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Maximal Function Characterizations of Hardy Spaces on $\mathbb{R}^n$ with Pointwise Variable Anisotropy

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Abstract: In 2011, Dekel et al. developed highly geometric Hardy spaces $H^p(\Theta)$, for the full range $0 < p \leq 1$, which were constructed by continuous multi-level ellipsoid covers Θ of $\mathbb{R}^n$ with high anisotropy in the sense that the ellipsoids can rapidly change shape from point to point and from level to level. In this article, when the ellipsoids in Θ rapidly change shape from level to level, the authors further obtain some real-variable characterizations of $H^p(\Theta)$ in terms of the radial, the non-tangential, and the tangential maximal functions, which generalize the known results on the anisotropic Hardy spaces of Bownik.

Keywords: anisotropy; Hardy space; continuous ellipsoid cover; maximal function

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1. Introduction

As a generalization of the classical isotropic Hardy spaces $H^p(\mathbb{R}^n)$ [1], anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$ were introduced and investigated by Bownik [2] in 2003. These spaces were defined on $\mathbb{R}^n$, associated with a fixed expansive matrix, which acts on an ellipsoid instead of Euclidean balls. In [3–8], many authors also studied Bownik's anisotropic Hardy spaces. In 2011, Dekel et al. [9] further generalized Bownik's spaces by constructing Hardy spaces with pointwise variable anisotropy $H^p(\Theta)$, $0 < p \leq 1$, associated with an ellipsoid cover Θ. The anisotropy in Bownik's Hardy spaces is the same one at each point in $\mathbb{R}^n$, while the anisotropy in $H^p(\Theta)$ can change rapidly from point to point and from level to level. Moreover, the ellipsoid cover Θ is a very general setting that includes the classical isotropic setting, non-isotropic setting of Calderón and Torchinsky [10], and the anisotropic setting of Bownik [2] as special cases; see more details in ([2], pp. 2–3) and ([11], p. 157).

On the other hand, maximal function characterizations are very fundamental characterizations of Hardy spaces, and they are crucial to conveniently apply the real-variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$. Maximal function characterizations were first shown for the classical isotropic Hardy spaces $H^p(\mathbb{R}^n)$ by Fefferman and Stein in their fundamental work [1], ([12], Chapter III). Analogous results were shown by Calderón and Torchinsky [10,13] for parabolic H^p spaces and Uchiyama [14] for H^p on a homogeneous-type space. In 2003, Bownik ([2], p. 42) obtained the maximal function characterizations of the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$. This was further extended to anisotropic Hardy spaces of the Musielak–Orlicz type in [15], to anisotropic Hardy–Lorentz spaces in [16], to variable anisotropic Hardy spaces in [17], and to anisotropic mixed-norm Hardy spaces in [18].

Motivated by the abovementioned facts, a natural question arises: Do the maximal function characterizations still hold for Hardy spaces $H^p(\Theta)$ with variable anisotropy? In this article, we answer this question affirmatively in the sense that the ellipsoids in Θ

can change shape rapidly from level to level, which is a variable anisotropic extension of Bownik's [2].

This article is organized as follows.

In Section 2, we recall some notation and definitions concerning anisotropic continuous ellipsoid cover Θ, several maximal functions, and anisotropic Hardy spaces $H^p(\Theta)$ defined via the grand radial maximal function. We also give some propositions about $H^p(\Theta)$, several classes of variable anisotropic maximal functions, and Schwartz functions since they provide tools for further work. In Section 3, we first state the main result: if the ellipsoids in Θ can rapidly change shape from level to level (see Definition 1), denoted as Θ_t, we may obtain some real-variable characterizations of $H^p(\Theta_t)$ in terms of the radial, the non-tangential, and the tangential maximal functions (see Theorem 1). Then, we present several lemmas that are isotropic extensions in the setting of variable anisotropy, and finally, we show the proof for the main result.

In the process of proving the main result, we used the methods from Stein [1] and Bownik [2]. However, it is worth pointing out that these ellipsoids of Bownik were images of the unit ball by powers of a fixed expansive matrix, whereas in our case, the ellipsoids of Dekel are images of the unit ball by powers of a group of matrices satisfying some "shape condition". This makes the proof complicated and needs many subtle estimates such as Propositions 5 and 6, and Lemma 1.

However, this article left an open question: if the maximal function characterizations of $H^p(\Theta)$ still hold true in the sense that the ellipsoids of Θ change rapidly from level to level and from point to point?

Finally, we note some conventions on notation. Let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\lceil t \rceil$ be the smallest integer no less than t. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. Throughout the whole paper, we denote by C a positive constant that is independent on the main parameters but may vary from line to line. For any sets E, $F \subset \mathbb{R}^n$, we use $E^{\complement}$ to denote the set $\mathbb{R}^n \setminus E$. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. Denote by $\mathcal{S}$ the space of all Schwartz functions and $\mathcal{S}'$ the space of all tempered distributions.

2. Preliminary and Some Basic Propositions

In this section, we first recall the notion of continuous ellipsoid covers Θ and we introduce the pointwise continuity for Θ. An *ellipsoid* ξ in $\mathbb{R}^n$ is an image of the Euclidean unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| < 1\}$ under an affine transform, i.e.,

$$\xi := M_\xi(\mathbb{B}^n) + c_\xi,$$

where M_ξ is a non-singular matrix and $c_\xi \in \mathbb{R}^n$ is the center.

Let us begin with the definition of continuous ellipsoid covers, which was introduced in ([11], Definition 2.4).

Definition 1. *We say that*

$$\Theta := \{\theta(x, t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

is a continuous ellipsoid cover of $\mathbb{R}^n$ or, in short, an ellipsoid cover if there exist positive constants $p(\Theta) := \{a_1, \ldots, a_6\}$ such that

(i) For every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, there exists an ellipsoid $\theta(x, t) := M_{x,t}(\mathbb{B}^n) + x$ satisfying

$$a_1 2^{-t} \le |\theta(x, t)| \le a_2 2^{-t}. \tag{1}$$

(ii) Intersecting ellipsoids from Θ satisfy a "shape condition", i.e., for any x, $y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $s \ge 0$, if $\theta(x, t) \cap \theta(y, t + s) \neq \emptyset$, then

$$a_3 2^{-a_4 s} \le \frac{1}{\|(M_{y,t+s})^{-1} M_{x,t}\|} \le \|(M_{x,t})^{-1} M_{y,t+s}\| \le a_5 2^{-a_6 s}. \tag{2}$$

where $\|\cdot\|$ *is the matrix norm given by* $\|M\| := \max_{|x|=1} |Mx|$ *for an* $n \times n$ *real matrix* M*.*

Particularly, for any $\theta(x,t) \in \Theta$*, when the related matrix function* $M_{x,t}$ *of* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$ *is reduced to the matrix function* M_t *of* $t \in \mathbb{R}$*, we call a cover* Θ *a t-continuous ellipsoid cover, denoted as* Θ_t*.*

The word continuous refers to the fact that ellipsoids $\theta x, t$ *are defined for all values of* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$*, and we say that a continuous ellipsoid cover* Θ *is pointwise continuous if, for every* $t \in \mathbb{R}$*, the matrix valued function* $x \mapsto M_{x,t}$ *is continuous:*

$$\|M_{x',t} - M_{x,t}\| \to 0 \ \text{ as } \ x' \to x. \tag{3}$$

Remark 1. *By ([19], Theorem 2.2), we know that the pointwise continuous assumption is not necessary since it is always possible to construct an equivalent ellipsoid cover*

$$\Xi := \{\zeta x, t : x \in \mathbb{R}^n, t \in \mathbb{R}\}$$

such that Ξ *is pointwise continuous and* Ξ *is equivalent to* Θ*. Here, we say that two ellipsoid covers* Θ *and* Ξ *are equivalent if there exists a constant* $C > 0$ *such that, for any* $x \in \mathbb{R}^n$ *and* $t \in \mathbb{R}$*, we have*

$$\frac{1}{C}\zeta x, t \subset \theta x, t \subset C\zeta x, t.$$

Taking $M_{y,\,t+s} = M_{x,\,t}$ in (2), we have

$$a_3 \le 1 \ \text{ and } \ a_5 \ge 1. \tag{4}$$

For more properties about ellipsoid covers, see [9,11].

