{array}\middle\vert\frac{1}{4s}\right)$
2	$\frac{1}{4}$	$\frac{16}{5\,\Gamma\left(\frac{1}{4}\right)s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{9}{8},\frac{13}{8}\end{array}\middle\vert\frac{1}{4s}\right)$
2	$\frac{1}{3}$	$\frac{9}{4\,\Gamma\left(\frac{1}{3}\right)s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{7}{6},\frac{5}{3}\end{array}\middle\vert\frac{1}{4s}\right)$
2	$\frac{1}{2}$	$\frac{4}{3\sqrt{\pi}s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{5}{4},\frac{7}{4}\end{array}\middle\vert\frac{1}{4s}\right)$
2	1	$\frac{1}{2s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{3}{2},2\end{array}\middle\vert\frac{1}{4s}\right)$
2	2	$\frac{1}{6s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{5}{2},2\end{array}\middle\vert\frac{1}{4s}\right)$
2	3	$\frac{1}{24s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{5}{2},3\end{array}\middle\vert\frac{1}{4s}\right)$
2	4	$\frac{1}{120s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{7}{2},3\end{array}\middle\vert\frac{1}{4s}\right)$
2	β	$\frac{1}{\Gamma(\beta+2)s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ 1+\frac{\beta}{2},\frac{\beta+3}{2}\end{array}\middle\vert\frac{1}{4s}\right)$

Table 4. The Laplace transforms of the integral Mittag-Leffler functions $\mathrm{Ei}_{\alpha,\beta}$ derived for some values of parameters α and β by using (15).

α	β	$\mathcal{L}[\mathbf{Ei}_{\alpha,\beta}(t)]$
3	$\frac{1}{5}$	$\frac{125}{66\,\Gamma\left(\frac{1}{5}\right)s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{16}{15},\frac{7}{5},\frac{26}{15}\end{array}\middle\vert\frac{1}{27s}\right)$
3	$\frac{1}{4}$	$\frac{64}{25\,\Gamma\left(\frac{1}{4}\right)s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{13}{12},\frac{17}{12},\frac{7}{4}\end{array}\middle\vert\frac{1}{27s}\right)$
3	$\frac{1}{3}$	$\frac{27}{28\,\Gamma\left(\frac{1}{3}\right)s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{10}{9},\frac{13}{9},\frac{10}{9}\end{array}\middle\vert\frac{1}{27s}\right)$
3	$\frac{1}{2}$	$\frac{27}{15\sqrt{\pi}s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{7}{6},\frac{3}{2},\frac{11}{6}\end{array}\middle\vert\frac{1}{27s}\right)$
3	1	$\frac{1}{6\,s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{4}{3},\frac{5}{3},2\end{array}\middle\vert\frac{1}{27s}\right)$
3	3	$\frac{1}{120\,s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ 2,\frac{7}{3},\frac{8}{3}\end{array}\middle\vert\frac{1}{27s}\right)$
3	β	$\frac{1}{\Gamma(\beta+3)s^2}\,{}_2F_3\left(\begin{array}{c}1,1\\ 1+\frac{\beta}{3},\frac{\beta+4}{3},\frac{\beta+5}{3}\end{array}\middle\vert\frac{1}{27s}\right)$

Table 4. *Cont.*

α	β	$\mathcal{L}\left[\mathbf{Ei}_{\alpha,\beta}(t)\right]$
4	1	$\frac{1}{24\,s^2}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{5}{4},\frac{3}{2},\frac{7}{4},2\end{matrix}\middle\vert\frac{1}{256\,s}\right)$
4	4	$\frac{1}{5040\,s^2}\,{}_2F_4\left(\begin{matrix}1,1\\ 2,\frac{9}{4},\frac{5}{2},\frac{11}{4}\end{matrix}\middle\vert\frac{1}{256\,s}\right)$
4	β	$\frac{1}{\Gamma(\beta+4)s^2}\,{}_2F_4\left(\begin{matrix}1,1\\ 1+\frac{\beta}{4},\frac{\beta+5}{3},\frac{3}{2}+\frac{\beta}{4},\frac{\beta+7}{4}\end{matrix}\middle\vert\frac{1}{256\,s}\right)$
5	1	$\frac{1}{120\,s^2}\,{}_2F_5\left(\begin{matrix}1,1\\ \frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}\end{matrix}\middle\vert\frac{1}{3125\,s}\right)$
5	β	$\frac{1}{\Gamma(\beta+5)s^2}\,{}_2F_5\left(\begin{matrix}1,1\\ 1+\frac{\beta}{5},\frac{\beta+6}{5},\frac{\beta+7}{5},\frac{\beta+8}{5},\frac{\beta+9}{5}\end{matrix}\middle\vert\frac{1}{3125\,s}\right)$

3. The Integral Whittaker Functions

In 1903, Whittaker [15] showed that it is possible to express some special functions such as Bessel functions, parabolic cylinder functions, error functions, incomplete gamma functions, and logarithm and cosine integrals in terms of a new function suggested by him, i.e., the Whittaker function. Two Whittaker functions are applied today, and they are defined by using the Kummer confluent hypergeometric function [3,4]:

$$\begin{aligned}\mathrm{M}_{\kappa,\mu}(x) &= x^{\mu-1/2}e^{-x/2}\,{}_1F_1\left(\begin{matrix}\mu-\kappa+\frac{1}{2}\\ 1+2\mu\end{matrix}\middle\vert x\right),\\ \mathrm{W}_{\kappa,\mu}(x) &= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2}-k-\mu\right)}\mathrm{M}_{\kappa,\mu}(x)+\frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2}-k+\mu\right)}\mathrm{M}_{\kappa,-\mu}(x).\end{aligned} \tag{18}$$

This permits us to introduce four integral Whittaker functions:

$$\begin{aligned}\mathrm{Mi}_{\kappa,\mu}(x) &= \int_0^x \frac{\mathrm{M}_{\kappa,\mu}(t)}{t}dt,\\ \mathrm{mi}_{\kappa,\mu}(x) &= \int_x^\infty \frac{\mathrm{M}_{\kappa,\mu}(t)}{t}dt.\end{aligned} \tag{19}$$

and

$$\begin{aligned}\mathrm{Wi}_{\kappa,\mu}(x) &= \int_0^x \frac{\mathrm{W}_{\kappa,\mu}(t)}{t}dt,\\ \mathrm{wi}_{\kappa,\mu}(x) &= \int_x^\infty \frac{\mathrm{W}_{\kappa,\mu}(t)}{t}dt.\end{aligned} \tag{20}$$

The integral Whittaker functions with particular values of parameters κ and μ can be expressed in terms of elementary and special functions. These cases, derived using the MATHEMATICA program, are presented in Tables 5–9. Several integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}(x)$, $\mathrm{mi}_{\kappa,\mu}(x)$, $\mathrm{Wi}_{\kappa,\mu}(x)$ and $\mathrm{wi}_{\kappa,\mu}(x)$ as a function of variable x at fixed values of parameters κ and μ are plotted in Figures 3–6. Similarly, a long list of the Whittaker functions $\mathrm{M}_{\kappa,\mu}(x)$ and $\mathrm{W}_{\kappa,\mu}(x)$ with integer and fractional parameters was prepared (see Appendix B). In some cases, it was possible to obtain for them their Laplace transforms, and they are also reported in Appendix B.

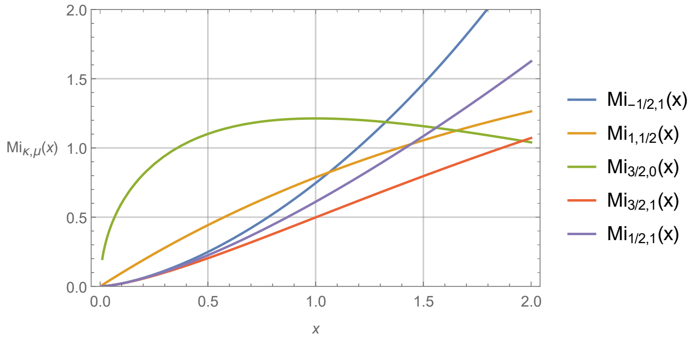

Figure 3. The integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}(x)$ as a function of variable x at fixed values of parameters κ and μ.

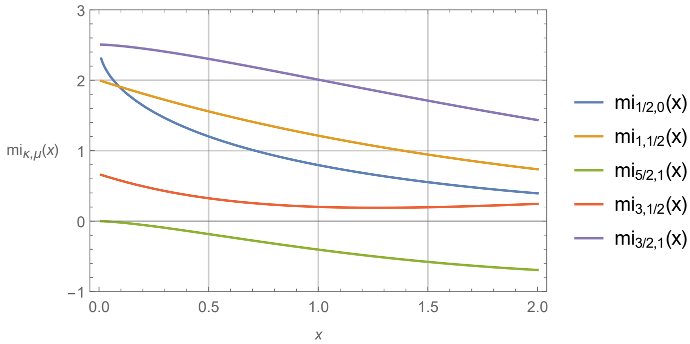

Figure 4. The integral Whittaker functions $\mathrm{mi}_{\kappa,\mu}(x)$ as a function of variable x at fixed values of parameters κ and μ.

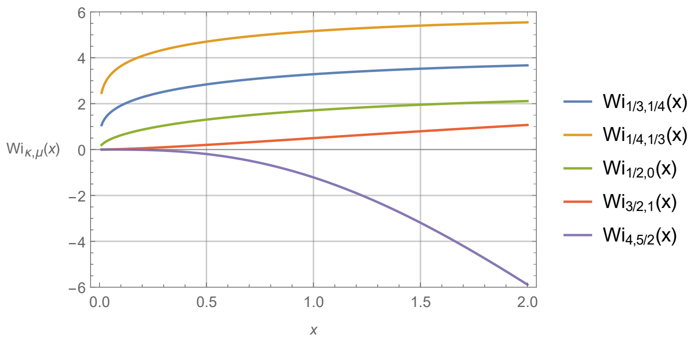

Figure 5. The integral Whittaker functions $\mathrm{Wi}_{\kappa,\mu}(x)$ as a function of variable x at fixed values of parameters κ and μ.

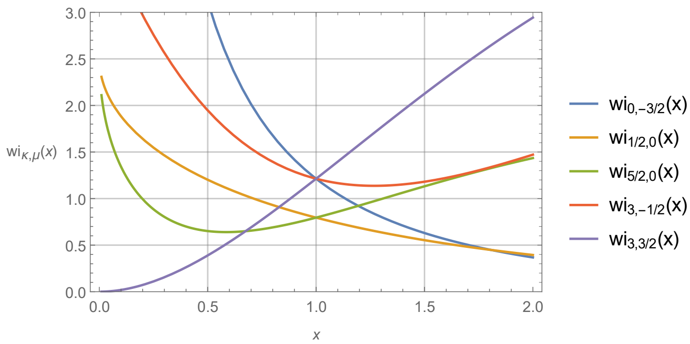

Figure 6. The integral Whittaker functions $\mathrm{wi}_{\kappa,\mu}(x)$ as a function of variable x at fixed values of parameters κ and μ.

Table 5. The integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18) and (19).

κ	μ	$\mathbf{Mi}_{\kappa,\mu}(x)$
$-\frac{5}{2}$	0	$e^{x/2}\left[\sqrt{x}(x+1)+2F\left(\sqrt{\frac{x}{2}}\right)\right], \quad F(x)=e^{-x^2}\int_0^x e^{t^2}dt$
$-\frac{3}{2}$	0	$2\sqrt{x}e^{x/2}$
$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{6}\left[\left(8x+3\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)\right)I_1\left(\frac{x}{2}\right)+\left(-\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+6x+4\right)I_0\left(\frac{x}{2}\right)-4\right]$
$-\frac{3}{2}$	1	$2e^{x/2}\left[\sqrt{x}-\sqrt{2}F\left(\sqrt{\frac{x}{2}}\right)\right]$
$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{2}{5}\left[\left(-8+8x+3\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)\right)I_1\left(\frac{x}{2}\right)+\left(-3\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x-12\right)I_0\left(\frac{x}{2}\right)+12\right]$
$-\frac{3}{2}$	2	$8\left[\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)+\frac{e^{-x/2}}{x^{-3/2}}\left(2(x-1)+e^x(x(x-4)+2)\right)\right]$
$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{16}{7}\left[20+\left(5\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+18x-44\right)I_0\left(\frac{x}{2}\right)+\frac{1}{x}\left(-5\pi x^2\,\mathbf{L}_0\left(\frac{x}{2}\right)+8x(x-9)+96\right)I_1\left(\frac{x}{2}\right)\right]$
$-\frac{3}{2}$	3	$30\left[x^{-5/2}e^{-x/2}\left(48+2e^x\left(x^3-12x^2+24x-24\right)\right)+3\sqrt{2\pi}\,\mathrm{erfi}\left(\sqrt{\frac{x}{2}}\right)\right]$
$-\frac{1}{2}$	0	$\sqrt{2\pi}\,\mathrm{erfi}\left(\sqrt{\frac{x}{2}}\right)$
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\left[-\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)I_1\left(\frac{x}{2}\right)+\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x+4\right)I_0\left(\frac{x}{2}\right)-4\right]$
$-\frac{1}{2}$	1	$8x^{-1/2}\sinh\left(\frac{x}{2}\right)-2\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)$
$-\frac{1}{2}$	$\frac{3}{2}$	$2\left(\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)+8\right)I_1\left(\frac{x}{2}\right)-2\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x-4\right)I_0\left(\frac{x}{2}\right)-8$
$-\frac{1}{2}$	2	$48x^{-3/2}e^{-x/2}\left[(x-1)e^x+1\right]-12\sqrt{2\pi}\,\mathrm{erfi}\left(\sqrt{\frac{x}{2}}\right)$
0	$\frac{1}{8}$	$\frac{8}{5}x^{5/8}\,{}_1F_2\left(\begin{matrix}\frac{5}{16}\\ \frac{9}{8},\frac{21}{16}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{7}$	$\frac{14}{9}x^{9/14}\,{}_1F_2\left(\begin{matrix}\frac{9}{28}\\ \frac{8}{7},\frac{37}{28}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{6}$	$\frac{3}{2}x^{2/3}\,{}_1F_2\left(\begin{matrix}\frac{1}{3}\\ \frac{7}{6},\frac{4}{3}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{5}$	$\frac{10}{7}x^{7/10}\,{}_1F_2\left(\begin{matrix}\frac{7}{20}\\ \frac{6}{5},\frac{27}{20}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{4}$	$\frac{4}{3}x^{3/4}\,{}_1F_2\left(\begin{matrix}\frac{3}{8}\\ \frac{5}{4},\frac{11}{8}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{3}$	$\frac{6}{5}x^{5/6}\,{}_1F_2\left(\begin{matrix}\frac{5}{12}\\ \frac{4}{3},\frac{17}{12}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{2}$	$2\,\mathrm{Shi}\left(\frac{x}{2}\right)$
0	1	$\frac{2}{3}x^{3/2}\,{}_1F_2\left(\begin{matrix}\frac{3}{4}\\ \frac{7}{4},2\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{3}{2}$	$\frac{24}{x}\sinh\left(\frac{x}{2}\right)-12$
0	2	$\frac{2}{5}x^{5/2}\,{}_1F_2\left(\begin{matrix}\frac{5}{4}\\ \frac{9}{4},3\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{5}{2}$	$60\left[x^{-2}\left(6x\cosh\left(\frac{x}{2}\right)-2\sinh\left(\frac{x}{2}\right)\right)-\mathrm{Shi}\left(\frac{x}{2}\right)\right]$

Table 6. The integral Whittaker functions $\mathrm{Mi}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18) and (19).