For any N, $\widetilde{N} \in \mathbb{N}_0$ with $N \le \widetilde{N}$, let

$$\mathcal{S}_{N,\widetilde{N}} := \left\{ \psi \in \mathcal{S} : \ \|\psi\|_{\mathcal{S}_{N,\widetilde{N}}} := \max_{\alpha \in \mathbb{N}_0^n,\, |\alpha| \le N} \sup_{y \in \mathbb{R}^n} (1 + |y|)^{\widetilde{N}} |\partial^\alpha \psi(y)| \le 1 \right\}.$$

For any $\varphi \in \mathcal{S}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\theta(x,\,t) = M_{x,\,t}(\mathbb{B}^n) + x$, denote

$$\varphi_{x,\,t}(y) := \left|\det(M_{x,t}^{-1})\right| \varphi(M_{x,\,t}^{-1} y), \quad y \in \mathbb{R}^n.$$

Particularly, when the matrix $M_{x,\,t}$ is reduced to M_t, $\varphi_{x,\,t}(y)$ is simply denoted as $\varphi_t(y)$.

Now, we give the notions of anisotropic variants of the non-tangential, the grand non-tangential, the radial, the grand radial, and the tangential maximal functions.

Definition 2. *Let* $f \in \mathcal{S}'$*,* $\varphi \in \mathcal{S}$ *and* N*,* $\widetilde{N} \in \mathbb{N}_0$ *with* $N \le \widetilde{N}$*. We define the non-tangential, the grand non-tangential, the radial, the rand radial, and the tangential maximal functions, respectively as*

$$M_\varphi f(x) := \sup_{t \in \mathbb{R}} \sup_{y \in \theta(x,\,t)} |f * \varphi_{x,\,t}(y)|, \quad x \in \mathbb{R}^n,$$

$$M_{N,\widetilde{N}} f(x) := \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}} M_\varphi f(x), \quad x \in \mathbb{R}^n,$$

$$M_\varphi^0 f(x) := \sup_{t \in \mathbb{R}} |f * \varphi_{x,\,t}(x)|, \quad x \in \mathbb{R}^n,$$

$$M_{N,\widetilde{N}}^0 f(x) := \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}} M_\varphi^\circ f(x), \quad x \in \mathbb{R}^n,$$

$$T_\varphi^N f(x) := \sup_{t \in \mathbb{R}} \sup_{y \in \mathbb{R}^n} |f * \varphi_{x,\,t}(y)| \left(1 + \left|M_{x,\,t}^{-1}(x - y)\right|\right)^{-N}, \quad x \in \mathbb{R}^n.$$

Here and hereafter, the symbol "$*$" always represents a convolution.

Remark 2. *We immediately have the following pointwise estimate among the radial, the non-tangential, and the tangential maximal functions:*

$$M_\varphi^0 f(x) \le M_\varphi f(x) \le 2^N\, T_\varphi^N f(x), \quad x \in \mathbb{R}^n.$$

Next, we recall the definition of Hardy spaces with pointwise variable anisotropy ([9], Definition 3.6) via the grand radial maximal function.

Let Θ be an ellipsoid cover of $\mathbb{R}^n$ with parameters $p(\Theta) = \{a_1, \cdots, a_6\}$ and $0 < p \le 1$. We define $N_p(\Theta)$ as the minimal integer satisfying

$$N_p := N_p(\Theta) > \frac{\max(1, a_4)n + 1}{a_6 p}, \tag{5}$$

and then $\widetilde{N}_p(\Theta)$ as the minimal integer satisfying

$$\widetilde{N}_p := \widetilde{N}_p(\Theta) > \frac{a_4 N_p(\Theta) + 1}{a_6}. \tag{6}$$

Definition 3. *Let Θ be a continuous ellipsoid cover and $0 < p \le 1$. Define $M^0 := M^0_{N_p, \widetilde{N}_p}$, and the anisotropic Hardy space is defined as*

$$H^p_{N_p, \widetilde{N}_p}(\Theta) := \{f \in \mathcal{S}' : M^0 f \in L^p\}$$

with the (quasi-)norm $\|f\|_{H^p(\Theta)} := \|M^0 f\|_{L^p}$.

Remark 3. *By Remark 1, we know that, for every continuous ellipsoid cover Θ, there exists an equivalent pointwise continuous ellipsoid cover Ξ. This implies that their corresponding (quasi-)norms $\rho_\Theta(\cdot,\cdot)$ and $\rho_\Xi(\cdot,\cdot)$ are also equivalent, and hence, the corresponding Hardy spaces $H^p(\Theta) = H^p(\Xi) (0 < p \le 1)$ with equivalent (quasi-)norms (see ([9], Theorem 5.8)). Therefore, here and hereafter, we always consider Θ of $H^p(\Theta)$ to be a pointwise continuous ellipsoid cover.*

Proposition 1. *Let Θ be an ellipsoid cover, $0 < p \le 1 \le q \le \infty$, $p < q$ and $l \ge N_p$ with N_p as in (5). If $N \ge N_p$ and $\widetilde{N} \ge (a_4 N + 1)/a_6$, then*

$$H^p_{N_p, \widetilde{N}_p}(\Theta) = H^p_{q,l}(\Theta) = H^p_{N, \widetilde{N}}(\Theta)$$

with equivalent (quasi-)norms, where $H^p_{q,l}(\Theta)$ denotes the atomic Hardy space with pointwise variable anisotropy; see ([9], Definition 4.2).

Proof. This proposition is a corollary of ([9], Theorems 4.4 and 4.19). Indeed, by Definition 3, we obtain that, for any $N \ge N_p$ and $\widetilde{N} \ge (a_4 N + 1)/a_6$,

$$H^p_{N_p, \widetilde{N}_p}(\Theta) \subseteq H^p_{N, \widetilde{N}}(\Theta).$$

Combining this and $H^p_{q,l}(\Theta) \subseteq H^p_{N_p, \widetilde{N}_p}(\Theta)$ (see ([9], Theorem 4.4)), we obtain

$$H^p_{q,l}(\Theta) \subseteq H^p_{N, \widetilde{N}}(\Theta). \tag{7}$$

By checking the definition of anisotropic (p, q, l)-atom (see ([9], Definition 4.1)), we know that every (p, ∞, l)-atom is also a (p, q, l)-atom and hence

$$H^p_{\infty, l}(\Theta) \subseteq H^p_{q,l}(\Theta).$$

Let $l' \geq \max(l,\, N)$. By a similar argument to the proof of ([9], Theorem 4.19), we obtain

$$H^p_{N,\widetilde{N}}(\Theta) \subseteq H^p_{\infty,l'}(\Theta),$$

where $N \geq N_p$ and $\widetilde{N} \geq (a_4 N + 1)/a_6$. Thus,

$$H^p_{N,\widetilde{N}}(\Theta) \subseteq H^p_{\infty,l'}(\Theta) \subseteq H^p_{\infty,l}(\Theta) \subseteq H^p_{q,l}(\Theta). \tag{8}$$

Combining (7) and (8), we conclude that

$$H^p_{N_p,\widetilde{N}_p}(\Theta) = H^p_{q,l}(\Theta) = H^p_{N,\widetilde{N}}(\Theta)$$

with equivalent (quasi-)norms. □

Remark 4. *From Proposition 1, we deduce that, for any integers $N \geq N_p$ and $\widetilde{N} \geq (a_4 N + 1)/a_6$, the definition of $H^p_{N,\widetilde{N}}(\Theta)$ is independent of N and $\widetilde{N}$. Therefore, from now on, we denote $H^p_{N,\widetilde{N}}(\Theta)$ with $N \geq N_p$ and $\widetilde{N} \geq (a_4 N + 1)/a_6$ simply by $H^p(\Theta)$.*

Proposition 2 ([9], Lemma 2.3)**.** *Let Θ be an ellipsoid cover. Then, there exists a constant $J := J(p(\Theta)) \geq 1$ such that, for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,*

$$2M_{x,t}(\mathbb{B}) + x \subset \theta(x, t - J).$$

Here and hereafter, let J always be as in Proposition 2.

Definition 4 ([9], Definition 3.1)**.** *Let Θ be an ellipsoid cover. For any locally integrable function f, the maximal function of the Hardy–Littlewood type of f is defined by*

$$M_\Theta f(x) := \sup_{t\in\mathbb{R}} \frac{1}{|\theta(x,t)|} \int_{\theta(x,t)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

Proposition 3 ([9], Theorem 3.3)**.** *Let Θ be an ellipsoid cover. Then,*

(i) *There exists a constant C depending only on $p(\Theta)$ and n such that for all $f \in L^1$ and $\alpha > 0$,*

$$|\{x : M_\Theta f(x) > \alpha\}| \leq C\alpha^{-1}\|f\|_{L^1}; \tag{9}$$

(ii) *For $1 < p < \infty$, there exists a constant C_p depending only on C and p such that, for all $f \in L^p$,*

$$\|M_\Theta f\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{10}$$

We give some useful results about variable anisotropic maximal functions with different apertures. They also play important roles in obtaining the maximal function characterizations of $H^p(\Theta)$. For any given $x \in \mathbb{R}^n$, suppose that $F : \mathbb{R}^n \times \mathbb{R} \to (0,\, \infty)$ is a Lebesgue measurable function. Let Θ be an ellipsoid cover. For fixed $l \in \mathbb{Z}$ and $t_0 < 0$, define the maximal function of F with aperture l as

$$F_l^{*\,t_0}(x) := \sup_{t \geq t_0} \sup_{y \in \theta(x, t-lJ)} F(y,t). \tag{11}$$

Proposition 4. *For any $l \in \mathbb{Z}$ and $t_0 < 0$, let $F_l^{*\,t_0}$ be as in (11). If the ellipsoid cover Θ is pointwise continuous, then $F_l^{*\,t_0} : \mathbb{R}^n \to (0, \infty]$ is lower semi-continuous, i.e.,*

$$\{x \in \mathbb{R}^n : F_l^{*\,t_0}(x) > \lambda\} \text{ is open for any } \lambda > 0.$$

Proof. If $F_l^{*\,t_0}(x) > \lambda$ for some $x \in \mathbb{R}^n$, then there exist $t \geq t_0$ and $y \in \theta(x,\, t - lJ)$ such that $F(y,t) > \lambda$. Since $\theta(x,t)$ is continuous for variable x (see Remark 1), there exists $\delta_1 > 0$ such that, for any $x' \in U(x,\delta) := \{z \in \mathbb{R}^n : |z - x| < \delta\}$, $y \in \theta(x',\, t - lJ)$ and hence $F_l^{*\,t_0}(x') > \lambda$. □

By Proposition 4, we obtain that $\{x \in \mathbb{R}^n : F_l^{*\,t_0}(x) > \lambda\}$ is Lebesgue measurable. Based on this and inspired by ([2], Lemma 7.2), the following Proposition 5 shows some estimates for maximal function $F_l^{*\,t_0}$.

Proposition 5. *Let Θ be an ellipsoid cover, $F_l^{*\,t_0}$ and $F_{l'}^{*\,t_0}$ as in (11) with integers $l > l'$ and $t_0 < 0$. Then, there exists a constant $C > 0$ that depends on parameters $p(\Theta)$ such that, for any functions $F_l^{*\,t_0}$, $F_{l'}^{*\,t_0}$ and $\lambda > 0$, we have*

$$\left|\left\{x \in \mathbb{R}^n : F_l^{*\,t_0}(x) > \lambda\right\}\right| \leq C 2^{(l-l')J} \left|\left\{x \in \mathbb{R}^n : F_{l'}^{*\,t_0}(x) > \lambda\right\}\right| \tag{12}$$

and

$$\int_{\mathbb{R}^n} F_l^{*\,t_0}(x)\,dx \leq C 2^{(l-l')J} \int_{\mathbb{R}^n} F_{l'}^{*\,t_0}(x)\,dx. \tag{13}$$

Proof. Let $\Omega := \{x \in \mathbb{R}^n : F_{l'}^{*\,t_0}(x) > \lambda\}$. We claim that

$$\left\{x \in \mathbb{R}^n : F_l^{*\,t_0}(x) > \lambda\right\} \subset \left\{x \in \mathbb{R}^n : M_\Theta(\chi_\Omega)(x) \geq C_1 2^{(l'-l)J}\right\}, \tag{14}$$

where C_1 is a positive constant to be fixed later. Assuming that the claim holds for the moment, from this and a weak type (1,1) of M_Θ (see (9)), we deduce

$$\begin{aligned}\left|\left\{x \in \mathbb{R}^n : F_l^{*\,t_0}(x) > \lambda\right\}\right| &\leq \left|\left\{x \in \mathbb{R}^n : M_\Theta(\chi_\Omega)(x) \geq C_1 2^{(l'-l)J}\right\}\right| \\ &\leq C_1^{-1} 2^{(l-l')J} \|\chi_\Omega\|_{L^1} \leq C 2^{(l-l')J} |\Omega|\end{aligned}$$

and hence (12) holds true, where $C := 1/C_1$. Furthermore, integrating (12) on $(0,\infty)$ with respect to λ yields (13). Therefore, (14) remains to be shown.

Suppose $F_l^{*\,t_0}(x) > \lambda$ for some $x \in \mathbb{R}^n$. Then, there exist t with $t \geq t_0$ and $y \in \theta(x,\, t - lJ)$ such that $F(y,\, t) > \lambda$. For any $l, l' \in \mathbb{Z}$ and $l \geq l'$, we first prove that the following holds true:

$$a_5{}^{-1}\,\theta(y,\, t - l'J) \subseteq \theta(x,\, t - (l+1)J) \cap \Omega. \tag{15}$$

For any $z \in a_5{}^{-1}\,\theta(y,\, t - l'J)$, by (4), we have $z \in \theta(y,\, t - l'J)$ and hence

$$\theta(z,\, t - l'J) \cap \theta(y,\, t - l'J) \neq \emptyset.$$

Thus, by (2), we have

$$\left\| M_{z,\,t-l'J}^{-1}\, M_{y,\,t-l'J} \right\| \leq a_5.$$

From this, it follows that

$$a_5{}^{-1} M_{z,\,t-l'J}^{-1}\, M_{y,\,t-l'J}(\mathbb{B}^n) \subseteq \mathbb{B}^n$$

and hence

$$a_5{}^{-1}\, M_{y,\,t-l'J}(\mathbb{B}^n) \subseteq M_{z,\,t-l'J}(\mathbb{B}^n).$$

By this and $y \in a_5^{-1} M_{y,t-l'J}(\mathbb{B}^n) + z$, we obtain $y \in \theta(z, t - l'J)$. From this and $F(y,t) > \lambda$ with $t \geq t_0$, we deduce that $F_{l'}^{*\,t_0}(z) > \lambda$, and hence, $z \in \Omega$, which implies

$$a_5^{-1}\,\theta(y,\, t - l'J) \subseteq \Omega. \tag{16}$$

Moreover, by $y \in \theta(x,\, t - lJ)$, (2), and $l \geq l'$, we have

$$\left\| M_{x,\,t-lJ}^{-1}\, M_{y,\,t-l'J} \right\| \leq a_5 2^{-a_6(l-l')J} \leq a_5.$$

From this, it follows that

$$a_5{}^{-1} M_{x,\,t-lJ}^{-1}\, M_{y,\,t-l'J}(\mathbb{B}^n) \subseteq \mathbb{B}^n$$

and hence

$$a_5{}^{-1}\, M_{y,\,t-l'J}(\mathbb{B}^n) \subseteq M_{x,\,t-lJ}(\mathbb{B}^n).$$

By this, (4), $y \in \theta(x,\, t - lJ)$, and Proposition 2, we obtain

$$a_5{}^{-1} M_{y,\,t-l'J}(\mathbb{B}^n) + y \subseteq 2M_{x,\,t-lJ}(\mathbb{B}^n) + x \subseteq \theta(x,\, t - (l+1)J).$$

From this and (16), we deduce that (15) holds true.