κ	μ	$\mathbf{Mi}_{\kappa,\mu}(x)$
$\frac{1}{2}$	0	$\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\left[-\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)I_1\left(\frac{x}{2}\right)+\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x-4\right)I_0\left(\frac{x}{2}\right)+4\right]$
$\frac{1}{2}$	1	$2\sqrt{2\pi}\,\mathrm{erfi}\left(\sqrt{\frac{x}{2}}\right)-8x^{-1/2}\sinh\left(\frac{x}{2}\right)$
$\frac{1}{2}$	$\frac{3}{2}$	$-2\left(\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)+8\right)I_1\left(\frac{x}{2}\right)+2\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x+4\right)I_0\left(\frac{x}{2}\right)+8$
$\frac{1}{2}$	$\frac{5}{2}$	$16\left[\left(\pi x\,\mathbf{L}_0\left(\frac{x}{2}\right)+8\right)I_1\left(\frac{x}{2}\right)-\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)+2x+4\right)I_0\left(\frac{x}{2}\right)+4\right]$ $+8x^2\,{}_2F_3\left(\begin{matrix}1,1\\2,2,2\end{matrix}\middle\vert\frac{x^2}{16}\right)-4x^2\,{}_2F_3\left(\begin{matrix}1,1\\2,2,3\end{matrix}\middle\vert\frac{x^2}{16}\right)$
1	0	$\frac{\sqrt{x}}{30}\left[60\,{}_1F_2\left(\begin{matrix}\frac{1}{4}\\1,\frac{5}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)+3x^2\,{}_1F_2\left(\begin{matrix}\frac{5}{4}\\2,\frac{9}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)-20x\,{}_1F_2\left(\begin{matrix}\frac{3}{4}\\1,\frac{7}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)\right]$
1	$\frac{1}{2}$	$2\left(1-e^{-x/2}\right)$
1	1	$-\frac{2x^{3/2}}{45}\left[-20\,{}_1F_2\left(\begin{matrix}\frac{3}{4}\\1,\frac{7}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)+5x^2\,{}_1F_2\left(\begin{matrix}\frac{3}{4}\\\frac{7}{4},\frac{3}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)+3x\,{}_1F_2\left(\begin{matrix}\frac{5}{4}\\2,\frac{9}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)\right]$
$\frac{3}{2}$	0	$2\sqrt{x}e^{-x/2}$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{6}\left[x\left(\pi\,\mathbf{L}_0\left(\frac{x}{2}\right)-8\right)I_1\left(\frac{x}{2}\right)+\left(\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)-6x+4\right)I_0\left(\frac{x}{2}\right)+4\right]$
$\frac{3}{2}$	1	$\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)-2\sqrt{x}e^{-x/2}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{5}\left[2\left(-3\pi\,\mathbf{L}_0\left(\frac{x}{2}\right)+8x+8\right)I_1\left(\frac{x}{2}\right)+\left(6\pi x\,\mathbf{L}_1\left(\frac{x}{2}\right)-4x+24\right)I_0\left(\frac{x}{2}\right)+24\right]$
2	$\frac{1}{2}$	$xe^{-x/2}$
2	$\frac{3}{2}$	$4-2(2+x)e^{-x/2}$
2	$\frac{5}{2}$	$-5x^{-2}\left(6(2+x)e^{x}-2x(3+x)^2-12\right)+30\,\mathrm{Shi}\left(\frac{x}{2}\right)$

Table 7. The integral Whittaker functions $\mathrm{mi}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18) and (19).

κ	μ	$\mathbf{mi}_{\kappa,\mu}(x)$
$\frac{1}{2}$	0	$\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
1	$\frac{1}{2}$	$2e^{-x/2}$
$\frac{3}{2}$	1	$2\sqrt{x}e^{-x/2}+\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
2	$\frac{1}{2}$	$-e^{-x/2}$
2	$\frac{3}{2}$	$2(2+x)e^{-x/2}$
$\frac{5}{2}$	1	$-\frac{2}{3}x^{3/2}e^{-x/2}$
$\frac{5}{2}$	2	$2\sqrt{x}(3+x)e^{-x/2}+3\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
3	$\frac{1}{2}$	$\frac{1}{3}[2+(x-2)x]e^{-x/2}$
3	$\frac{3}{2}$	$-\frac{1}{2}x^2e^{-x/2}$
4	$\frac{1}{2}$	$-\frac{x}{12}[12+(x-6)x]e^{-x/2}$
4	$\frac{3}{2}$	$\frac{1}{10}\left[8+(x-2)^2x\right]e^{-x/2}$

Table 8. The integral Whittaker functions $\mathrm{Wi}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18) and (20).

κ	μ	$\mathbf{Wi}_{\kappa,\mu}(x)$
$-\frac{3}{2}$	0	$-2\sqrt{x}e^{-x/2}\mathrm{Ei}(-x)+\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
$\frac{1}{4}$	$\frac{1}{4}$	$2^{1/4}\gamma\left(\frac{1}{4},\frac{x}{2}\right)$
$\frac{1}{2}$	0	$\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)$
1	$-\frac{1}{2}$	$2\left(1-e^{-x/2}\right)$
1	$\frac{1}{2}$	$2\left(1-e^{-x/2}\right)$
$\frac{3}{2}$	0	$-2\sqrt{x}e^{-x/2}$
$\frac{3}{2}$	1	$\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)-2\sqrt{x}e^{-x/2}$
2	$-\frac{1}{2}$	$-2xe^{-x/2}$
2	$\frac{1}{2}$	$-2xe^{-x/2}$
2	$\frac{3}{2}$	$4-2(2+x)e^{-x/2}$
$\frac{5}{2}$	0	$\sqrt{2\pi}\,\mathrm{erf}\left(\sqrt{\frac{x}{2}}\right)-2\sqrt{x}(x-1)e^{-x/2}$
3	$\frac{3}{2}$	$-2x^2e^{-x/2}$
3	$\frac{5}{2}$	$16-2[8+(4+x)x]e^{-x/2}$
4	$-\frac{1}{2}$	$-2x[12+(x-6)x]e^{-x/2}$
4	$\frac{1}{2}$	$-2x[12+(x-6)x]e^{-x/2}$
4	$\frac{3}{2}$	$16-2\left[8+x(x-2)^2\right]e^{-x/2}$
4	$\frac{5}{2}$	$-2x^3e^{-x/2}$

Table 9. The integral Whittaker functions $\mathrm{wi}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18) and (20).

κ	μ	$\mathbf{wi}_{\kappa,\mu}(x)$
$-\frac{1}{2}$	1	$2x^{-1/2}e^{-x/2}-\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
$-\frac{1}{2}$	2	$2x^{-3/2}e^{-x/2}$
$-\frac{1}{2}$	3	$\frac{1}{3}\left[-2(x-6)(x+2)x^{-5/2}e^{-x/2}+\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)\right]$
0	$-\frac{5}{2}$	$\frac{1}{2}\mathrm{Ei}\left(-\frac{x}{2}\right)+3x^{-2}(x+2)e^{-x/2}$
0	$-\frac{3}{2}$	$2x^{-1}e^{-x/2}$
0	$-\frac{1}{2}$	$-\mathrm{Ei}\left(-\frac{x}{2}\right)$
$\frac{1}{2}$	0	$\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)$
1	$-\frac{1}{2}$	$2e^{-x/2}$
1	$\frac{1}{2}$	$2e^{-x/2}$
1	$\frac{5}{2}$	$2x^{-2}[6+x(6+x)]e^{-x/2}$
$\frac{3}{2}$	0	$2\sqrt{x}e^{-x/2}$

Table 9. *Cont.*

κ	μ	$\mathbf{wi}_{\kappa,\mu}(x)$
$\frac{3}{2}$	1	$\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)+2\sqrt{x}e^{-x/2}$
2	$\frac{1}{2}$	$2xe^{-x/2}$
2	$\frac{3}{2}$	$2(2+x)e^{-x/2}$
$\frac{5}{2}$	0	$\sqrt{2\pi}\,\mathrm{erfc}\left(\sqrt{\frac{x}{2}}\right)+2\sqrt{x}(x-1)e^{-x/2}$
$\frac{5}{2}$	1	$2x^{3/2}e^{-x/2}$
3	$-\frac{1}{2}$	$2[2+x(x-2)]e^{-x/2}$
3	$\frac{1}{2}$	$2[2+x(x-2)]e^{-x/2}$
3	$\frac{3}{2}$	$2x^2e^{-x/2}$
3	$\frac{5}{2}$	$2[8+x(4+x)]e^{-x/2}$
4	$-\frac{1}{2}$	$2x[12+(x-6)x]e^{-x/2}$
4	$\frac{1}{2}$	$2x[12+(x-6)x]e^{-x/2}$
4	$\frac{3}{2}$	$2\left[8+(x-2)^2x\right]e^{-x/2}$

There is a number of recurrence relations between the Whittaker functions, for example [3,4]

$$\begin{aligned} 2\mu\left[\mathrm{M}_{\kappa-1/2,\mu-1/2}(t)-\mathrm{M}_{\kappa+1/2,\mu-1/2}(t)\right] &= t^{1/2}\mathrm{M}_{\kappa,\mu}(t), \\ (\kappa+\mu)\mathrm{W}_{\kappa-1/2,\mu}(t)+\mathrm{W}_{\kappa+1/2,\mu}(t) &= t^{1/2}\mathrm{W}_{\kappa,\mu+1/2}(t), \end{aligned} \tag{21}$$

and this leads to integrals that are expressed in terms of the integral Whittaker functions

$$\begin{aligned} \int_0^x \frac{\mathrm{M}_{\kappa,\mu}(t)}{t^{1/2}}dt &= 2\mu\left[\mathrm{Mi}_{\kappa-1/2,\mu-1/2}(t)-\mathrm{Mi}_{\kappa+1/2,\mu-1/2}(t)\right], \\ \int_0^x \frac{\mathrm{W}_{\kappa,\mu+1/2}(t)}{t^{1/2}}dt &= (\kappa+\mu)\mathrm{Wi}_{\kappa-1/2,\mu}(t)+\mathrm{Wi}_{\kappa+1/2,\mu}(t). \end{aligned} \tag{22}$$

Using the following representation of the Whittaker functions [5]

$$\mathrm{M}_{\kappa,\mu}(t) = t^{\mu+1/2}\sum_{n=0}^{\infty} {}_2F_1\left(\left.\begin{matrix} -n,\mu-\kappa+\frac{1}{2} \\ 1+2\mu \end{matrix}\right| 2\right)\frac{(-t/2)^n}{n!}, \tag{23}$$

it is possible to obtain the integral Whittaker functions in terms of a rapidly convergent alternating series as follows:

$$\mathrm{Mi}_{\kappa,\mu}(x) = x^{\mu+1/2}\sum_{n=0}^{\infty} {}_2F_1\left(\left.\begin{matrix} -n,\mu-\kappa+\frac{1}{2} \\ 1+2\mu \end{matrix}\right| 2\right)\frac{(-x/2)^n}{n!\left(\frac{1}{2}+\mu+n\right)}. \tag{24}$$

There is a number of particular cases where the integral Whittaker functions can be written in a closed-form, for example, from [5]

$$\mathrm{M}_{\kappa,\kappa-1/2}(x) = x^{\kappa}e^{-x/2}, \tag{25}$$

we have

$$\mathrm{Mi}_{\kappa,\kappa-1/2}(x) = 2^{\kappa}\,\gamma\left(\kappa,\frac{x}{2}\right), \tag{26}$$

but [5]

$$\mathrm{M}_{\kappa,\kappa-1/2}(x) = \mathrm{W}_{\kappa,\kappa-1/2}(x) = \mathrm{W}_{\kappa,-\kappa+1/2}(x) = e^{-x/2}x^{\kappa}, \tag{27}$$

and therefore

$$\mathrm{Mi}_{\kappa,\kappa-1/2}(x) = \mathrm{Wi}_{\kappa,\kappa-1/2}(x) = \mathrm{Wi}_{\kappa,-\kappa+1/2}(x) = 2^{\kappa}\,\gamma\left(\kappa, \frac{x}{2}\right). \tag{28}$$

Furthermore, from [5]

$$\mathrm{M}_{0,\mu}(x) = 2^{2\mu+1/2}\Gamma(\mu+1)\sqrt{\frac{t}{2}}\, I_{\mu}\left(\frac{x}{2}\right), \tag{29}$$

follows that

$$\mathrm{Mi}_{0,\mu}(x) = \frac{x^{\mu+1/2}}{\mu+1/2}\,{}_1F_2\left(\begin{array}{c} \frac{2\mu+1}{4} \\ \mu+1, \frac{2\mu+5}{4} \end{array}\middle| \frac{x^2}{16}\right). \tag{30}$$

Similary from

$$\mathrm{W}_{0,\mu}(x) = \sqrt{\frac{x}{\pi}} K_{\mu}\left(\frac{x}{2}\right), \tag{31}$$

we have

$$\mathrm{Wi}_{0,\mu}(x) = \frac{\sqrt{\pi}}{2\sin \pi\mu}\left[\frac{4^{\mu}\,\mathrm{Mi}_{0,-\mu}(x)}{\Gamma(1-\mu)} - \frac{4^{-\mu}\,\mathrm{Mi}_{0,\mu}(x)}{\Gamma(1+\mu)}\right], \tag{32}$$

and in a general case

$$\mathrm{Wi}_{\kappa,\mu}(x) = \frac{\Gamma(-2\mu)\mathrm{Mi}_{\kappa,\mu}(x)}{\Gamma\left(\frac{1}{2}-\kappa-\mu\right)} + \frac{\Gamma(2\mu)\mathrm{Mi}_{\kappa,-\mu}(x)}{\Gamma\left(\frac{1}{2}-\kappa+\mu\right)}. \tag{33}$$

For $\kappa = \pm 1/2$, it is possible to obtain

$$\mathrm{Wi}_{\pm\frac{1}{2},\mu}(x) = \mathrm{F}^{\pm}_{\mu}(x) + \mathrm{F}^{\pm}_{-\mu}(x), \tag{34}$$

where we have set

$$\begin{aligned} \mathrm{F}^{\pm}_{\mu}(x) &= \frac{2x^{1/2+\mu}\Gamma(-2\mu)}{(1+2\mu)\Gamma\left(\frac{1}{2}\mp\frac{1}{2}-\mu\right)} \\ &\quad \left[{}_1F_2\left(\begin{array}{c} \frac{1}{4}+\frac{\mu}{2} \\ \frac{1}{2}+\mu, \frac{3}{4}+\frac{\mu}{2} \end{array}\middle| \frac{x^2}{16}\right) \mp \frac{x/2}{3+2\mu}\,{}_1F_2\left(\begin{array}{c} \frac{3}{4}+\frac{\mu}{2} \\ \frac{3}{2}+\mu, \frac{7}{4}+\frac{\mu}{2} \end{array}\middle| \frac{x^2}{16}\right)\right]. \end{aligned} \tag{35}$$

Since [16]

$$\begin{aligned} &{}_2F_1\left(\begin{array}{c} -n, \lambda \\ 2\lambda+1 \end{array}\middle| 2\right) \\ &= \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\sqrt{\pi}}\left[\left(\frac{1+(-1)^n}{2}\right)\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\lambda+\frac{n+1}{2}\right)} + \left(\frac{1-(-1)^n}{2}\right)\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\lambda+\frac{n}{2}+1\right)}\right], \end{aligned} \tag{36}$$

and

$$\begin{aligned} &{}_2F_1\left(\begin{array}{c} -n, \lambda \\ 2\lambda-1 \end{array}\middle| 2\right) \\ &= \frac{\Gamma\left(\lambda-\frac{1}{2}\right)}{\sqrt{\pi}}\left[\left(\frac{1+(-1)^n}{2}\right)\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\lambda+\frac{n-1}{2}\right)} - \left(\frac{1-(-1)^n}{2}\right)\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\lambda+\frac{n}{2}\right)}\right], \end{aligned} \tag{37}$$

by introducing $\lambda = \mu$ and $\lambda = \mu + 1$, after some steps, it leads to

$$\begin{aligned} &\mathrm{Mi}_{\pm\frac{1}{2},\mu}(x) \\ &= \frac{x^{\mu+1/2}}{\mu+1/2}\left[{}_1F_2\left(\begin{array}{c} \frac{\mu}{2}+\frac{1}{4} \\ \mu+\frac{1}{2}, \frac{\mu}{2}+\frac{3}{4} \end{array}\middle| \frac{x^2}{16}\right) \mp \frac{x/2}{2\mu+3}\,{}_1F_2\left(\begin{array}{c} \frac{\mu}{2}+\frac{3}{4} \\ \mu+\frac{3}{2}, \frac{\mu}{2}+\frac{7}{4} \end{array}\middle| \frac{x^2}{16}\right)\right]. \end{aligned} \tag{38}$$

4. The Integral Wright Functions

In 1933 [17] and in 1940 [18], Wright introduced new special functions that were considered as a kind of generalization of the Bessel functions. However, today they play a significant independent role in mathematics and in solutions of physical problems by modeling space diffusion, stochastic processes, probability distributions and other diverse natural phenomena [19,20]. The Wright functions are defined by the following series

$$\mathrm{W}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!\,\Gamma(\alpha k+\beta)}. \tag{39}$$

If the parameter α is a positive real number, they are called the Wright functions of the first kind, and when $-1 < \alpha < 0$, the Wright functions of the second kind.

Furthermore, consider the following functions:

$$\begin{aligned} &\mathrm{F}_\alpha(x) = \mathrm{W}_{-\alpha,0}(-x), \quad 0 < \alpha < 1, \\ &\mathrm{M}_\alpha(x) = \mathrm{W}_{-\alpha,1-\alpha}(-x), \quad 0 < \alpha < 1, \\ &\mathrm{F}_\alpha(x) = \alpha\, x\, M_\alpha(x). \end{aligned} \tag{40}$$

These functions with negative arguments x and with particular values of parameters are frequently named as the Mainardi functions and are denoted as $\mathrm{F}_\alpha(x)$ and $\mathrm{M}_\alpha(x)$ [19,20].

Their explicit form is

$$\begin{aligned} \mathrm{F}_\alpha(x) &= \sum_{k=1}^{\infty} \frac{(-x)^k}{k!\Gamma(-\alpha k)} \\ &= -\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-x)^k}{k!}\Gamma(\alpha k+1)\sin(\pi\alpha k), \\ \mathrm{M}_\alpha(x) &= \sum_{k=0}^{\infty} \frac{(-x)^k}{k!\Gamma(-\alpha(k+1)+1)} \\ &= \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\Gamma(\alpha(k+1))\sin(\pi\alpha(k+1)). \end{aligned} \tag{41}$$

For positive rational $\alpha = p/q$, where p, q are positive coprimes, we have obtained reduction formulas for $\mathrm{F}_{p/q}(x)$ and $\mathrm{M}_{p/q}(x)$ in Appendix C. Furthermore, by applying the MATHEMATICA program to sums of infinite series in (39), it is possible to obtain the Wright functions of the first and second kinds for particular values of parameters α and β in an explicit form (Appendix C). The Laplace transforms of these functions are expressed in terms of the Mittag-Leffler functions, so they are omitted here [19–21].

The two-parameter $\mathrm{E}_{\alpha,\beta}(t)$ Mittag-Leffler functions defined in (9) differ only by the absence of factorials from the Wright functions and, therefore, the form of series in (39) leads to the integral Wright function, which is similar to that introduced in (11) and (12).

$$\mathrm{Wi}_{\alpha,\beta}(x) = \int_0^x \frac{\mathrm{W}_{\alpha,\beta}(t) - 1/\Gamma(\beta)}{t}\,dt. \tag{42}$$

Unfortunately, the notation is the same as the integral Whittaker functions. In an explicit form from (39), we have

$$\mathrm{Wi}_{\alpha,\beta}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k\,k!\,\Gamma(\alpha k+\beta)}. \tag{43}$$

For p and q positive coprimes, applying (A1) and (A2), the corresponding expression to (14) is

$$\begin{aligned} &\mathrm{Wi}_{p/q,\beta}(x) \\ = &\sum_{k=1}^{q-1} \frac{x^k}{k\,k!\Gamma\left(\frac{p}{q}k+\beta\right)}\,{}_2F_{p+q}\left(\begin{array}{c} 1, k/q \\ b_0,\ldots,b_{p-1},c_0,\ldots,c_{q-1} \end{array}\middle|\frac{x^q}{p^p q^q}\right), \\ &b_j = \frac{k}{q}+\frac{\beta+j}{p}, \quad c_j = \frac{k+1+j}{q}. \end{aligned} \tag{44}$$

In the case of the Mainardi functions, we have

$$\begin{aligned} \mathrm{Fi}_{p/q}(x) &= -\frac{1}{\pi}\sum_{k=1}^{q} \frac{(-x)^k}{k\,k!}\Gamma\left(\frac{p}{q}k+1\right)\sin\left(\pi\frac{p}{q}k\right)S_k(x), \\ \mathrm{Mi}_{p/q}(x) &= \frac{1}{\pi}\sum_{k=1}^{q} \frac{(-x)^k}{k\,k!}\sin\left(\pi\frac{p}{q}(k+1)\right)\Gamma\left(\frac{p}{q}(k+1)\right)S_k(x), \end{aligned} \tag{45}$$

where

$$\begin{aligned} S_k(x) &= {}_{p+2}F_{q+1}\left(\begin{array}{c} 1, \frac{k}{q}, a_0,\ldots,a_{p-1} \\ \frac{k}{q}+1, b_0,\ldots,b_{q-1} \end{array}\middle|\frac{(-1)^{p+q}x^q p^p}{q^q}\right), \\ a_j &= \frac{k}{q}+\frac{j+1}{p}, \quad b_j = \frac{k+1+j}{q}. \end{aligned} \tag{46}$$

In Tables 10 and 11, the integral Wright functions derived with the help of MATHEMATICA program for some values of parameters α and β are derived. There are many other expressions for these functions, which are available using this program, but being long and complex, they were omitted. The integral Mainardi fuctions $\mathrm{Fi}_\alpha(x)$ and $\mathrm{Mi}_\alpha(x)$ for $0<\alpha<1$, are presented in Tables 12 and 13. As can be expected, most of these integral functions are expressed in terms of generalized hypergeometric functions.

Table 10. The integral Wright functions $\mathrm{Wi}_{\alpha,\beta}$ derived for some values of parameters α and β by using (43).

α	β	$\mathbf{Wi}_{\alpha,\beta}(x)$
-1	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}}\left[\ln 4 - 2\ln\left(\sqrt{1+x}+1\right)\right]$
-1	$\frac{3}{2}$	$\frac{x}{\sqrt{\pi}}\,{}_3F_2\left(\begin{array}{c}\frac{1}{2},1,1\\2,2\end{array}\middle|-x\right)$
-1	β	$\frac{x}{\Gamma(\beta-1)}\,{}_3F_2\left(\begin{array}{c}1,1,2-\beta\\2,2\end{array}\middle|-x\right)$
0	$-\frac{4}{3}$	$\frac{-\gamma-\ln x+\mathrm{Chi}(x)+\mathrm{Shi}(x)}{\Gamma(-4/3)}$
0	β	$\frac{-\gamma-\ln x+\mathrm{Chi}(x)+\mathrm{Shi}(x)}{\Gamma(\beta)}$
$\frac{1}{2}$	0	$-\frac{1}{2}+\frac{1}{2}\,{}_0F_2\left(\begin{array}{c}-\\\frac{1}{2},1\end{array}\middle|\frac{x^2}{4}\right)+\frac{x}{\sqrt{\pi}}\,{}_0F_2\left(\begin{array}{c}-\\\frac{1}{2},\frac{3}{2}\end{array}\middle|\frac{x^2}{4}\right)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{x^2}{2\sqrt{\pi}}\,{}_2F_4\left(\begin{array}{c}1,1\\\frac{3}{2},\frac{3}{2},2,2\end{array}\middle|\frac{x^2}{4}\right)+x\,{}_1F_3\left(\begin{array}{c}\frac{1}{2}\\1,\frac{3}{2},\frac{3}{2}\end{array}\middle|\frac{x^2}{4}\right)$

Table 10. *Cont.*

α	β	$\mathbf{Wi}_{\alpha,\beta}(x)$
$\frac{1}{2}$	1	$\frac{x}{4}\left[\frac{8}{\sqrt{\pi}}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert\frac{x^2}{4}\right)+x\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},\frac{3}{2},2,2\end{matrix}\middle\vert\frac{x^2}{4}\right)\right]$
$\frac{1}{2}$	2	$\frac{x^2}{8}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},2,2,3\end{matrix}\middle\vert\frac{x^2}{4}\right)+\frac{4x}{3\sqrt{\pi}}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\frac{5}{2}\end{matrix}\middle\vert\frac{x^2}{4}\right)$
$\frac{1}{2}$	β	$\frac{x^2}{4\,\Gamma(1+\beta)}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},2,2,\beta+1\end{matrix}\middle\vert\frac{x^2}{4}\right)+\frac{x}{\Gamma(\frac{1}{2}+\beta)}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{3}{2},\frac{3}{2},\beta+\frac{1}{2}\end{matrix}\middle\vert\frac{x^2}{4}\right)$
1	$-\frac{3}{2}$	$-\frac{x}{2\sqrt{\pi}}\,{}_2F_3\left(\begin{matrix}1,1\\ -\frac{1}{2},2,2\end{matrix}\middle\vert x\right)$
3	$\frac{3}{2}$	$\frac{x}{\sqrt{\pi}}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{1}{2},2,2\end{matrix}\middle\vert x\right)$
1	0	$-1+I_0(2\sqrt{x})$
1	$\frac{1}{4}$	$\frac{x}{\Gamma(5/4)}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{5}{2},2,2\end{matrix}\middle\vert x\right)$
1	$\frac{1}{2}$	$-\frac{2\gamma+\ln 4+\ln x-2\,\mathrm{Chi}(2\sqrt{x})}{\sqrt{\pi}}$
1	1	$x\,{}_2F_3\left(\begin{matrix}1,1\\ 2,2,2\end{matrix}\middle\vert x\right)$
1	$\frac{3}{2}$	$-\frac{1}{\sqrt{\pi x}}\left[2\sinh(2\sqrt{x})-2\sqrt{x}(2\gamma-2+\ln 4-2\,\mathrm{Chi}(2\sqrt{x}))\right]$
1	β	$x\,{}_2F_3\left(\begin{matrix}1,1\\ 2,2,\beta+1\end{matrix}\middle\vert x\right)$

Table 11. The integral Wright functions $\mathrm{Wi}_{\alpha,\beta}$ derived for some values of parameters α and β by using (43).