Next, let us prove (14). By (15) and (1), we obtain

$$\begin{aligned} |\theta(x,\, t - (l+1)J) \cap \Omega| &\geq (a_5)^{-n} |\theta(y,\, t - l'J)| \\ &\geq \frac{a_1}{(a_5)^n} 2^{l'J - t}. \end{aligned} \tag{17}$$

Taking $b_0 := t - (l+1)J$, by (1) and (17), we have

$$\frac{1}{|\theta(x,\, b_0)|} \int_{\theta(x,\, b_0)} |\chi_\Omega(y)|dy \geq a_2{}^{-1} 2^{b_0} |\theta(x,\, b_0) \cap \Omega| \geq \frac{a_1}{(a_5)^n a_2} 2^{(l'-l-1)J},$$

which implies $M_\Theta(\chi_\Omega)(x) \geq C_1 2^{(l'-l)J}$ and hence (14) holds true, where $C_1 := 2^{-J} a_1 / [(a_5)^n a_2]$. $\square$

The following result enables us to pass from one function in $\mathcal{S}$ to the sum of dilates of another function in $\mathcal{S}$ with nonzero mean, which is a variable anisotropic extension of ([12], p. 93, Lemma 2) of Stein and ([2], Lemma 7.3) of Bownik.

Proposition 6. *Let Θ be an ellipsoid cover of $\mathbb{R}^n$ and $\varphi \in \mathcal{S}$, with $\int_{\mathbb{R}^n} \varphi(x)\, dx \neq 0$. Then, for any $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$, and $t \in \mathbb{R}$, there exists a sequence $\{\eta^k\}_{k=0}^\infty$ and $\eta^k \in \mathcal{S}$, such that*

$$\psi = \sum_{k=0}^\infty \eta^k * \varphi^k \tag{18}$$

converges in $\mathcal{S}$, where

$$\varphi^k := |\det(M_{x,\,t+kJ}^{-1} M_{x,t})| \varphi(M_{x,\,t+kJ}^{-1} M_{x,t} \cdot), \quad k > 0,$$

where $J > 0$ is as in Proposition 2.

Furthermore, for any positive integers N, $\widetilde{N}$ and L, there exists a constant $C > 0$ depending on φ, L, N, $\widetilde{N}$, and $p(\Theta)$ but not ψ, such that

$$\|\eta^k\|_{\mathcal{S}_{N,\widetilde{N}}} \leq C 2^{-kL} \|\psi\|_{\mathcal{S}_{N+n+1+\lceil L/(a_6 J) \rceil,\, \widetilde{N}+n+1}}. \tag{19}$$

Proof. The following simplified proof is accomplished by Dekel. By scaling φ, we can assume that $\int_{\mathbb{R}^n}\varphi(x)dx=1$ and $|\widehat{\varphi}(\xi)|\ge 1/2$, for $|\xi|\le 2$. This assumption only impacts the constant in (19). Let $\zeta\in\mathcal{S}$ such that $0\le\zeta\le 1$ on $\mathbb{B}^n$ and supp $(\zeta)\subset 2\mathbb{B}^n$. We fix $x\in\mathbb{R}^n$ and $t\in\mathbb{R}$, denote $M_k:=M_{x,t+kJ}$, and define the sequence of functions $\{\zeta_k\}_{k=0}^{\infty}$, where $\zeta_0:=\zeta$, and

$$\zeta_k:=\zeta\left(\left(M_{x,t}^{-1}M_k\right)^T\cdot\right)-\zeta\left(\left(M_{x,t}^{-1}M_{k-1}\right)^T\cdot\right),\quad k\ge 1,$$

where M^T denotes the transpose of a matrix M. We claim that

$$\operatorname{supp}(\zeta_k)\subset\left\{\xi\in\mathbb{R}^n:a_5^{-1}2^{-a_6J}2^{a_6kJ}\le|\xi|\le 2a_3^{-1}2^{a_4kJ}\right\}.\tag{20}$$

Indeed, by the properties of ζ, Proposition 2 and (2),

$$\begin{aligned}\xi\in\operatorname{supp}(\zeta_k)&\Rightarrow\left(M_{x,t}^{-1}M_k\right)^T(\xi)\in 2\mathbb{B}^n\vee\left(M_{x,t}^{-1}M_{k-1}\right)^T(\xi)\in 2\mathbb{B}^n\\&\Rightarrow\xi\in 2\left(M_k^{-1}M_{x,t}\right)^T(\mathbb{B}^n)\vee\xi\in 2\left(M_{k-1}^{-1}M_{x,t}\right)^T(\mathbb{B}^n)\\&\Rightarrow\xi\in 2a_3^{-1}2^{a_4kJ}\mathbb{B}^n.\end{aligned}$$

In the other direction, Proposition 2 and the properties of ζ yield

$$\begin{aligned}\xi\in\left(M_{k-1}^{-1}M_{x,t}\right)^T(\mathbb{B}^n)&\Rightarrow\left(M_{x,t}^{-1}M_k\right)^T(\xi)\in\mathbb{B}^n,\left(M_{x,t}^{-1}M_{k-1}\right)^T(\xi)\in\mathbb{B}^n\\&\Rightarrow\zeta_k(\xi)=0.\end{aligned}$$

Applying (2), we have

$$\xi\notin\left(M_{k-1}^{-1}M_{x,t}\right)^T(\mathbb{B}^n)\Rightarrow|\xi|\ge 2a_5^{-1}2^{a_6(k-1)J}.$$

This proves (20). Additionally, by (2), for any $\xi\in\mathbb{R}^n$,

$$\left|\left(M_{x,t}^{-1}M_k\right)^T\xi\right|\le\left\|M_{x,t}^{-1}M_k\right\||\xi|\le a_5 2^{-a_6kJ}|\xi|\to 0,\quad k\to\infty.$$

From this, we deduce that, for any $\xi\in\mathbb{R}^n$, for a large enough k, $(M_{x,t}^{-1}M_k)^T\xi\in\mathbb{B}^n$. This implies that

$$\sum_{k=0}^{\infty}\zeta_k(\xi)=1,\quad\forall\xi\in\mathbb{R}^n.$$

Thus, formally, a Fourier transform of (18) is given by

$$\widehat{\psi}=\sum_{k=0}^{\infty}\widehat{\eta^k}\widehat{\varphi}\left(\left(M_{x,t}^{-1}M_k\right)^T\cdot\right),\quad\widehat{\eta^k}:=\frac{\zeta_k}{\widehat{\varphi}\left(\left(M_{x,t}^{-1}M_k\right)^T\cdot\right)}\widehat{\psi}.$$

Observe that η^k is well defined and in $\mathcal{S}$. Indeed, $\widehat{\eta^k}$ is well defined with $0/0:=0$, since by our assumption on φ,

$$\begin{aligned}\xi\in supp(\zeta_k)&\Rightarrow\xi\in 2\left(M_k^{-1}M_{x,t}\right)^T(\mathbb{B}^n)\\&\Rightarrow\left|\left(M_{x,t}^{-1}M_k\right)^T\xi\right|\le 2\\&\Rightarrow\widehat{\varphi}\left(\left(M_{x,t}^{-1}M_k\right)^T\xi\right)\ge\frac{1}{2}.\end{aligned}$$

From this, it is obvious that $\widehat{\eta^k} \in \mathcal{S}$, and therefore, $\eta^k \in \mathcal{S}$. We now proceed to prove (19). First, observe that, for any $\eta \in \mathcal{S}$, N, $\widetilde{N} \in \mathbb{N}$,

$$\|\eta\|_{\mathcal{S}_{N,\widetilde{N}}} \le C(N, \widetilde{N}, n)\|\widehat{\eta}\|_{\mathcal{S}_{\widetilde{N},N+n+1}}. \tag{21}$$

Next, we claim that, for any $K \in \mathbb{N}$,

$$\max_{|\alpha|\le K}\left\|\partial^\alpha\left(\zeta_k/\widehat{\varphi}\left(\left(M_k^{-1}M_{x,t}\right)^T\cdot\right)\right)\right\|_\infty \le C(K, n, \varphi). \tag{22}$$

Indeed, on its support, any partial derivative of $\zeta_k/\widehat{\varphi}((M_{x,t}^{-1}M_k)^T\cdot)$ has a denominator with its absolute value bounded from below and a numerator that is a superposition of compositions of partial derivatives of η and φ with contractive matrices of the type $(M_{x,t}^{-1}M_k)^T$. Using (20)–(22), we obtain

$$\begin{aligned}
\left\|\eta^k\right\|_{\mathcal{S}_{N,\widetilde{N}}} &\le C\left\|\widehat{\eta^k}\right\|_{\mathcal{S}_{\widetilde{N},N+n+1}} \\
&\le C \sup_{|\xi|\ge a_5^{-1}2^{-a_6 J}2^{a_6 kJ}} \max_{|\alpha|\le\widetilde{N}}\left|\partial^\alpha\widehat{\eta^k}(\xi)\right|(1+|\xi|)^{N+n+1} \\
&\le C \sup_{|\xi|\ge a_5^{-1}2^{-a_6 J}2^{a_6 kJ}} \max_{|\alpha|\le\widetilde{N}}\left|\partial^\alpha\widehat{\psi}(\xi)\right|(1+|\xi|)^{N+n+1} \\
&\le C \sup_{|\xi|\ge a_5^{-1}2^{-a_6 J}2^{a_6 kJ}} \max_{|\alpha|\le\widetilde{N}}\left|\partial^\alpha\widehat{\psi}(\xi)\right|(1+|\xi|)^{N+n+1+\lceil L/(a_6 J)\rceil} \\
&\qquad \times (1+|\xi|)^{-\lceil L/(a_6 J)\rceil} \\
&\le C2^{-kL}\|\widehat{\psi}\|_{\mathcal{S}_{\widetilde{N},N+n+1+\lceil L/(a_6 J)\rceil}} \\
&\le C2^{-kL}\|\psi\|_{\mathcal{S}_{N+n+1+\lceil L/(a_6 J)\rceil,\widetilde{N}+n+1}}.
\end{aligned}$$

□

3. Maximal Function Characterizations of $H^p(\Theta_t)$

In this section, we show the maximal function characterizations of $H^p(\Theta_t)$ using the radial, the non-tangential, and the tangential maximal functions of a single test function $\varphi \in \mathcal{S}$.

Theorem 1. *Let Θ_t be a t-continuous ellipsoid cover, $0 < p \le 1$, and $\varphi \in \mathcal{S}$ satisfy $\int_{\mathbb{R}^n}\varphi(x)\,dx \ne 0$. Then, for any $f \in \mathcal{S}'$, the following are mutually equivalent:*

$$f \in H^p(\Theta_t); \tag{23}$$

$$M_\varphi f \in L^p; \tag{24}$$

$$M_\varphi^0 f \in L^p; \tag{25}$$

$$T_\varphi^N f \in L^p, \quad N > \frac{1}{a_6 p}. \tag{26}$$

In this case,

$$\|f\|_{H^p(\Theta_t)} = \left\|M^0 f\right\|_{L^p} \le C_1\left\|T_\varphi^N f\right\|_{L^p} \le C_2\|M_\varphi f\|_{L^p} \le C_3\left\|M_\varphi^0 f\right\|_{L^p} \le C_4\|f\|_{H^p(\Theta_t)},$$

where the positive constants C_1, C_2, C_3 and C_4 are independent of f.

The framework to prove Theorem 1 is motivated by Fefferman and Stein [1], ([12], Chapter III), and Bownik ([2], p. 42, Theorem 7.1).

Inspired by Fefferman and Stein ([12], p. 97), and Bownik ([2], p. 47), we now start with maximal functions obtained from truncation with an additional extra decay term. Namely, for $t_0 < 0$ representing the truncation level and real number $L \geq 0$ representing the decay level, we define the *radial*, the *non-tangential*, the *tangential*, the *grand radial*, and the *grand non-tangential maximal functions*, respectively, as

$$M_\varphi^{0\,(t_0,L)} f(x) := \sup_{t \geq t_0} |(f * \varphi_{x,t})(x)| \left(1 + \left|M_{x,t_0}^{-1}\, x\right|\right)^{-L} (1 + 2^{t+t_0})^{-L},$$

$$M_\varphi^{(t_0,L)} f(x) := \sup_{t \geq t_0} \sup_{y \in \theta(x,t)} |(f * \varphi_{x,t})(y)| \left(1 + \left|M_{x,t_0}^{-1}\, y\right|\right)^{-L} (1 + 2^{t+t_0})^{-L},$$

$$T_\varphi^{N\,(t_0,L)} f(x) := \sup_{t \geq t_0} \sup_{y \in \mathbb{R}^n} \frac{|(f * \varphi_{x,t})(y)|}{\left[1 + \left|M_{x,t}^{-1}\,(x-y)\right|\right]^N} \frac{1}{(1+2^{t+t_0})^L \left(1 + \left|M_{x,t_0}^{-1}\, y\right|\right)^L},$$

$$M_{N,\widetilde{N}}^{0\,(t_0,L)} f(x) := \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}} M_\varphi^{0\,(t_0,L)} f(x)$$

and

$$M_{N,\widetilde{N}}^{(t_0,L)} f(x) := \sup_{\varphi \in \mathcal{S}_{N,\widetilde{N}}} M_\varphi^{(t_0,L)} f(x).$$

The following Lemma 1 guarantees control of the tangential by the non-tangential maximal function in $L^p(\mathbb{R}^n)$ independent of t_0 and L.

Lemma 1. *Let Θ_t be a t-continuous ellipsoid cover. Suppose $p > 0$, $N > 1/(a_6\, p)$, and $\varphi \in \mathcal{S}$. Then, there exists a positive constant C such that, for any $t_0 < 0$, $L \geq 0$ and $f \in \mathcal{S}'$,*

$$\left\| T_\varphi^{N\,(t_0,L)} f \right\|_{L^p} \leq C \left\| M_\varphi^{(t_0,L)} f \right\|_{L^p}.$$

Proof. Consider the function $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow [0, \infty)$ given by

$$F(y,\, t) := |(f * \varphi_t)(y)|^p \left(1 + \left|M_{t_0}^{-1}\, y\right|\right)^{-pL} (1 + 2^{t+t_0})^{-pL}.$$

Let $F_l^{*\,t_0}$ be as in (11) with $l = 0$. When $y \in \theta(x,\, t)$, we have $M_t^{-1}\,(x-y) \in \mathbb{B}^n$ and hence $|M_t^{-1}\,(x-y)| < 1$. If $t \geq t_0$, then

$$F(y,\, t)\left[1 + \left|M_t^{-1}\,(x-y)\right|\right]^{-pN} \leq F_0^{*\,t_0}(x).$$

When $y \in \theta(x,\, t-kJ) \backslash \theta(x,\, t-(k-1)J)$ for some $k \geq 1$, we have

$$M_t^{-1}\,(x-y) \notin M_t^{-1}\, M_{t-(k-1)J}\,(\mathbb{B}^n). \tag{27}$$

By (2), we obtain

$$\left\| M_{t-(k-1)J}^{-1}\, M_t \right\| \leq a_5 2^{-a_6(k-1)J}$$

and hence,

$$M_{t-(k-1)J}^{-1}\, M_t(\mathbb{B}^n) \subseteq a_5 2^{-a_6(k-1)J}\mathbb{B}^n,$$

which implies

$$(2^{a_6(k-1)J}/a_5)\mathbb{B}^n \subseteq M_t^{-1}\, M_{t-(k-1)J}(\mathbb{B}^n).$$

From this and (27), it follows that $|M_t^{-1}(x-y)| \geq 2^{a_6(k-1)J}/a_5$. Thus, for any $t \geq t_0$, we have

$$F(y,t)\left[1+\left|M_t^{-1}(x-y)\right|\right]^{-pN} \leq a_5{}^{pN} 2^{-pNa_6(k-1)J} F_k^{*\,t_0}(x).$$

By taking the supremum over all $y \in \mathbb{R}^n$ and $t \geq t_0$, we know that

$$\left[T_\varphi^{N\,(t_0,L)} f(x)\right]^p \leq a_5{}^{pN} \sum_{k=0}^{\infty} 2^{-pNa_6(k-1)J} F_k^{*\,t_0}(x).$$

Therefore, using this and Proposition 5, we obtain

$$\begin{aligned}
\left\|T_\varphi^{N\,(t_0,L)} f\right\|_{L^p(\mathbb{R}^n)}^p &\leq a_5{}^{pN} \sum_{k=0}^{\infty} 2^{-pNa_6(k-1)J} \int_{\mathbb{R}^n} F_k^{*\,t_0}(x)dx \\
&\leq Ca_5{}^{pN} \sum_{k=0}^{\infty} 2^{-pNa_6(k-1)J} 2^{kJ} \int_{\mathbb{R}^n} F_0^{*\,t_0}(x)dx \\
&= C'\left\|M_\varphi^{(t_0,L)} f\right\|_{L^p(\mathbb{R}^n)}^p,
\end{aligned}$$

where $C' := Ca_5{}^{pN} 2^{pNa_6J} \sum_{k=0}^{\infty} 2^{(1-pNa_6)kJ} = Ca_5{}^{pN} 2^J/(1-2^{(1-pNa_6)J})$. $\square$

The following Lemma 2 gives the pointwise majorization of the grand radial maximal function by the tangential one, which is a variable anisotropic extension of ([2], Lemma 7.5).