α	β	$\mathbf{Wi}_{\alpha,\beta}(x)$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{2x^2}{15\sqrt{\pi}}\,{}_2F_6\left(\begin{matrix}1,1\\ \frac{7}{2},\frac{3}{2},\frac{3}{2},\frac{11}{6},2,2\end{matrix}\middle\vert\frac{x^2}{108}\right)+x\,{}_1F_5\left(\begin{matrix}\frac{1}{2}\\ \frac{2}{3},1,\frac{4}{3},\frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert\frac{x^2}{108}\right)$
2	$\frac{1}{4}$	$\frac{16\,x}{5\,\Gamma(1/4)}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{9}{8},\frac{13}{8},2,2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	$\frac{1}{3}$	$\frac{9\,x}{4\,\Gamma(1/3)}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{7}{6},\frac{5}{3},2,2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	$\frac{1}{2}$	$\frac{4\,x}{3\sqrt{\pi}}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{5}{4},\frac{7}{4},2,2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	1	$\frac{x}{2}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},2,2,2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	2	$\frac{x}{6}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{5}{2},2,2,2\end{matrix}\middle\vert\frac{x}{4}\right)$
2	β	$\frac{x}{\Gamma(\beta+2)}\,{}_2F_4\left(\begin{matrix}1,1\\ 2,2,\frac{\beta}{2}+1,\frac{\beta+3}{2}\end{matrix}\middle\vert\frac{x}{4}\right)$
3	1	$\frac{x}{6}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{4}{3},\frac{5}{3},2,2\end{matrix}\middle\vert\frac{x}{27}\right)$
3	β	$\frac{x}{\Gamma(\beta+3)}\,{}_2F_5\left(\begin{matrix}1,1\\ 2,2,\frac{\beta}{3}+1,\frac{\beta+4}{3},\frac{\beta+5}{3}\end{matrix}\middle\vert\frac{x}{27}\right)$
4	β	$\frac{x}{\Gamma(\beta+4)}\,{}_2F_6\left(\begin{matrix}1,1\\ 2,2,\frac{\beta}{4}+1,\frac{\beta+5}{4},\frac{\beta+6}{4},\frac{\beta+7}{4}\end{matrix}\middle\vert\frac{x}{256}\right)$
5	β	$\frac{x}{\Gamma(\beta+5)}\,{}_2F_7\left(\begin{matrix}1,1\\ 2,2,\frac{\beta}{4}+1,\frac{\beta+6}{5},\frac{\beta+7}{5},\frac{\beta+8}{5},\frac{\beta+9}{5}\end{matrix}\middle\vert\frac{x}{3125}\right)$

Table 12. The integral Mainardi function Fi_α derived for some values of parameter α by using (45).

α	$\mathrm{Fi}_\alpha(x)$
$\frac{3}{4}$	$-x\left[\frac{1}{\Gamma\left(-\frac{3}{4}\right)}\,{}_3F_3\left(\begin{matrix}\frac{1}{4},\frac{7}{12},\frac{11}{12}\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{matrix}\middle\vert -\frac{27x^4}{256}\right)+\frac{x}{144}\left(\frac{8x}{\Gamma\left(-\frac{9}{4}\right)}\,{}_3F_3\left(\begin{matrix}\frac{3}{4},\frac{13}{12},\frac{17}{12}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{matrix}\middle\vert -\frac{27x^4}{256}\right)-\frac{27}{\sqrt{\pi}}\,{}_3F_4\left(\begin{matrix}\frac{1}{2},\frac{5}{6},\frac{7}{6}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{matrix}\middle\vert -\frac{27x^4}{256}\right)\right)\right]$
$\frac{2}{3}$	$\frac{x}{4}\left[\frac{x}{\Gamma\left(-\frac{4}{3}\right)}\,{}_2F_2\left(\begin{matrix}\frac{2}{3},\frac{7}{6}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\middle\vert -\frac{4x^3}{27}\right)-\frac{4}{\Gamma\left(-\frac{2}{3}\right)}\,{}_2F_2\left(\begin{matrix}\frac{1}{3},\frac{5}{6}\\ \frac{2}{3},\frac{4}{3}\end{matrix}\middle\vert -\frac{4x^3}{27}\right)\right]$
$\frac{1}{2}$	$\frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2}\right)$
$\frac{1}{3}$	$\frac{x}{4}\left[\frac{x}{\Gamma\left(-\frac{2}{3}\right)}\,{}_1F_2\left(\begin{matrix}\frac{2}{3}\\ \frac{4}{3},\frac{5}{3}\end{matrix}\middle\vert \frac{x^3}{27}\right)-\frac{4}{\Gamma\left(-\frac{1}{3}\right)}\,{}_1F_2\left(\begin{matrix}\frac{1}{3}\\ \frac{2}{3},\frac{4}{3}\end{matrix}\middle\vert \frac{x^3}{27}\right)\right]$
$\frac{1}{4}$	$-x\left[\frac{1}{\Gamma\left(-\frac{1}{4}\right)}\,{}_1F_3\left(\begin{matrix}\frac{1}{4}\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{matrix}\middle\vert -\frac{x^4}{256}\right)+\frac{x}{72}\left(\frac{9}{\sqrt{\pi}}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{matrix}\middle\vert -\frac{x^4}{256}\right)+\frac{4x}{\Gamma\left(-\frac{3}{4}\right)}\,{}_3F_4\left(\begin{matrix}\frac{3}{4}\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{matrix}\middle\vert -\frac{x^4}{256}\right)\right)\right]$

Table 13. The integral Mainardi function Mi_α derived for some values of parameter α by using (45).

α	$\mathrm{Mi}_\alpha(x)$
$\frac{3}{4}$	$\frac{x}{96}\left[\frac{48}{\sqrt{\pi}}\,{}_3F_3\left(\begin{matrix}\frac{1}{4},\frac{5}{6},\frac{7}{6}\\ \frac{3}{4},\frac{5}{4},\frac{5}{4}\end{matrix}\middle\vert -\frac{27x^4}{256}\right)+\frac{24x}{\Gamma\left(-\frac{5}{4}\right)}\,{}_3F_3\left(\begin{matrix}\frac{1}{2},\frac{13}{12},\frac{17}{12}\\ \frac{5}{4},\frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert -\frac{27x^4}{256}\right)+\frac{x^3}{\Gamma\left(-\frac{11}{4}\right)}\,{}_4F_4\left(\begin{matrix}1,1,\frac{19}{12},\frac{23}{12}\\ \frac{3}{2},\frac{7}{4},2,2\end{matrix}\middle\vert -\frac{27x^4}{256}\right)\right]$
$\frac{2}{3}$	$-\frac{x^3}{18\,\Gamma\left(-\frac{5}{3}\right)}\,{}_3F_3\left(\begin{matrix}1,1,\frac{11}{6}\\ \frac{5}{3},2,2\end{matrix}\middle\vert -\frac{4x^3}{27}\right)-\frac{x}{\Gamma\left(-\frac{1}{3}\right)}\,{}_2F_2\left(\begin{matrix}\frac{1}{3},\frac{7}{6}\\ \frac{4}{3},\frac{4}{3}\end{matrix}\middle\vert -\frac{4x^3}{27}\right)$
$\frac{1}{2}$	$\frac{1}{2\sqrt{\pi}}\left[\mathrm{Chi}\left(\frac{x^2}{4}\right)-\mathrm{Shi}\left(\frac{x^2}{4}\right)-\ln\left(\frac{x^2}{4}\right)-\gamma\right]$
$\frac{1}{3}$	$-\frac{x^3}{18\,\Gamma\left(-\frac{1}{3}\right)}\,{}_2F_3\left(\begin{matrix}1,1\\ \frac{5}{3},2,2\end{matrix}\middle\vert \frac{x^3}{27}\right)-\frac{x}{\Gamma\left(\frac{1}{3}\right)}\,{}_1F_2\left(\begin{matrix}\frac{1}{3}\\ \frac{4}{3},\frac{4}{3}\end{matrix}\middle\vert \frac{x^3}{27}\right)$
$\frac{1}{4}$	$-\frac{x}{\sqrt{\pi}}\,{}_1F_3\left(\begin{matrix}\frac{1}{4}\\ \frac{3}{4},\frac{5}{4},\frac{5}{4}\end{matrix}\middle\vert -\frac{x^4}{256}\right)+\frac{x^2}{4\,\Gamma\left(\frac{1}{4}\right)}\,{}_1F_3\left(\begin{matrix}\frac{1}{2}\\ \frac{5}{4},\frac{3}{2},\frac{3}{2}\end{matrix}\middle\vert -\frac{x^4}{256}\right)+\frac{x^4}{96\,\Gamma\left(-\frac{1}{4}\right)}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{3}{2},\frac{7}{4},2,2\end{matrix}\middle\vert -\frac{x^4}{256}\right)\Big]$

5. Conclusions

For the first time, three new special functions are presented in this investigation: the integral Mittag-Leffler functions, the integral Whittaker functions, and the integral Wright functions. These functions are defined in the mathematical literature in the same manner as other elementary and special integral functions. It is feasible to generate these functions in an explicit form for certain parameters values using the MATHEMATICA application. These integral functions are often represented in terms of generalized hypergeometric functions. The behavior of some of them is shown graphically. In the Appendices, a large number of Mittag-Leffler, Whittaker, and Wright functions with integral and fractional parameters, as well as their Laplace transforms, are presented in tabular form.

It may be observed that, generally, it is highly possible to make general integral functions such as (19) and (20) by using generalized hypergeometric ${}_pF_p(t)$, because they converge in the whole complex t-plane, or, for every real number t.

Author Contributions: Conceptualization, A.A. and J.L.G.-S.; Methodology, A.A. and J.L.G.-S.; Resources, A.A.; Writing—original draft, A.A. and J.L.G.-S.; Writing—review & editing, A.A. and J.L.G.-S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We are grateful to Armando Consiglio from Institut für Theoretische Physik und Astrophysik of University of Würzburg, Germany, who performed numerical evaluations of the integral functions presented in Figures 1 and 2, and to Francesco Mainardi from the Department of Physics and Astronomy, University of Bologna, Bologna, Italy, for his kind encouragement and interest in our work.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Representations of the One- and Two-Parameter Mittag-Leffler Functions and Their Laplace Transforms

The Mittag-Leffler functions are defined by the sums of infinite series presented in (9) and their Laplace transforms in (10). For positive variable x and some values of parameters α and β, these sums can be expressed in terms of elementary and special functions, especially in terms of generalized hypergeometric functions. They were derived by using the MATHEMATICA program and presented in Tables A1 and A2 for the Mittag-Leffler functions, as well as Tables A3 and A4 for the Laplace transforms. These results, given in terms of infinite series, are mostly new, and only they are only partly known in the mathematical literature. Knowing that any infinite sum can be split as

$$\sum_{k=0}^{\infty} a(k) = \sum_{j=0}^{q-1} \sum_{k=0}^{\infty} a(qk+j), \tag{A1}$$

and applying the multiplication formula of the gamma function ([5] (Eqn. 5.5.6)), for $nt \neq 0, -1, -2, \ldots$

$$\Gamma(nt) = (2\pi)^{(1-n)/2} n^{nt-1/2} \prod_{j=0}^{n-1} \Gamma\left(t + \frac{j}{n}\right), \tag{A2}$$

it is possible to express from (9) the Mittag-Leffler function in the case of positive rational $\alpha = p/q$ with p and q positive coprimes,

$$\mathrm{E}_{p/q,\beta}(x) = \sum_{k=0}^{q-1} \frac{x^k}{\Gamma\left(\frac{p}{q}k + \beta\right)} \, {}_1F_p\left(\begin{array}{c} 1 \\ b_0, \ldots, b_{p-1} \end{array} \middle| \frac{x^q}{p^p} \right), \tag{A3}$$

where

$$b_j = \frac{k}{q} + \frac{\beta + j}{p}.$$

The corresponding Laplace transforms are

$$\begin{aligned} &\mathcal{L}\left[\mathrm{E}_{p/q,\beta}(t)\right] \\ &= \sum_{k=0}^{q-1} \frac{s^{-k-1}}{\Gamma\left(\frac{p}{q}k + \beta\right)} \, {}_{q+1}F_p\left(\begin{array}{c} 1, a_0, \ldots, a_{q-1} \\ b_0, \ldots, b_{p-1} \end{array} \middle| \frac{(q/s)^q}{p^p} \right), \end{aligned} \tag{A4}$$

where

$$
\begin{aligned}
a_j &= \frac{k+1+j}{q}, \\
b_j &= \frac{k}{q} + \frac{\beta+j}{p}.
\end{aligned}
$$

Table A1. The Mittag-Leffler functions derived for some values of parameters α and β by using (9).

α	β	$\mathbf{E}_{\alpha,\beta}(x)$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}} + x\, e^{x^2}[\mathrm{erf}(x)+1]$
$\frac{1}{2}$	1	$e^{x^2}[\mathrm{erf}(x)+1]$
$\frac{1}{2}$	$\frac{3}{2}$	$\frac{e^{x^2}[\mathrm{erf}(x)+1]-1}{x}$
$\frac{1}{2}$	2	$\frac{1}{x^2}\left[e^{x^2}[\mathrm{erf}(x)+1] - 1 - \frac{2x}{\sqrt{\pi}}\right]$
$\frac{1}{2}$	3	$-\frac{1}{3x^4}\left[3e^{x^2}[\mathrm{erfc}(x)-2] + \frac{4x^3+6x}{\sqrt{\pi}} + 3\left(1+x^2\right)\right]$
$\frac{1}{2}$	4	$\frac{1}{30x^4}\left[30\, e^{x^2}[\mathrm{erf}(x)+1] - \frac{4x\left(4x^4+10x^2+15\right)}{\sqrt{\pi}} - 15\left(x^4+2x^2+2\right)\right]$
$\frac{1}{2}$	β	$e^{x^2}x^{2(1-\beta)}\left[2 - \frac{\Gamma\left(\beta-1,x^2\right)}{\Gamma(\beta-1)} - \frac{\Gamma\left(\beta-\frac{1}{2},x^2\right)}{\Gamma\left(\beta-\frac{1}{2}\right)}\right]$
1	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}} + \sqrt{x}\, e^x\, \mathrm{erf}\left(\sqrt{x}\right)$
1	1	e^x
1	$\frac{3}{2}$	$\frac{e^x\,\mathrm{erf}\left(\sqrt{x}\right)}{\sqrt{x}}$
1	2	$\frac{e^x-1}{x}$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}}\, {}_1F_3\left(\begin{matrix}1\\ \frac{1}{6},\frac{1}{2},\frac{5}{6}\end{matrix}\middle\vert \frac{x^2}{27}\right) + \frac{x^{1/3}}{3}\left[e^{x^{2/3}} - 2\, e^{-x^{2/3}/2}\sin\left(\frac{\pi-3\sqrt{3}x^{2/3}}{6}\right)\right]$
$\frac{3}{2}$	1	$\frac{1}{3}\left[\frac{4x}{\sqrt{\pi}}\, {}_1F_3\left(\begin{matrix}1\\ \frac{5}{6},\frac{7}{6},\frac{3}{2}\end{matrix}\middle\vert \frac{x^2}{27}\right) + e^{x^{2/3}} + 2\, e^{-x^{2/3}/2}\cos\left(\frac{\sqrt{3}}{2}x^{2/3}\right)\right]$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{2}{\sqrt{\pi}}\, {}_1F_3\left(\begin{matrix}1\\ \frac{1}{2},\frac{5}{6},\frac{7}{6}\end{matrix}\middle\vert \frac{x^2}{27}\right) + \frac{x^{-1/3}}{3}\left[e^{x^{2/3}} - 2\, e^{-x^{2/3}/2}\sin\left(\frac{\pi+3\sqrt{3}x^{2/3}}{6}\right)\right]$
$\frac{3}{2}$	2	$\frac{8x}{15\sqrt{\pi}}\, {}_1F_3\left(\begin{matrix}1\\ \frac{7}{6},\frac{3}{2},\frac{11}{6}\end{matrix}\middle\vert \frac{x^2}{27}\right) + \frac{x^{-2/3}}{3}\left[e^{x^{2/3}} - 2\, e^{-x^{2/3}/2}\sin\left(\frac{\pi-3\sqrt{3}x^{2/3}}{6}\right)\right]$
$\frac{3}{2}$	β	$\frac{x}{\Gamma\left(\beta+\frac{1}{2}\right)}\, {}_1F_3\left(\begin{matrix}1\\ \frac{2\beta+3}{6},\frac{2\beta+5}{6},\frac{2\beta+7}{6}\end{matrix}\middle\vert \frac{x^2}{27}\right) + \frac{1}{\Gamma(\beta)}\, {}_1F_3\left(\begin{matrix}1\\ \frac{\beta+1}{3},\frac{\beta+2}{3},\frac{\beta}{3}\end{matrix}\middle\vert \frac{x^2}{27}\right)$
2	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}}\, {}_1F_2\left(\begin{matrix}1\\ \frac{1}{4},\frac{3}{4}\end{matrix}\middle\vert \frac{x}{4}\right)$
2	1	$\cosh\left(\sqrt{x}\right)$
2	2	$\frac{\sinh\left(\sqrt{x}\right)}{\sqrt{x}}$
2	3	$\frac{\cosh\left(\sqrt{x}\right)-1}{x}$
2	4	$\frac{\sinh\left(\sqrt{x}\right)}{x^{3/2}} - \frac{1}{x}$
2	β	$\frac{1}{\Gamma(\beta)}\, {}_1F_2\left(\begin{matrix}1\\ \frac{\beta+1}{2},\frac{\beta}{2}\end{matrix}\middle\vert \frac{x}{4}\right)$

Table A2. The Mittag-Leffler functions derived for some values of parameters α and β by using (9).