Lemma 2. *Let Θ be an ellipsoid cover of $\mathbb{R}^n$, $\varphi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$, and $f \in \mathcal{S}'$. For any given positive integers N and L, there exist integers $0 < U \leq \widetilde{U}$, $U \geq N_p$, and $\widetilde{U} \geq \widetilde{N_p}$ that are large enough and constant $C > 0$ such that, for any $t_0 < 0$,*

$$M_{U,\widetilde{U}}^{0\,(t_0,L)} f(x) \leq CT_\varphi^{N\,(t_0,L)} f(x), \quad \forall x \in \mathbb{R}^n.$$

Proof. The simplified proof of this final version is from Dekel (Lemma 6.20). By Proposition 6, for any $\psi \in \mathcal{S}$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, there exists a sequence $\{\eta^k\}_{k=0}^{\infty}$, $\eta^k \in \mathcal{S}$ that satisfies

$$\psi = \sum_{k=0}^{\infty} \eta^k * \varphi^k$$

converging in $\mathcal{S}$, where

$$\varphi^k := |\det(M_{x,t+kJ}^{-1} M_{x,t})|\varphi(M_{x,t+kJ}^{-1} M_{x,t}\cdot), \quad k \geq 0.$$

Furthermore, for any positive integers U, $\widetilde{U}$ and V,

$$\|\eta^k\|_{\mathcal{S}_{U,\widetilde{U}}} \leq C2^{-kV} \|\psi\|_{\mathcal{S}_{U+n+1+\lceil V/(a_6J)\rceil,\widetilde{U}+n+1}}. \tag{28}$$

where the constant depends on φ, U, $\widetilde{U}$, V, $p(\Theta)$ but not ψ. Denoting $M_k := M_{x,t+kJ}$, for $t \geq t_0$, implies

$$
\begin{aligned}
|(f*\psi_{x,t})(x)| &= \left|\left[f*\sum_{k=0}^{\infty}\left(\eta^k*\varphi^k\right)_{x,t}\right](x)\right| \\
&\le C\left|\left[f*\sum_{k=0}^{\infty}\left|\det\left(M_k^{-1}\right)\right|\int_{\mathbb{R}^n}\eta^k(y)\,\varphi\left(M_k^{-1}(\cdot-M_{x,t}y)\right)dy\right](x)\right| \\
&= C\left|\left[f*\sum_{k=0}^{\infty}\left|\det\left(M_k^{-1}M_{x,t}^{-1}\right)\right|\int_{\mathbb{R}^n}\eta^k\left(M_{x,t}^{-1}y\right)\varphi\left(M_k^{-1}(\cdot-y)\right)dy\right](x)\right| \\
&\le C\sum_{k=0}^{\infty}\left|\left[f*\left(\eta^k\right)_{x,t}*\varphi_{x,t+kJ}\right](x)\right| \\
&\le C\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}|f*\varphi_{x,t+kJ}(x-y)|\left|\left(\eta^k\right)_{x,t}(y)\right|dy \\
&\le CT_{\varphi}^{N\,(t_0,L)}f(x)\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}\left(1+\left|M_k^{-1}y\right|\right)^N \\
&\quad\times\left(1+\left|M_{x,t_0}^{-1}(x-y)\right|\right)^L\left(1+2^{t+t_0+kJ}\right)^L\left|\left(\eta^k\right)_{x,t}(y)\right|dy.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
M_{\psi}^{0\,(t_0,L)}f(x) &\le T_{\varphi}^{N\,(t_0,L)}f(x)\sup_{t\ge t_0}\sum_{k=0}^{\infty}\int_{\mathbb{R}^n}\left(1+\left|M_k^{-1}y\right|\right)^N \\
&\quad\times\frac{\left(1+\left|M_{x,t_0}^{-1}(x-y)\right|\right)^L\left(1+2^{t+t_0+kJ}\right)^L}{\left(1+\left|M_{x,t_0}^{-1}x\right|\right)^L(1+2^{t+t_0})^L}\left|\left(\eta^k\right)_{x,t}(y)\right|dy \\
&=: T_{\varphi}^{N\,(t_0,L)}f(x)\sup_{t\ge t_0}\sum_{k=0}^{\infty}\mathrm{I}_{t,k}.
\end{aligned}
\tag{29}
$$

Let us now estimate $\mathrm{I}_{t,k}$ for $t\ge t_0$, $k\ge 0$. We begin with the simple observations that

$$
\frac{1+2^{t+t_0+kJ}}{1+2^{t+t_0}}=\frac{2^{kJ}(2^{-kJ}+2^{t+t_0})}{1+2^{t+t_0}}\le C2^{kJ}
$$

and

$$
1+|x+y|\le 1+|x|+|y|\le(1+|x|)(1+|y|),\ \ x,\,y\in\mathbb{R}^n. \tag{30}
$$

Therefore, we may obtain

$$
\begin{aligned}
\mathrm{I}_{t,k} &\le C2^{t+kJL}\int_{\mathbb{R}^n}\left(1+\left|M_k^{-1}y\right|\right)^N\left(1+\left|M_{x,t_0}^{-1}y\right|\right)^L\left|\eta^k\left(M_{x,t}^{-1}y\right)\right|dy \\
&\le C2^{kJL}\int_{\mathbb{R}^n}\left(1+\left\|M_k^{-1}M_{x,t}\right\|\,|y|\right)^N\left(1+\left\|M_{x,t_0}^{-1}M_{x,t}\right\|\,|y|\right)^L\left|\eta^k(y)\right|dy,
\end{aligned}
$$

which, together with

$$
\|M_k^{-1}M_{x,t}\|\le a_3 2^{a_4 kJ}\ \text{ and }\ \|M_{x,t_0}^{-1}M_{x,t}\|\le a_5 2^{-a_6(t-t_0)}\le a_5\ (\text{by }\ t\ge t_0 \text{ and } (2)),
$$

further implies that

$$\begin{aligned} \mathrm{I}_{t,k} &\leq C2^{kJ(L+a_4N)} \int_{\mathbb{R}^n} (1+|y|)^{N+L} \left|\eta^k(y)\right| dy \\ &\leq C2^{kJ(L+a_4N)} \left\|\eta^k\right\|_{\mathcal{S}_{0,\widetilde{N}+n+L}}. \end{aligned} \tag{31}$$

We now apply (28) with $V := \lceil J(L+a_4N) \rceil + 1$, which gives

$$\mathrm{I}_{t,k} \leq C2^{-kV} \|\psi\|_{\mathcal{S}_{n+1+\lceil V/(a_6J)\rceil,\, N+L+2n+2}}. \tag{32}$$

This yields for any $\psi \in \mathcal{S}_{U,\widetilde{U}}$, $U := \max(N_p,\, n+1+\lceil V/(a_6J)\rceil)$, $\widetilde{U} := \max(\widetilde{N}_p,\, N+L+2n+2)$

$$M^{0\,(t_0,L)}_{U,\widetilde{U}} f(x) = \sup_{\psi\in\mathcal{S}_{U,\widetilde{U}}} M^{0\,(t_0,L)}_{\psi} f(x) \leq C\, T^{N\,(t_0,L)}_{\varphi} f(x).$$

This finishes the proof of Lemma 2. □

The following Lemma 3 shows that the radial and the grand non-tangential maximal functions are pointwise equivalent, which is a variable anisotropic extension of ([2], Proposition 3.10).

Lemma 3 ([19], Theorem 3.4)**.** *For any $N,\, \widetilde{N} \in \mathbb{N}$ with $N \leq \widetilde{N}$, there exists a positive constant $C := C(\widetilde{N})$ such that, for any $f \in \mathcal{S}'$,*

$$M^0_{N,\widetilde{N}} f(x) \leq M_{N,\widetilde{N}} f(x) \leq C M^0_{N,\widetilde{N}} f(x), \quad x \in \mathbb{R}^n.$$

The following Lemma 4 is a variable anisotropic extension of ([2], p. 46, Lemma 7.6).

Lemma 4. *Let Θ_t be a t-continuous ellipsoid cover, $\varphi \in \mathcal{S}$, and $f \in \mathcal{S}'$. Then, for every $M > 0$ and $t_0 < 0$, there exist $L > 0$ and $N' > 0$ large enough such that*

$$M^{(t_0,L)}_{\varphi} f(x) \leq C2^{-t_0(2a_4N'+2L+a_4L)} (1+|x|)^{-M}, \quad x \in \mathbb{R}^n, \tag{33}$$

where C is a positive constant dependent on $p(\Theta)$, N', f, and φ.

Proof. For any $\varphi \in \mathcal{S}$, there exist an integer $N > 0$ and positive constant $C := C(\varphi)$ such that, for any $N' \geq N$ and $y \in \mathbb{R}^n$,

$$|f * \varphi(y)| \leq C\|\varphi\|_{\mathcal{S}_{N,N'}} (1+|y|)^{N'}. \tag{34}$$

Therefore, for any $t_0 < 0$, $t \geq t_0$ and $x \in \mathbb{R}^n$, by (34), we have

$$\begin{aligned} &|(f * \varphi_t)(y)| \left(1+\left|M^{-1}_{t_0} y\right|\right)^{-L} (1+2^{t+t_0})^{-L} \\ &\leq C2^{-L(t+t_0)} \|\varphi_t\|_{\mathcal{S}_{N,N'}} (1+|y|)^{N'} \left(1+\left|M^{-1}_{t_0} y\right|\right)^{-L}. \end{aligned} \tag{35}$$

Let us first estimate $\|\varphi_t\|_{\mathcal{S}_{N,N'}}$. By the chain rule and (1), we have

$$\begin{aligned}\|\varphi_t\|_{\mathcal{S}_{N,N'}} &= |\det M_t^{-1}| \sup_{z\in\mathbb{R}^n} \sup_{|\alpha|\le N} (1+|z|)^{N'} \left|\partial^\alpha \left(\varphi\left(M_t^{-1}\cdot\right)\right)(z)\right| \\ &\le C2^t \sup_{z\in\mathbb{R}^n} \sup_{|\alpha|\le N} (1+|z|)^{N'} \left\|M_t^{-1}\right\|^{|\alpha|} \left|(\partial^\alpha\varphi)\left(M_t^{-1}z\right)\right| \\ &\le C2^t \sup_{z\in\mathbb{R}^n} \sup_{|\alpha|\le N} (1+|M_t z|)^{N'} \left\|M_t^{-1}\right\|^{|\alpha|} |\partial^\alpha \varphi(z)|. \end{aligned} \tag{36}$$

Now, let us further estimate (36) in the following two cases.
Case 1: $t \ge 0$. By (2), we have

$$\left\|M_t^{-1}\right\| = \left\|M_t^{-1}M_0M_0^{-1}\right\| \le \left\|M_t^{-1}M_0\right\|\left\|M_0^{-1}\right\| \le \left\|M_0^{-1}\right\| a_3^{-1}2^{a_4 t} = C2^{a_4 t}$$

and

$$\begin{aligned}|M_t z| &= \left|M_0M_0^{-1}M_t z\right| \le \|M_0\|\left|M_0^{-1}M_t z\right| \le \|M_0\|\left\|M_0^{-1}M_t\right\||z| \\ &\le \|M_0\| a_5 2^{-a_6 t}|z| \le C|z|.\end{aligned}$$

Inserting the above two estimates into (36) with $t \ge 0$, we know that

$$\begin{aligned}\|\varphi_t\|_{\mathcal{S}_{N,N'}} &\le C2^t \sup_{z\in\mathbb{R}^n} \sup_{|\alpha|\le N} (1+|M_t z|)^{N'} \left\|M_t^{-1}\right\|^{|\alpha|} |\partial^\alpha \varphi(z)| \\ &\le C2^t 2^{a_4 tN} \|\varphi\|_{\mathcal{S}_{N,N'}}.\end{aligned} \tag{37}$$

Case 2: $t_0 \le t < 0$. By (2), we have

$$\left\|M_t^{-1}\right\| = \left\|M_t^{-1}M_0M_0^{-1}\right\| \le \left\|M_t^{-1}M_0\right\|\left\|M_0^{-1}\right\| \le \left\|M_0^{-1}\right\| a_5 2^{a_6 t} \le C$$

and

$$\begin{aligned}|M_t z| &= \left|M_0M_0^{-1}M_t z\right| \le \|M_0\|\left|M_0^{-1}M_t z\right| \le \|M_0\|\left\|M_0^{-1}M_t\right\||z| \\ &\le \|M_0\| a_3^{-1} 2^{-a_4 t}|z| = C2^{-a_4 t_0}|z|.\end{aligned}$$

Inserting the above two estimates into (36) with $t_0 \le t < 0$, we know that

$$\begin{aligned}\|\varphi_t\|_{\mathcal{S}_{N,N'}} &\le C2^t \sup_{z\in\mathbb{R}^n} \sup_{|\alpha|\le N} (1+|M_t z|)^{N'} \left\|M_t^{-1}\right\|^{|\alpha|} |\partial^\alpha \varphi(z)| \\ &\le C2^{-a_4 t_0 N'} \|\varphi\|_{\mathcal{S}_{N,N'}}.\end{aligned} \tag{38}$$

For any $M > 0$, let $L := M + N'$. For any $t_0 < 0$, $t \ge t_0$ and taking some integer $N' > 0$ large enough, by (37) and (38), we obtain

$$2^{-L(t+t_0)} \|\varphi_t\|_{\mathcal{S}_{N,N'}} \le C2^{-t_0(a_4 N'+2L)} \|\varphi\|_{\mathcal{S}_{N,N'}}. \tag{39}$$

Inserting (39) into (35), we further obtain

$$\begin{aligned}&|(f*\varphi_t)(y)| \left(1+\left|M_{t_0}^{-1}y\right|\right)^{-L} \left(1+2^{t+t_0}\right)^{-L} \\ &\le C2^{-t_0(a_4 N'+2L)} \|\varphi\|_{\mathcal{S}_{N,N'}} (1+|y|)^{N'} \left(1+\left|M_{t_0}^{-1}y\right|\right)^{-L}.\end{aligned} \tag{40}$$

For any $y \in \theta(x, t)$, there exists $z \in \mathbb{B}^n$ such that $y = x + M_t z$. By (30), we have

$$1 + |y| = 1 + |x + M_t z| \le (1 + |x|)(1 + |M_t z|). \tag{41}$$

If $t \ge 0$, by (2), then

$$\begin{aligned}|M_t z| &= \left|M_0 M_0^{-1} M_t z\right| \le \|M_0\| \left|M_0^{-1} M_t z\right| \le \|M_0\| \left\|M_0^{-1} M_t\right\| |z| \\ &\le \|M_0\| a_5 2^{-a_6 t} |z| \le C.\end{aligned}$$

If $t_0 \le t < 0$, by (2), then

$$\begin{aligned}|M_t z| &= \left|M_0 M_0^{-1} M_t z\right| \le \|M_0\| \left|M_0^{-1} M_t z\right| \le \|M_0\| \left\|M_0^{-1} M_t\right\| |z| \\ &\le \|M_0\| a_3^{-1} 2^{-a_4 t} |z| = C 2^{-a_4 t_0}.\end{aligned}$$