α	β	$\mathrm{E}_{\alpha,\beta}(x)$
3	1	$\frac{1}{3}\left[e^{x^{1/3}} + 2\,e^{-x^{1/3}/2}\cos\left(\frac{\sqrt{3}}{2}x^{1/3}\right)\right]$
3	2	$\frac{x^{-1/3}}{3}\left[e^{x^{1/3}} - 2\,e^{-x^{1/3}/2}\sin\left(\frac{\pi-3\sqrt{3}x^{1/3}}{6}\right)\right]$
3	3	$\frac{x^{-2/3}}{3}\left[e^{x^{1/3}} - 2\,e^{-x^{1/3}/2}\sin\left(\frac{\pi+3\sqrt{3}x^{1/3}}{6}\right)\right]$
3	β	$\frac{1}{\Gamma(\beta)}\,{}_1F_3\left(\begin{matrix}1\\ \frac{\beta+1}{3},\frac{\beta+2}{3},\frac{\beta}{3}\end{matrix}\middle\vert\,\frac{x}{27}\right)$
4	1	$\frac{1}{2}\left[\cos\left(x^{1/4}\right) + \cosh\left(x^{1/4}\right)\right]$
4	2	$\frac{\sin\left(x^{1/4}\right)+\sinh\left(x^{1/4}\right)}{2x^{1/4}}$
4	3	$\frac{\cosh\left(x^{1/4}\right)-\cos\left(x^{1/4}\right)}{2\sqrt{x}}$
4	4	$\frac{\sinh\left(x^{1/4}\right)-\sin\left(x^{1/4}\right)}{2x^{3/4}}$
4	β	$\frac{1}{\Gamma(\beta)}\,{}_1F_4\left(\begin{matrix}1\\ \frac{\beta+1}{4},\frac{\beta+2}{4},\frac{\beta+3}{4},\frac{\beta}{4}\end{matrix}\middle\vert\,\frac{x}{256}\right)$
5	1	${}_0F_4\left(\begin{matrix}-\\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$
5	2	${}_0F_4\left(\begin{matrix}-\\ \frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$
5	3	$\frac{1}{2}\,{}_0F_4\left(\begin{matrix}-\\ \frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$
5	4	$\frac{1}{6}\,{}_0F_4\left(\begin{matrix}-\\ \frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$
5	5	$\frac{1}{24}\,{}_0F_4\left(\begin{matrix}-\\ \frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$
5	β	$\frac{1}{\Gamma(\beta)}\,{}_1F_5\left(\begin{matrix}1\\ \frac{\beta+1}{5},\frac{\beta+2}{5},\frac{\beta+3}{5},\frac{\beta+4}{5},\frac{\beta}{5}\end{matrix}\middle\vert\,\frac{x}{3125}\right)$

Table A3. The Laplace transforms Mittag-Leffler functions derived for some values of parameters α and β by using (10).

α	β	$\mathcal{L}\left[\mathrm{E}_{\alpha,\beta}(t)\right]$
1	$\frac{1}{2}$	$\frac{\sqrt{s-1}+\csc^{-1}\left(\sqrt{s}\right)}{\sqrt{\pi}(s-1)^{3/2}}$
$\frac{1}{2}$	1	$\frac{1}{s-1}$
1	$\frac{3}{2}$	$\frac{2\csc^{-1}\left(\sqrt{s}\right)}{\sqrt{\pi}\sqrt{s-1}}$
1	2	$\ln\left(\frac{s}{s-1}\right)$
1	β	$\frac{1}{s\,\Gamma(\beta)}\,{}_2F_1\left(\begin{matrix}1,1\\ \beta\end{matrix}\middle\vert\,\frac{1}{s}\right)$
$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi s}}\,{}_2F_2\left(\begin{matrix}1,1\\ \frac{1}{6},\frac{5}{6}\end{matrix}\middle\vert\,\frac{4}{27s^2}\right) + \frac{1}{s^2}\,{}_2F_2\left(\begin{matrix}1,\frac{3}{2}\\ \frac{2}{3},\frac{4}{3}\end{matrix}\middle\vert\,\frac{4}{27s^2}\right)$

Table A3. *Cont.*

α	β	$\mathcal{L}\left[\mathbf{E}_{\alpha,\beta}(t)\right]$
$\frac{3}{2}$	1	$\frac{4}{3\sqrt{\pi}s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{5}{6},\frac{7}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)+\frac{1}{s}\,{}_2F_2\left(\begin{array}{c}\frac{1}{2},1\\ \frac{1}{3},\frac{2}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{2}{\sqrt{\pi}s}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{5}{6},\frac{7}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)+\frac{1}{2s^2}\,{}_2F_2\left(\begin{array}{c}1,\frac{3}{2}\\ \frac{4}{3},\frac{5}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)$
$\frac{3}{2}$	2	$\frac{8}{15\sqrt{\pi}s^2}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{7}{6},\frac{11}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)+\frac{1}{s}\,{}_2F_2\left(\begin{array}{c}\frac{1}{2},1\\ \frac{2}{3},\frac{4}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)$
$\frac{3}{2}$	β	$\frac{1}{\Gamma(\beta)s}\,{}_3F_3\left(\begin{array}{c}\frac{1}{2},1,1\\ \frac{\beta+1}{3},\frac{\beta+2}{3},\frac{\beta}{3}\end{array}\middle\vert\frac{4}{27s^2}\right)+$ $\frac{1}{\Gamma(\beta+\frac{3}{2})s^2}\,{}_3F_3\left(\begin{array}{c}1,1,\frac{3}{2}\\ \frac{2\beta+3}{6},\frac{2\beta+5}{6},\frac{2\beta+7}{6}\end{array}\middle\vert\frac{4}{27s^2}\right)$
2	$\frac{1}{2}$	$\frac{1}{\sqrt{\pi}s}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{1}{4},\frac{3}{4}\end{array}\middle\vert\frac{1}{4s}\right)$
2	1	$\frac{1}{s}+\frac{\sqrt{\pi}}{2}s^{-3/2}e^{1/(4s)}\mathrm{erf}\left(\frac{1}{2\sqrt{s}}\right)$
2	2	$\sqrt{\pi}s^{-1/2}e^{1/(4s)}\mathrm{erf}\left(\frac{1}{2\sqrt{s}}\right)$
2	3	$\frac{1}{2s}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{3}{2},2\end{array}\middle\vert\frac{1}{4s}\right)$
2	4	$\frac{1}{6s}\,{}_2F_2\left(\begin{array}{c}1,1\\ 2,\frac{5}{2}\end{array}\middle\vert\frac{1}{4s}\right)$
2	β	$\frac{1}{\Gamma(\beta)s}\,{}_2F_2\left(\begin{array}{c}1,1\\ \frac{\beta+1}{2},\frac{\beta}{2}\end{array}\middle\vert\frac{1}{4s}\right)$

Table A4. The Laplace transforms Mittag-Leffler functions derived for some values of parameters α and β by using (10).

α	β	$\mathcal{L}\left[\mathbf{E}_{\alpha,\beta}(t)\right]$
3	1	$\frac{1}{s}\,{}_1F_2\left(\begin{array}{c}1\\ \frac{1}{3},\frac{2}{3}\end{array}\middle\vert\frac{1}{27s}\right)$
3	2	$\frac{1}{s}\,{}_1F_2\left(\begin{array}{c}1\\ \frac{2}{3},\frac{4}{3}\end{array}\middle\vert\frac{1}{27s}\right)$
3	3	$\frac{1}{2s}\,{}_1F_2\left(\begin{array}{c}1\\ \frac{4}{3},\frac{5}{3}\end{array}\middle\vert\frac{1}{27s}\right)$
3	β	$\frac{1}{\Gamma(\beta)s}\,{}_2F_3\left(\begin{array}{c}1,1\\ \frac{\beta+1}{3},\frac{\beta+2}{3},\frac{\beta}{3}\end{array}\middle\vert\frac{1}{27s}\right)$
4	1	$\frac{1}{s}\,{}_1F_3\left(\begin{array}{c}1\\ \frac{1}{4},\frac{1}{2},\frac{3}{4}\end{array}\middle\vert\frac{1}{256s}\right)$
4	2	$\frac{1}{s}\,{}_1F_3\left(\begin{array}{c}1\\ \frac{1}{2},\frac{3}{4},\frac{5}{4}\end{array}\middle\vert\frac{1}{256s}\right)$
4	3	$\frac{1}{2s}\,{}_1F_3\left(\begin{array}{c}1\\ \frac{3}{4},\frac{5}{4},\frac{3}{2}\end{array}\middle\vert\frac{1}{256s}\right)$
4	4	$\frac{1}{6s}\,{}_1F_3\left(\begin{array}{c}1\\ \frac{5}{4},\frac{3}{2},\frac{7}{4}\end{array}\middle\vert\frac{1}{256s}\right)$

Table A4. *Cont.*

α	β	$\mathcal{L}\left[\mathbf{E}_{\alpha,\beta}(t)\right]$
4	β	$\frac{1}{\Gamma(\beta)s}\,{}_2F_4\left(\begin{matrix}1,1\\ \frac{\beta+1}{4},\frac{\beta+2}{4},\frac{\beta+3}{4},\frac{\beta}{4}\end{matrix}\middle\vert\frac{1}{256s}\right)$
5	1	$\frac{1}{s}\,{}_1F_4\left(\begin{matrix}1\\ \frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$
5	2	$\frac{1}{s}\,{}_1F_4\left(\begin{matrix}1\\ \frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$
5	3	$\frac{1}{2s}\,{}_1F_4\left(\begin{matrix}1\\ \frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$
5	4	$\frac{1}{6s}\,{}_1F_4\left(\begin{matrix}1\\ \frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$
5	5	$\frac{1}{24s}\,{}_1F_4\left(\begin{matrix}1\\ \frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$
5	β	$\frac{1}{\Gamma(\beta)s}\,{}_2F_5\left(\begin{matrix}1,1\\ \frac{\beta+1}{5},\frac{\beta+2}{5},\frac{\beta+3}{5},\frac{\beta+4}{5},\frac{\beta}{5}\end{matrix}\middle\vert\frac{1}{3125s}\right)$

Appendix B. Representations of the Whittaker Functions and Their Laplace Transforms

The Whittaker functions $\mathrm{M}_{\kappa,\mu}(x)$ and $\mathrm{W}_{\kappa,\mu}(x)$ defined in (18) were derived by using the MATHEMATICA program, and they are presented in Tables A5, A6, A10 and A11. The corresponding Laplace transforms are in Tables A7–A9 and A12. Most of the reported results in these tables are unknown in the mathematical reference literature.

Table A5. The Whittaker functions $\mathrm{M}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18).

κ	μ	$\mathbf{M}_{\kappa,\mu}(x)$
$-\frac{5}{2}$	0	$\frac{\sqrt{x}}{2}e^{x/2}[x(x+4)+2]$
$-\frac{3}{2}$	0	$\sqrt{x}e^{x/2}(x+1)$
$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{x}{3}\left[(2x+3)I_0\left(\frac{x}{2}\right)+(2x+1)I_1\left(\frac{x}{2}\right)\right]$
$-\frac{3}{2}$	1	$e^{x/2}x^{3/2}$
$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{4}{5}\left[x(2x-3)I_0\left(\frac{x}{2}\right)+[x(2x-3)+4]I_1\left(\frac{x}{2}\right)\right]$
$-\frac{3}{2}$	2	$4x^{-3/2}e^{-x/2}\left[e^x\left(x^3-3x^2+6x-6\right)+6\right]$
$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{32}{7x}\left[x\left(2x^2-9x+24\right)I_0\left(\frac{x}{2}\right)+\left(2x^3-11x^2+36x-96\right)I_1\left(\frac{x}{2}\right)\right]$
$-\frac{3}{2}$	3	$30x^{-5/2}e^{-x/2}\left[e^x\left(x^4-8x^3+36x^2-96x+120\right)-24(x+5)\right]$
$-\frac{1}{6}$	0	$e^{-x/2}\sqrt{x}L_{-2/3}(x)$
$-\frac{1}{4}$	0	$e^{-x/2}\sqrt{x}L_{-3/4}(x)$
$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{\sqrt{\pi}}{2}e^{x/2}x^{1/4}\mathrm{erf}\left(\sqrt{x}\right)$
$-\frac{1}{3}$	0	$e^{-x/2}\sqrt{x}L_{-5/6}(x)$
$-\frac{1}{2}$	$\frac{1}{2}$	$x\left[I_0\left(\frac{x}{2}\right)+I_1\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	1	$x^{-1/2}e^{-x/2}[2e^x(x-1)+2]$
$-\frac{1}{2}$	$\frac{3}{2}$	$4\left[xI_0\left(\frac{x}{2}\right)+(x-4)I_1\left(\frac{x}{2}\right)\right]$
$-\frac{1}{2}$	2	$12x^{-3/2}e^{-x/2}\left[e^x\left(x^2-4x+6\right)-2(x+3)\right]$

Table A5. *Cont.*

κ	μ	$\mathbf{M}_{\kappa,\mu}(x)$
0	$\frac{1}{8}$	$x^{5/8}\,{}_0F_1\left(\begin{matrix}-\\\frac{9}{8}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{7}$	$x^{9/14}\,{}_0F_1\left(\begin{matrix}-\\\frac{8}{7}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{6}$	$x^{2/3}\,{}_0F_1\left(\begin{matrix}-\\\frac{7}{6}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{5}$	$x^{7/10}\,{}_0F_1\left(\begin{matrix}-\\\frac{6}{5}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{4}$	$x^{3/4}\,{}_0F_1\left(\begin{matrix}-\\\frac{5}{4}\end{matrix}\middle\vert\frac{x^2}{16}\right)$
0	$\frac{1}{3}$	$x^{5/6}\,{}_0F_1\left(\begin{matrix}-\\\frac{4}{3}\end{matrix}\middle\vert\frac{x^2}{16}\right)$

Table A6. The Whittaker functions $\mathrm{M}_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18).

κ	μ	$\mathbf{M}_{\kappa,\mu}(x)$
0	$\frac{1}{2}$	$2\sinh\left(\frac{x}{2}\right)$
0	1	$4\sqrt{x}I_1\left(\frac{x}{2}\right)$
0	$\frac{3}{2}$	$12\left[\cosh\left(\frac{x}{2}\right)-\frac{2}{x}\sinh\left(\frac{x}{2}\right)\right]$
0	2	$32\sqrt{x}I_2\left(\frac{x}{2}\right)$
0	$\frac{5}{2}$	$\frac{120}{x^2}\left[\left(x^2+12\right)\sinh\left(\frac{x}{2}\right)-6x\cosh\left(\frac{x}{2}\right)\right]$
$\frac{1}{6}$	0	$e^{-x/2}\sqrt{x}L_{-1/3}(x)$
$\frac{1}{4}$	$-\frac{5}{4}$	$x^{-3/4}e^{-x/2}\left(\frac{2x}{3}+1\right)$
$\frac{1}{4}$	$-\frac{3}{4}$	$x^{-1/4}e^{x/2}$
$\frac{1}{4}$	$-\frac{1}{4}$	$x^{1/4}e^{-x/2}$
$\frac{1}{4}$	0	$e^{-x/2}\sqrt{x}L_{-1/4}(x)$
$\frac{1}{3}$	0	$e^{-x/2}\sqrt{x}L_{-1/6}(x)$
$\frac{1}{2}$	0	$e^{-x/2}\sqrt{x}$
$\frac{1}{2}$	$\frac{1}{2}$	$x\left[I_0\left(\frac{x}{2}\right)-I_1\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	1	$2x^{-1/2}e^{-x/2}(e^x-x-1)$
$\frac{1}{2}$	$\frac{3}{2}$	$4\left[-xI_0\left(\frac{x}{2}\right)+(x+4)I_1\left(\frac{x}{2}\right)\right]$
$\frac{1}{2}$	$\frac{5}{2}$	$32\left[(x+8)I_0\left(\frac{x}{2}\right)-\left(x+4+\frac{32}{x}\right)I_1\left(\frac{x}{2}\right)\right]$
1	0	$\sqrt{x}\left[-(x-1)I_0\left(\frac{x}{2}\right)+xI_1\left(\frac{x}{2}\right)\right]$
1	$\frac{1}{2}$	$x\,e^{-x/2}$
1	1	$\frac{4}{3}\sqrt{x}\left[xI_0\left(\frac{x}{2}\right)-(x+1)I_1\left(\frac{x}{2}\right)\right]$
$\frac{3}{2}$	0	$-\sqrt{x}e^{-x/2}(x-1)$
$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{x}{3}\left[(2x-3)I_0\left(\frac{x}{2}\right)+(1-2x)I_1\left(\frac{x}{2}\right)\right]$

Table A6. *Cont.*

κ	μ	$\mathbf{M}_{\kappa,\mu}(x)$
$\frac{3}{2}$	1	$x^{3/2}\, e^{-x/2}$
$\frac{3}{2}$	$\frac{3}{2}$	$\frac{4}{5}\left[x(2x+1)\, I_0\left(\frac{x}{2}\right) - \left(2x^2+3x+4\right) I_1\left(\frac{x}{2}\right)\right]$
2	0	$\frac{1}{3}\sqrt{x}\left[\left(2x^2-6x+3\right) I_0\left(\frac{x}{2}\right) - 2x(x-2) I_1\left(\frac{x}{2}\right)\right]$
2	$\frac{1}{2}$	$-\frac{1}{2}e^{-x/2}x(x-2)$
2	1	$-\frac{4}{15}\sqrt{x}\left[2x(x-2) I_0\left(\frac{x}{2}\right) + \left(-2x^2+2x+1\right) I_1\left(\frac{x}{2}\right)\right]$
2	$\frac{3}{2}$	$x^2\, e^{-x/2}$
2	2	$\frac{32}{35\sqrt{x}}\left[x\left(2x^2+2x+3\right) I_0\left(\frac{x}{2}\right) - \left(x^3+2x^2+4x+6\right) I_1\left(\frac{x}{2}\right)\right]$
$\frac{5}{2}$	0	$\frac{1}{2}e^{-x/2}\sqrt{x}\left(x^2-4x+2\right)$
$\frac{5}{2}$	$\frac{1}{2}$	$\frac{x}{15}\left[\left(4x^2-18x+15\right) I_0\left(\frac{x}{2}\right) + \left(-4x^2+14x-3\right) I_1\left(\frac{x}{2}\right)\right]$
$\frac{5}{2}$	1	$-\frac{1}{3}e^{-x/2}x^{3/2}(x-3)$
$\frac{5}{2}$	2	$x^{5/2}\, e^{-x/2}$
3	$\frac{1}{2}$	$\frac{1}{6}e^{-x/2}x\left(x^2-6x+6\right)$
3	1	$\frac{4\sqrt{x}}{105}\left[x\left(4x^2-24x+27\right) I_0\left(\frac{x}{2}\right) - \left(4x^3+20x^2+9x+3\right) I_1\left(\frac{x}{2}\right)\right]$
3	$\frac{3}{2}$	$-\frac{1}{4}e^{-x/2}x^2(x-4)$
3	$\frac{5}{2}$	$x^3\, e^{-x/2}$
$\frac{7}{2}$	0	$-\frac{1}{6}e^{-x/2}\sqrt{x}\left(x^3-9x^2+18x-6\right)$
4	$\frac{1}{2}$	$-\frac{1}{24}e^{-x/2}x\left(x^3-12x^2+36x-24\right)$
4	$\frac{3}{2}$	$\frac{1}{20}e^{-x/2}x^2\left(x^2-10x+20\right)$

Table A7. The Laplace transforms of the Whittaker function $\mathrm{M}_{\kappa,\mu}$ derived for some values of parameters κ and μ.

κ	μ	$\mathcal{L}\left[\mathbf{M}_{\kappa,\mu}(t)\right]$
$-\frac{5}{2}$	0	$\sqrt{\frac{\pi}{2}}\,\frac{8s^2+16s+5}{(2s-1)^{7/2}}$
$-\frac{3}{2}$	0	$\frac{2\sqrt{2\pi}(s+1)}{(2s-1)^{5/2}}$
$-\frac{3}{2}$	$\frac{1}{2}$	$\begin{cases} \frac{4\sqrt{4s^2-1}}{(2s-1)^3}, & s > \frac{1}{2} \\ 0, & s < \frac{1}{2} \end{cases}$
$-\frac{3}{2}$	1	$\frac{2\sqrt{2\pi}}{(2s-1)^{5/2}}$
$-\frac{3}{2}$	$\frac{3}{2}$	$\begin{cases} \frac{16\left\{-4s\left[4s^2-2s\left(\sqrt{4s^2-1}+3\right)+3\left(\sqrt{4s^2-1}+1\right)\right]+7\sqrt{4s^2-1}+2\right\}}{5(2s-1)^3}, & s > \frac{1}{2} \\ -\frac{32}{5}, & s < \frac{1}{2} \end{cases}$
$-\frac{1}{6}$	0	$\begin{cases} \frac{\sqrt{2\pi}\left[(6s+3)\,{}_2F_1\left(\left.\begin{array}{c} -\frac{1}{2}, \frac{2}{3} \\ 1 \end{array}\right\vert \frac{2}{2s+1}\right)+2\,{}_2F_1\left(\left.\begin{array}{c} \frac{1}{2}, \frac{2}{3} \\ 1 \end{array}\right\vert \frac{2}{2s+1}\right)\right]}{5(2s-1)^3}, & s > \frac{1}{2} \\ -\frac{\pi\,{}_2F_1\left(\left.\begin{array}{c} \frac{2}{3}, \frac{2}{3} \\ \frac{1}{6} \end{array}\right\vert s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^{5/6}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}, & s < \frac{1}{2} \end{cases}$

Table A7. *Cont.*

κ	μ	$\mathcal{L}\left[\mathbf{M}_{\kappa,\mu}(t)\right]$
$-\frac{1}{4}$	$\frac{1}{4}$	$\dfrac{2^{15/4}\Gamma\left(\frac{11}{4}\right)\left[(6s+3)\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{4}, \frac{1}{2}\\ \frac{3}{2}\end{matrix}\right\vert \frac{2}{1-2s}\right)-4\left(\frac{2}{2s+1}+1\right)^{1/4}\right]}{21(2s-1)^{7/4}(2s+1)}$
$-\frac{1}{3}$	0	$\begin{cases} \dfrac{\sqrt{2\pi}\left[(6s+3)\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}, \frac{5}{6}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)+4\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{2}, \frac{5}{6}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)\right]}{3(2s-1)(2s+1)^{3/2}}, & s > \frac{1}{2} \\ -\dfrac{\pi\,{}_2F_1\left(\left.\begin{matrix}\frac{5}{6}, \frac{5}{6}\\ \frac{1}{3}\end{matrix}\right\vert s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^{2/3}\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}, & s < \frac{1}{2} \end{cases}$
$-\frac{1}{2}$	0	$\dfrac{2\sqrt{2\pi}}{(2s-1)^{3/2}}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\begin{cases} \dfrac{4}{(2s-1)\sqrt{4s^2-1}}, & s > \frac{1}{2} \\ 0, & s < \frac{1}{2} \end{cases}$
$-\frac{1}{2}$	1	$2\sqrt{2\pi}\left[\dfrac{1}{\sqrt{2s+1}} - \dfrac{1}{\sqrt{2s-1}} + \dfrac{1}{(2s-1)^{3/2}}\right]$
0	1	$\begin{cases} \dfrac{2^{3/2}\left[(1-2s)\mathrm{K}\left(\frac{2}{2s+1}\right)+2s\,\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{\sqrt{\pi}(2s-1)\sqrt{2s+1}}, & s > \frac{1}{2} \\ \dfrac{8\left[(1-2s)\mathrm{K}\left(s+\frac{1}{2}\right)+4s\,\mathrm{E}\left(s+\frac{1}{2}\right)\right]}{\sqrt{\pi}(4s^2-1)}, & s < \frac{1}{2} \end{cases}$
0	2	$\begin{cases} \dfrac{64\left[8s(1-2s)\mathrm{K}\left(\frac{2}{2s+1}\right)+\left(16s^2-3\right)\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{\sqrt{\pi}(2s-1)\sqrt{s+\frac{1}{2}}}, & s > \frac{1}{2} \\ \dfrac{64\left[\left(-16s^2+2s+3\right)\mathrm{K}\left(s+\frac{1}{2}\right)+\left(32s^2-6\right)\mathrm{E}\left(s+\frac{1}{2}\right)\right]}{\sqrt{\pi}(4s^2-1)}, & s < \frac{1}{2} \end{cases}$
$\frac{1}{6}$	0	$\begin{cases} \dfrac{\sqrt{2\pi}\left[(6s+3)\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}, \frac{1}{3}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)-2\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3}, \frac{1}{2}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)\right]}{3(2s-1)(2s+1)^{3/2}}, & s > \frac{1}{2} \\ \dfrac{\sqrt{2\pi}\Gamma\left(\frac{7}{6}\right)\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{3}, \frac{1}{3}\\ -\frac{1}{6}\end{matrix}\right\vert s+\frac{1}{2}\right)}{(2s+1)^{7/6}\Gamma\left(\frac{5}{6}\right)\Gamma\left(\frac{1}{3}\right)}, & s < \frac{1}{2} \end{cases}$

Table A8. The Laplace transforms of the Whittaker functions $\mathrm{M}_{\kappa,\mu}$ derived for some values of parameters κ and μ.

κ	μ	$\mathcal{L}\left[\mathbf{M}_{\kappa,\mu}(t)\right]$
$\frac{1}{4}$	$-\frac{5}{4}$	$\dfrac{3\left(s+\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)+2\,\Gamma\left(\frac{5}{4}\right)}{3\left(s+\frac{1}{2}\right)^{5/4}}$
$\frac{1}{4}$	$-\frac{3}{4}$	$\dfrac{\Gamma\left(\frac{3}{4}\right)}{\left(s-\frac{1}{2}\right)^{3/4}}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\dfrac{\Gamma\left(\frac{5}{4}\right)}{\left(s+\frac{1}{2}\right)^{5/4}}$
0	$\frac{1}{2}$	$\dfrac{4}{4s^2-1}$
$\frac{1}{3}$	0	$\begin{cases} \dfrac{\sqrt{2\pi}\left[(6s+3)\,{}_2F_1\left(\left.\begin{matrix}-\frac{1}{2}, \frac{1}{6}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)-4\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{6}, \frac{1}{2}\\ 1\end{matrix}\right\vert \frac{2}{2s+1}\right)\right]}{3(2s-1)(2s+1)^{3/2}}, & s > \frac{1}{2} \\ \dfrac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)\,{}_2F_1\left(\left.\begin{matrix}\frac{1}{6}, \frac{1}{6}\\ -\frac{1}{3}\end{matrix}\right\vert s+\frac{1}{2}\right)}{2^{2/3}\sqrt{3}\pi(2s+1)^{4/3}}, & s < \frac{1}{2} \end{cases}$
$\frac{1}{2}$	$\frac{1}{2}$	$\begin{cases} \dfrac{4}{(2s+1)\sqrt{4s^2-1}}, & s > \frac{1}{2} \\ 0, & s < \frac{1}{2} \end{cases}$

Table A8. *Cont.*

κ	μ	$\mathcal{L}\left[\mathbf{M}_{\kappa,\mu}(t)\right]$
$\frac{1}{2}$	0	$\frac{\sqrt{2\pi}}{(2s+1)^{3/2}}$
$\frac{1}{2}$	1	$2\sqrt{2\pi}\left[-\frac{1}{\sqrt{2s+1}}-\frac{1}{(2s+1)^{3/2}}+\frac{1}{\sqrt{2s-1}}\right]$
1	0	$\begin{cases} \frac{2^{3/2}\left[-\mathrm{K}\left(\frac{2}{2s+1}\right)+2\,\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{\sqrt{\pi}(2s+1)^{3/2}}, & s>\frac{1}{2} \\ \frac{2(2s-3)\mathrm{K}\left(s+\frac{1}{2}\right)+8\,\mathrm{E}\left(s+\frac{1}{2}\right)}{\sqrt{\pi}(2s+1)^{2}}, & s<\frac{1}{2} \end{cases}$
1	$\frac{1}{2}$	$\frac{4}{(2s+1)^2}$
1	1	$\begin{cases} -\frac{2^{7/2}\left[-2(s+1)\mathrm{K}\left(\frac{2}{2s+1}\right)+(2s+3)\,\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{3\sqrt{\pi}(2s+1)^{3/2}}, & s>\frac{1}{2} \\ \frac{8(2s+5)\mathrm{K}\left(s+\frac{1}{2}\right)-8(4s+6)\,\mathrm{E}\left(s+\frac{1}{2}\right)}{3\sqrt{\pi}(2s+1)^{2}}, & s<\frac{1}{2} \end{cases}$
$\frac{3}{2}$	0	$\frac{2\sqrt{2\pi}(s-1)}{(2s+1)^{5/2}}$
$\frac{3}{2}$	$\frac{1}{2}$	$\begin{cases} \frac{4\sqrt{4s^2-1}}{(2s+1)^3}, & s>\frac{1}{2} \\ 0, & s<\frac{1}{2} \end{cases}$
$\frac{3}{2}$	1	$\frac{3\sqrt{2\pi}}{(2s+1)^{5/2}}$
2	0	$\begin{cases} \frac{2^{3/2}\left[4(1-2s)\mathrm{K}\left(\frac{2}{2s+1}\right)+(14s-9)\,\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{3\sqrt{\pi}(2s+1)^{5/2}}, & s>\frac{1}{2} \\ \frac{2(2s-1)(6s-13)\mathrm{K}\left(s+\frac{1}{2}\right)+4(14s-9)\,\mathrm{E}\left(s+\frac{1}{2}\right)}{3\sqrt{\pi}(2s+1)^{3}}, & s<\frac{1}{2} \end{cases}$
2	$\frac{1}{2}$	$\frac{8s-4}{(2s+1)^3}$
2	1	$\begin{cases} \frac{2^{7/2}\left[[-5+4s(s+2)]\mathrm{K}\left(\frac{2}{2s+1}\right)-2[(2s+5)-6]\,\mathrm{E}\left(\frac{2}{2s+1}\right)\right]}{15\sqrt{\pi}(2s+1)^{5/2}}, & s>\frac{1}{2} \\ \frac{8(2s-1)(2s+17)\mathrm{K}\left(s+\frac{1}{2}\right)-32[s(2s+5)-6]\,\mathrm{E}\left(s+\frac{1}{2}\right)}{15\sqrt{\pi}(2s+1)^{3}}, & s<\frac{1}{2} \end{cases}$
2	$\frac{3}{2}$	$\frac{16}{(2s+1)^3}$

Table A9. The Laplace transforms of the Whittaker functions $\mathrm{M}_{\kappa,\mu}$ derived for some values of parameters κ and μ.

κ	μ	$\mathcal{L}\left[\mathbf{M}_{\kappa,\mu}(t)\right]$
$\frac{5}{2}$	1	$\frac{2\sqrt{2\pi}(3s-1)}{(2s+1)^{7/2}}$
$\frac{5}{2}$	2	$\frac{15\sqrt{2\pi}}{(2s+1)^{7/2}}$
3	$\frac{1}{2}$	$\frac{4(1-2s)^2}{(2s+1)^4}$
3	$\frac{3}{2}$	$\frac{8(4s-1)}{(2s+1)^4}$
3	$\frac{5}{2}$	$\frac{96}{(2s+1)^4}$
$\frac{7}{2}$	0	$\frac{\sqrt{2\pi}\left(8s^3-24s^2+15s-3\right)}{(2s+1)^{9/2}}$
4	$\frac{1}{2}$	$\frac{4(2s-1)^3}{(2s+1)^5}$
4	$\frac{3}{2}$	$\frac{32\left(10s^2-5s+1\right)}{5(2s+1)^5}$

Table A10. The Whittaker functions $W_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18).

κ	μ	$\mathbf{W}_{\kappa,\mu}(x)$
$-\frac{5}{2}$	0	$\frac{\sqrt{x}}{4}e^{-x/2}\left[e^x\left(x^2+4x+2\right)\Gamma(0,x)-x-3\right]$
$-\frac{3}{2}$	0	$\sqrt{x}e^{-x/2}\left[e^x(x+1)\Gamma(0,x)-1\right]$
$-\frac{3}{2}$	1	$x^{3/2}e^{x/2}\Gamma(-2,x)$
$-\frac{1}{4}$	$\frac{1}{4}$	$x^{1/4}e^{x/2}\Gamma\left(\frac{1}{2},x\right)$
$-\frac{1}{2}$	0	$x^{1/2}e^{x/2}\Gamma(0,x)$
$-\frac{1}{2}$	1	$x^{-1/2}e^{-x/2}$
$-\frac{1}{2}$	2	$x^{-3/2}e^{-x/2}(x+3)$
$-\frac{1}{2}$	3	$x^{-5/2}e^{-x/2}\left(x^2+8x+20\right)$
$-\frac{3}{4}$	$\frac{3}{4}$	$\frac{e^{-x/2}}{2x^{-1/4}}\left[2\sqrt{x}-\sqrt{\pi}e^x(2x-1)\mathrm{erfc}\left(\sqrt{x}\right)\right]$
0	β	$\sqrt{\frac{x}{\pi}}K_\beta\left(\frac{x}{2}\right)$
0	$\frac{1}{2}$	$e^{-x/2}$
0	$\frac{3}{2}$	$e^{-x/2}\left(1+\frac{2}{x}\right)$
0	$\frac{5}{2}$	$e^{-x/2}\left(1+\frac{6}{x}+\frac{12}{x^2}\right)$
$\frac{1}{4}$	$-\frac{5}{4}$	$x^{-3/4}e^{-x/2}\left(x+\frac{3}{2}\right)$
$\frac{1}{4}$	$-\frac{3}{4}$	$x^{-1/4}e^{x/2}\Gamma\left(\frac{3}{2},x\right)$
$\frac{1}{4}$	$-\frac{1}{4}$	$x^{1/4}e^{-x/2}$
$\frac{1}{2}$	0	$x^{1/2}e^{-x/2}$
$\frac{1}{2}$	1	$x^{-1/2}e^{x/2}\Gamma(2,x)$
$\frac{3}{4}$	$\frac{1}{4}$	$x^{3/4}e^{-x/2}$
$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}x^{-1/4}e^{-x/2}(2x+1)$

Table A11. The Whittaker functions $W_{\kappa,\mu}$ derived for some values of parameters κ and μ by using (18).

κ	μ	$\mathbf{W}_{\kappa,\mu}(x)$
$\frac{3}{4}$	$\frac{5}{4}$	$x^{-3/4}e^{x/2}\Gamma\left(\frac{5}{2},x\right)$
1	$\frac{1}{2}$	$x\,e^{-x/2}$
$\frac{3}{2}$	0	$\sqrt{x}e^{-x/2}(x-1)$
$\frac{3}{2}$	1	$x^{3/2}e^{-x/2}$
$\frac{3}{2}$	2	$x^{-3/2}e^{x/2}\Gamma(4,x)$
2	$\frac{1}{2}$	$x(x-2)e^{-x/2}$
2	$\frac{3}{2}$	$x^2e^{-x/2}$
2	$\frac{5}{2}$	$x^{-2}e^{x/2}\Gamma(5,x)$
$\frac{5}{2}$	0	$\sqrt{x}e^{-x/2}\left(x^2-4x+2\right)$
$\frac{5}{2}$	1	$x^{3/2}e^{-x/2}(x-3)$
$\frac{5}{2}$	2	$x^{5/2}e^{-x/2}$

Table A11. *Cont.*

κ	μ	$\mathbf{W}_{\kappa,\mu}(x)$
$\frac{5}{2}$	3	$x^{-5/2}e^{x/2}\Gamma(6,x)$
3	$\frac{1}{2}$	$e^{-x/2}x\left(x^2-6x+6\right)$
3	$\frac{3}{2}$	$e^{-x/2}x^2(x-4)$
3	$\frac{5}{2}$	$x^3e^{-x/2}$
$\frac{7}{2}$	0	$e^{-x/2}\sqrt{x}\left(x^3-9x^2+18x-6\right)$
4	$\frac{1}{2}$	$e^{-x/2}x\left(x^3-12x^2+36x-24\right)$
4	$\frac{3}{2}$	$e^{-x/2}x^2\left(x^2-10x+20\right)$

Table A12. The Laplace transforms of the Whittaker function $\mathrm{W}_{\kappa,\mu}$ derived for some values of parameters κ and μ.

κ	μ	$\mathcal{L}\left[\mathbf{W}_{\kappa,\mu}(t)\right]$
$-\frac{1}{2}$	0	$\sqrt{2\pi}\left[\frac{\ln\left(\sqrt{4s^2-1}+2s\right)}{(2s-1)^{3/2}}+\frac{2}{(1-2s)\sqrt{2s+1}}\right]$
$-\frac{1}{2}$	1	$\sqrt{\frac{2\pi}{2s+1}}$
$\frac{1}{4}$	$-\frac{5}{4}$	$\frac{2^{1/4}\Gamma\left(\frac{1}{4}\right)(3s+1)}{(2s+1)^{5/4}}$
$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{\Gamma\left(\frac{5}{4}\right)}{\left(s+\frac{1}{2}\right)^{5/4}}$
0	$\frac{1}{2}$	$\frac{2}{2s+1}$
0	1	$\frac{8s\,\mathrm{E}\left(\frac{1}{2}-s\right)-2(2s+1)\mathrm{K}\left(\frac{1}{2}-s\right)}{4s^2-1}$
$\frac{1}{2}$	0	$\frac{\sqrt{2\pi}}{(2s+1)^{3/2}}$
$\frac{3}{4}$	$\frac{1}{4}$	$\frac{\Gamma\left(\frac{7}{4}\right)}{\left(s+\frac{1}{2}\right)^{7/4}}$
$\frac{3}{4}$	$\frac{3}{4}$	$\frac{2^{3/4}\Gamma\left(\frac{3}{4}\right)(s+2)}{(2s+1)^{7/4}}$
$\frac{3}{4}$	$\frac{5}{4}$	$4\,\Gamma\left(\frac{11}{4}\right)\,{}_2F_1\left(\begin{array}{c}\frac{1}{4},\frac{11}{4}\\\frac{5}{4}\end{array}\middle\vert\,\frac{1}{2}-s\right)$
1	$\frac{1}{2}$	$\frac{4}{(2s+1)^2}$
$\frac{3}{2}$	0	$\frac{2\sqrt{2\pi}(1-s)}{(2s+1)^{5/2}}$
$\frac{3}{2}$	1	$\frac{3\sqrt{2\pi}}{(2s+1)^{5/2}}$
2	$\frac{1}{2}$	$\frac{8-16s}{(2s+1)^3}$
2	$\frac{3}{2}$	$\frac{16}{(2s+1)^3}$
$\frac{5}{2}$	0	$\frac{\sqrt{2\pi}\left(8s^2-16s+5\right)}{(2s+1)^{7/2}}$
$\frac{5}{2}$	1	$\frac{6\sqrt{2\pi}(1-3s)}{(2s+1)^{7/2}}$
$\frac{5}{2}$	2	$\frac{15\sqrt{2\pi}}{(2s+1)^{7/2}}$

Table A12. *Cont.*

κ	μ	$\mathcal{L}[\mathrm{W}_{\kappa,\mu}(t)]$
3	$\frac{1}{2}$	$\frac{24(1-2s)^2}{(2s+1)^4}$
3	$\frac{3}{2}$	$\frac{32(4s-1)}{(2s+1)^4}$
3	$\frac{5}{2}$	$\frac{96}{(2s+1)^4}$
$\frac{7}{2}$	0	$-\frac{6\sqrt{2\pi}\left(8s^3-24s^2+15s-3\right)}{(2s+1)^{9/2}}$
4	$\frac{1}{2}$	$\frac{96(1-2s)^3}{(2s+1)^5}$
4	$\frac{3}{2}$	$\frac{128\left(10s^2-5s+1\right)}{(2s+1)^5}$

Appendix C. Representations of the Wright Functions

The Wright functions $\mathrm{W}_{\alpha,\beta}(x)$, defined in (39), and presented in Tables A13 and A14, as well as the Mainardi functions $\mathrm{F}_{\alpha}(x)$ and $\mathrm{M}_{\alpha}(x)$, defined in (40), and presented in Tables A15 and A16, were derived by using the MATHEMATICA program. Only a small part of these Wright functions is known in the mathematical reference literature.

In the case of positive rational $\alpha = p/q$ with p and q positive coprimes, applying (A1) and (A2), it is possible to express the Wright function by

$$\begin{aligned} &\mathrm{W}_{p/q,\beta}(x) \\ &= \sum_{k=0}^{q-1} \frac{x^k}{k!\,\Gamma\left(\frac{p}{q}k+\beta\right)}\, {}_0F_{p+q-1}\left(\begin{array}{c} - \\ b_0,\ldots,b_{p-1},c_0^*,\ldots,c_{q-2} ^* \end{array} \middle| \frac{x^q}{p^p q^q} \right), \end{aligned} \tag{A5}$$

where

$$\begin{aligned} b_j &= \frac{k}{q} + \frac{\beta+j}{p}, \\ c_j &= \frac{k+1+j}{q}, \end{aligned} \tag{A6}$$

and the set of numbers $\left\{c_j^*\right\} = \left\{c_j\right\}\backslash\{1\}$.

For the Mainardi functions, we have the following reduction formulas for positive rational $\alpha = p/q$ with p and q positive coprimes:

$$\begin{aligned} &\mathrm{F}_{p/q}(x) \\ &= -\frac{1}{\pi}\sum_{k=1}^{q} \frac{(-x)^h}{k!}\Gamma\left(\frac{p}{q}k+1\right)\sin\left(\pi\frac{p}{q}k\right) \\ &\quad {}_pF_{q-1}\left(\begin{array}{c} a_0,\ldots,a_{p-1} \\ b_0^*,\ldots,b_{q-2}^* \end{array} \middle| \frac{(-1)^{p+q}x^q p^p}{q^q} \right), \end{aligned} \tag{A7}$$

and

$$\mathrm{M}_{p/q}(x) = \frac{q}{px}\mathrm{F}_{p/q}(x), \tag{A8}$$

where

$$\begin{aligned} a_j &= \frac{k}{q} + \frac{j+1}{p}, \\ b_j &= \frac{k+1+j}{q}, \end{aligned}$$

and the set of numbers $\left\{b_j^*\right\} = \{b_j\}\backslash\{1\}$.

Table A13. The Wright functions $W_{\alpha,\beta}$ derived for some values of parameters α and β by using (39).

α	β	$W_{\alpha,\beta}(x)$
-1	$\frac{1}{2}$	$\frac{1}{2\sqrt{\pi}(x+1)^{3/2}}$
-1	$\frac{3}{2}$	$\frac{1}{\sqrt{\pi}(x+1)^{1/2}}$
-1	β	$\frac{(x+1)^{\beta-1}}{\Gamma(\beta)}$
$-\frac{1}{2}$	β	$\frac{1}{\Gamma(\beta)}\,{}_1F_1\left(\begin{array}{c}1-\beta\\ \frac{1}{2}\end{array}\middle\vert -\frac{x^2}{4}\right)+\frac{x}{\Gamma(\beta-\frac{1}{2})}\,{}_1F_1\left(\begin{array}{c}\frac{3}{2}-\beta\\ \frac{3}{2}\end{array}\middle\vert -\frac{x^2}{4}\right)$
$-\frac{1}{2}$	-1	$\frac{1}{8\sqrt{\pi}}\left[x\left(6-x^2\right)e^{-x^2/4}\right]$
$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4\sqrt{\pi}}\left[\left(x^2-2\right)e^{-x^2/4}\right]$
$-\frac{1}{2}$	0	$-\frac{x\,e^{-x^2/4}}{2\sqrt{\pi}}$
$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{e^{-x^2/4}}{\sqrt{\pi}}$
$-\frac{1}{2}$	1	$\operatorname{erf}\left(\frac{x}{2}\right)+1$
$-\frac{1}{2}$	$\frac{3}{2}$	$x\left[\operatorname{erf}\left(\frac{x}{2}\right)+1\right]+\frac{2}{\sqrt{\pi}}e^{-x^2/4}$
0	$-\frac{3}{2}$	$\frac{3\,e^{x}}{4\sqrt{\pi}}$
0	$-\frac{1}{2}$	$\frac{e^{x}}{2\sqrt{\pi}}$
0	1	e^{x}
0	β	$\frac{e^{x}}{\Gamma(\beta)}$
$\frac{1}{3}$	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_3\left(\begin{array}{c}-\\ \frac{1}{3},\frac{2}{3},\beta\end{array}\middle\vert \frac{x^3}{27}\right)+\frac{x}{\Gamma(\beta+\frac{1}{3})}\,{}_0F_3\left(\begin{array}{c}-\\ \frac{2}{3},\frac{4}{3},\beta+\frac{1}{3}\end{array}\middle\vert \frac{x^3}{27}\right)$ $+\frac{x^2}{2\,\Gamma(\beta+\frac{2}{3})}\,{}_0F_3\left(\begin{array}{c}-\\ \frac{4}{3},\frac{5}{3},\beta+\frac{2}{3}\end{array}\middle\vert \frac{x^3}{27}\right)$
$\frac{1}{2}$	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_2\left(\begin{array}{c}-\\ \frac{1}{2},\beta\end{array}\middle\vert \frac{x^2}{4}\right)+\frac{x}{\Gamma(\beta+\frac{1}{2})}\,{}_0F_3\left(\begin{array}{c}-\\ \frac{3}{2},\beta+\frac{1}{2}\end{array}\middle\vert \frac{x^2}{4}\right)$
1	β	$x^{(1-\beta)/2}I_{\beta-1}\left(2\sqrt{x}\right)$
1	$-\frac{3}{2}$	$\frac{(4x+3)\cosh\left(2\sqrt{x}\right)-6\sqrt{x}\sinh\left(2\sqrt{x}\right)}{4\sqrt{\pi}}$
1	$-\frac{1}{2}$	$\frac{2\sqrt{x}\sinh\left(2\sqrt{x}\right)-\cosh\left(2\sqrt{x}\right)}{2\sqrt{\pi}}$
1	0	$\sqrt{x}I_1\left(2\sqrt{x}\right)$
1	$\frac{1}{2}$	$\frac{\cosh\left(2\sqrt{x}\right)}{\sqrt{\pi}}$
1	1	$I_0\left(2\sqrt{x}\right)$
1	$\frac{3}{2}$	$\frac{\sinh\left(2\sqrt{x}\right)}{\sqrt{\pi x}}$
1	$\frac{5}{2}$	$\frac{2\sqrt{x}\cosh\left(2\sqrt{x}\right)-\sinh\left(2\sqrt{x}\right)}{2\sqrt{\pi}x^{3/2}}$

Table A14. The Wright functions $\mathrm{W}_{\alpha,\beta}$ derived for some values of parameters α and β by using (39).

α	β	$\mathrm{W}_{\alpha,\beta}(x)$
$\frac{3}{2}$	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_4\left(\begin{matrix} - \\ \frac{1}{2}, \frac{\beta+1}{3}, \frac{\beta+2}{3}, \frac{\beta}{3} \end{matrix}\,\middle|\, \frac{x^2}{108}\right) + \frac{x}{\Gamma(\beta+\frac{3}{2})}\,{}_0F_4\left(\begin{matrix} - \\ \frac{3}{2}, \frac{2\beta+3}{6}, \frac{2\beta+5}{6}, \frac{2\beta+7}{6} \end{matrix}\,\middle|\, \frac{x^2}{108}\right)$
2	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_2\left(\begin{matrix} - \\ \frac{\beta+1}{2}, \frac{\beta}{2} \end{matrix}\,\middle|\, \frac{x}{4}\right)$
3	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_3\left(\begin{matrix} - \\ \frac{\beta+1}{3}, \frac{\beta+2}{3}, \frac{\beta}{3} \end{matrix}\,\middle|\, \frac{x}{27}\right)$
4	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_4\left(\begin{matrix} - \\ \frac{\beta+1}{4}, \frac{\beta+2}{4}, \frac{\beta+3}{4}, \frac{\beta}{4} \end{matrix}\,\middle|\, \frac{x}{256}\right)$
5	β	$\frac{1}{\Gamma(\beta)}\,{}_0F_5\left(\begin{matrix} - \\ \frac{\beta+1}{5}, \frac{\beta+2}{5}, \frac{\beta+3}{5}, \frac{\beta+4}{5}, \frac{\beta}{5} \end{matrix}\,\middle|\, \frac{x}{3125}\right)$

Table A15. The Mainardi function F_α derived for some values of parameter α by using (A7).

α	$\mathrm{F}_\alpha(x)$
$\frac{3}{4}$	$\frac{x\,\Gamma(\frac{7}{4})}{\sqrt{2}\pi}\,{}_2F_2\left(\begin{matrix} \frac{7}{12}, \frac{11}{12} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right) + \frac{3x^2}{8\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix} \frac{5}{6}, \frac{7}{6} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right) + \frac{x^3\,\Gamma(\frac{13}{4})}{6\sqrt{2}\pi}\,{}_2F_2\left(\begin{matrix} \frac{13}{12}, \frac{17}{12} \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right)$
$\frac{2}{3}$	$\frac{\sqrt{3}x}{4\pi}\left[2\,\Gamma(\frac{5}{3})\,{}_1F_1\left(\begin{matrix} \frac{5}{6} \\ \frac{2}{3} \end{matrix}\,\middle|\, -\frac{4x^3}{27}\right) + x\,\Gamma(\frac{7}{3})\,{}_1F_1\left(\begin{matrix} \frac{7}{6} \\ \frac{4}{3} \end{matrix}\,\middle|\, -\frac{4x^3}{27}\right)\right]$
$\frac{1}{2}$	$\frac{x\,e^{-x^2/4}}{2\sqrt{\pi}}$
$\frac{1}{3}$	$3^{-1/3}x\,\mathrm{Ai}\left(3^{-1/3}x\right)$
$\frac{1}{4}$	$\frac{x\,\Gamma(\frac{5}{4})}{\sqrt{2}\pi}\,{}_0F_2\left(\begin{matrix} - \\ \frac{1}{2}, \frac{3}{4} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right) - \frac{x^2}{4\sqrt{\pi}}\,{}_0F_2\left(\begin{matrix} - \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right) + \frac{x^3\,\Gamma(\frac{7}{4})}{6\sqrt{2}\pi}\,{}_0F_2\left(\begin{matrix} - \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right)$

Table A16. The Mainardi function M_α derived for some values of parameter α by using (A8).

α	$\mathrm{M}_\alpha(x)$
$\frac{3}{4}$	$\frac{1}{\Gamma(\frac{1}{4})}\,{}_2F_2\left(\begin{matrix} \frac{7}{12}, \frac{11}{12} \\ \frac{1}{2}, \frac{3}{4} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right) + \frac{x}{2\sqrt{\pi}}\,{}_2F_2\left(\begin{matrix} \frac{5}{6}, \frac{7}{6} \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right) + \frac{x^2}{2\,\Gamma(-\frac{5}{4})}\,{}_2F_2\left(\begin{matrix} \frac{13}{12}, \frac{17}{12} \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\,\middle|\, -\frac{27x^4}{256}\right)$
$\frac{2}{3}$	$3^{-2/3}e^{-2x^3/27}\left[3^{1/3}\mathrm{Ai}\left(3^{-4/3}x^2\right) - 3\mathrm{Ai}'\left(3^{-4/3}x^2\right)\right]$
$\frac{1}{2}$	$\frac{e^{-x^2/4}}{\sqrt{\pi}}$
$\frac{1}{3}$	$3^{2/3}\,\mathrm{Ai}\left(3^{-1/3}x\right)$
$\frac{1}{4}$	$\frac{2\sqrt{2}\Gamma(\frac{5}{4})}{\pi}\,{}_0F_2\left(\begin{matrix} - \\ \frac{1}{2}, \frac{3}{4} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right) - \frac{x}{\sqrt{\pi}}\,{}_0F_2\left(\begin{matrix} - \\ \frac{3}{4}, \frac{5}{4} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right) + \frac{\sqrt{2}x^2\,\Gamma(\frac{7}{4})}{3\pi}\,{}_0F_2\left(\begin{matrix} - \\ \frac{5}{4}, \frac{3}{2} \end{matrix}\,\middle|\, -\frac{x^4}{256}\right)$

References

1. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA, 1953; Volume 1.
2. Erdélyi, A.; Magnus, W.; Oberhettinger, F.; Tricomi, F. *Tables of Integral Transforms*; McGraw-Hill: New York, NY, USA, 1954.
3. Abramowitz, M.; Stegun, I. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*; U.S. National Bureau of Standards: Washington, DC, USA, 1964; Volume 55.
4. Magnus, W.; Oberhettinger, F.; Soni, R. *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed.; Springer: Berlin, Germany, 1966.
5. Olver, F.; Lozier, D.; Boisvert, R.; Clark, C. *NIST Handbook of Mathematical Functions*; Cambridge University Press: Cambridge, UK, 2010.
6. Roberts, G.; Kaufman, H. *Table of Laplace Transforms*; WB Saunders Co.: Philadelphia, PA, USA, 1966.
7. Oberhettinger, F.; Badii, L. *Tables of Laplace Transforms*; Springer: Berlin, Germany, 1970.
8. Apelblat, A. *Laplace Transforms and Their Applications*; Nova Science Publishers Inc.: New York, NY, USA, 2012.
9. Van der Pol, B. On the operational solution of linear differential equations and an investigation of the properties of these solutions. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1929**, *8*, 861–898. [CrossRef]
10. Van der Pol, B.; Bremmer, H. *Operational Calculus: Based on the Two-Sided Laplace Integral*; Taylor & Francis: Abingdon, UK, 1987; Volume 327.
11. Humbert, P. Bessel-integral functions. *Proc. Edinb. Math. Soc.* **1933**, *3*, 276–285. [CrossRef]
12. Apelblat, A.; Kravitsky, N. Integral representations of derivatives and integrals with respect to the order of the Bessel functions $J_\nu(t)$, $I_\nu(t)$, the Anger function $\mathbf{J}_\nu(t)$ and the integral Bessel function $Ji_\nu(t)$. *IMA J. Appl. Math.* **1985**, *34*, 187–210. [CrossRef]
13. Apelblat, A. *Bessel and Related Functions. Mathematical Operations with Respect to the Order*; Theoretical Aspects; Walter de Gruyter GmbH: Berlin, Germany, 2020; Volume 1.
14. Gorenflo, R.; Kilbas, A.; Mainardi, F.; Rogosin, S. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin, Germany, 2014; Volume 2.
15. Whittaker, E. An expression of certain known functions as generalized hypergeometric functions. *Bull. Am. Math. Soc.* **1903**, *10*, 125–134. [CrossRef]
16. Prudnikov, A.; Brychkov, Y.; Marichev, O. *Integrals and Series: More Specific Function*; CRC Press: Boca Raton, FL, USA, 1986; Volume 3.
17. Wright, E. On the coefficients of power series having exponential singularities. *J. Lond. Math. Soc.* **1933**, *1*, 71–79. [CrossRef]
18. Wright, E. The generalized Bessel function of order greater than one. *Q. J. Math.* **1940**, *1*, 36–48. [CrossRef]
19. Gorenflo, R.; Luchko, Y.; Mainardi, F. Analytical properties and applications of the Wright function. *Fract. Calc. Appl. Anal.* **1999**, *2*, 383–414.
20. Mainardi, F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*; World Scientific: Singapore, 2010.
21. Apelblat, A. Differentiation of the Mittag-Leffler functions with respect to parameters in the Laplace transform approach. *Mathematics* **2020**, *8*, 657. [CrossRef]