Therefore, for any $t \ge t_0$, by using the above two estimates, we have

$$|M_t z| \le C 2^{-a_4 t_0}.$$

From this and (41), it follows that

$$(1 + |y|) \le C 2^{-a_4 t_0} (1 + |x|). \tag{42}$$

Moreover, for any $t_0 < 0$, by (2), we have

$$1 + |x| \le 1 + \|M_0\| \left\|M_0^{-1} M_{t_0}\right\| \left|M_{t_0}^{-1} x\right| \le C 2^{-a_4 t_0} \left(1 + \left|M_{t_0}^{-1} x\right|\right).$$

Furthermore, for any $y \in \theta(x, t)$, we have $x \in M_t(\mathbb{B}^n) + y$. Thus, there exists $z \in \mathbb{B}^n$ such that $x = M_t z + y$. Hence, for any $t \ge t_0$, by (30) and (2), we obtain

$$\begin{aligned}\left(1 + \left|M_{t_0}^{-1} x\right|\right) &= \left(1 + \left|M_{t_0}^{-1}(y + M_t z)\right|\right) \le \left(1 + \left|M_{t_0}^{-1} y\right|\right)\left(1 + \left\|M_{t_0}^{-1} M_t\right\| |z|\right) \\ &\le \left(1 + \left|M_{t_0}^{-1} y\right|\right)\left(1 + a_5 2^{-a_6(t - t_0)} |z|\right) \le C\left(1 + \left|M_{t_0}^{-1} y\right|\right).\end{aligned}$$

Combining with the above two inequalities, we have

$$(1 + |M_{t_0}^{-1} y|) \ge C 2^{a_4 t_0} (1 + |x|). \tag{43}$$

Thus, for any $t \ge t_0$ and $y \in \theta(x, t)$, inserting (42) and (43) into (40) with $L = M + N'$, we obtain

$$|(f * \varphi_t)(y)| \left(1 + \left|M_{t_0}^{-1} y\right|\right)^{-L} \left(1 + 2^{t + t_0}\right)^{-L} \le C 2^{-t_0(2a_4 N' + 2L + a_4 L)} (1 + |x|)^{-M},$$

which implies that (33) holds true and hence completes the proof of Lemma 4. $\square$

Note that the above argument gives the same estimate for the truncated grand maximal function $M_{N,\tilde{N}}^{0\,(t_0, L)} f(x)$. As a consequence of Lemma 4, we obtain that, for any choice of $t_0 < 0$ and any $f \in \mathcal{S}'$, we can find an appropriate $L > 0$ so that the maximal function, say $M_\varphi^{(t_0, L)} f$, is bounded and belongs to $L^p(\mathbb{R}^n)$. This becomes crucial in the proof of Theorem 1, where we work with truncated maximal functions, The complexity of the preceding argument stems from the fact that, a priori, we do not know whether $M_\varphi^0 f \in L^p$ implies $M_\varphi f \in L^p$. Instead, we must work with variants of maximal functions for which this is satisfied.

Proof of Theorem 1. Suppose that Θ_t is a t-continuous ellipsoid cover and $\varphi \in \mathcal{S}$ satisfying $\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0$. From Remark 2 and the definition of the grand radial maximal function, it follows that

$$(26) \Rightarrow (24) \Rightarrow (25)$$

and

$$(23) \Rightarrow (25).$$

By Lemma 1 applied for $L = 0$, we have

$$\left\| T_\varphi^{N(t_0,0)} f \right\|_{L^p} \leq C \left\| M_\varphi^{(t_0,0)} f \right\|_{L^p} \quad \text{for any } f \in \mathcal{S}' \text{ and } t_0 < 0.$$

As $t_0 \to -\infty$, by the monotone convergence theorem, we obtain

$$\left\| T_\varphi^N f \right\|_{L^p} \leq C \| M_\varphi f \|_{L^p},$$

which shows (24) $\Rightarrow$ (26).

Combining Lemma 2 applied for $N > 1/(a_6 p)$ and $L = 0$ and Lemma 1 applied for $L = 0$, we conclude that there exist integers $0 < U \leq \widetilde{U}$, $U > N_p$, $\widetilde{U} \geq \widetilde{N_p}$ that are large enough and a positive constant C such that

$$\left\| M_{U,\widetilde{U}}^{0(t_0,0)} f \right\|_{L^p} \leq C \left\| M_\varphi^{(t_0,0)} f \right\|_{L^p} \quad \text{for any } f \in \mathcal{S}' \text{ and } t_0 < 0.$$

As $t_0 \to -\infty$, by the monotone convergence theorem, we obtain

$$\left\| M_{U,\widetilde{U}}^0 f \right\|_{L^p} \leq C \| M_\varphi f \|_{L^p}.$$

From this and Proposition 1, we deduce that

$$\|f\|_{H^p(\Theta_t)} = \left\| M_{N_p,\widetilde{N}_p}^0 f \right\|_{L^p} \leq C \left\| M_{U,\widetilde{U}}^0 f \right\|_{L^p} \leq C \| M_\varphi f \|_{L^p}$$

and hence (24) $\Rightarrow$ (23). (25) $\Rightarrow$ (24) remain to be shown.

Suppose now $M_\varphi^\circ f \in L^p$. By Lemma 4, we can find a $L > 0$ large enough such that (33) holds true, which implies $M_\varphi^{(t_0,L)} f \in L^p$ for all $t_0 < 0$. Combining Lemmas 1 and 2, we obtain that there exist $0 < U \leq \widetilde{U}$, $U > N_p$, and $\widetilde{U} \geq \widetilde{N_p}$ large enough such that

$$\left\| M_{U,\widetilde{U}}^{0(t_0,L)} f \right\|_p \leq C_1 \left\| M_\varphi^{(t_0,L)} f \right\|_p, \tag{44}$$

where constant C_1 is independent of $t_0 < 0$. For a given $t_0 < 0$, let

$$\Omega_{t_0} := \left\{ x \in \mathbb{R}^n : M_{U,\widetilde{U}}^{0\,(t_0,L)} f(x) \leq C_2 M_\varphi^{(t_0,L)} f(x) \right\}, \tag{45}$$

where $C_2 := 2^{1/p} C_1$. We claim that

$$\int_{\mathbb{R}^n} \left[M_\varphi^{(t_0,L)} f(x) \right]^p dx \leq 2 \int_{\Omega_{t_0}} \left[M_\varphi^{(t_0,L)} f(x) \right]^p dx. \tag{46}$$

Indeed, this follows from (44), $M_\varphi^{(t_0,L)} f \in L^p$ and

$$\int_{\Omega_{t_0}^c} \left[M_\varphi^{(t_0,L)} f(x) \right]^p dx \leq C_2^{-p} \int_{\Omega_{t_0}^c} \left[M_{U,\widetilde{U}}^{0\,(t_0,L)} f(x) \right]^p dx \leq (C_1/C_2)^p \int_{\mathbb{R}^n} \left[M_\varphi^{(t_0,L)} f(x) \right]^p dx,$$

where $(C_1/C_2)^p = 1/2$.

We also claim that, for $0 < q < p$, there exists a constant $C_3 > 0$ such that, for any $t_0 < 0$,

$$M_\varphi^{(t_0,L)} f(x) \le C_3 \Big[M_\Theta \Big(M_\varphi^{0\,(t_0,L)} f \Big)^q (x) \Big]^{1/q}, \tag{47}$$

where M_Θ is as in Definition 4. Indeed, let $t \ge t_0$, $y \in \theta(x,t)$ and

$$F(y,\,t) := |(f * \varphi_t)(y)|\,(1 + |M_{t_0}^{-1} y|)^{-L} (1 + 2^{t+t_0})^{-L}.$$

Suppose that $x \in \Omega_{t_0}$ and let $F_l^{*\,t_0}(x)$ be as in (11) with $l = 0$. Then, there exist $t' \in \mathbb{R}$ with $t' \ge t_0$ and $y' \in \theta(x,\,t')$ such that

$$F(y',\,t') \ge F_0^{*\,t_0}(x)/2 = M_\varphi^{(t_0,L)} f(x)/2. \tag{48}$$

Consider $x' \in y' + M_{t'+lJ}(\mathbb{B}^n)$ for some integer $l \ge 1$ to be specified later. Let $\Phi(z) := \varphi\Big(z + M_{t'}^{-1}(x' - y')\Big) - \varphi(z)$. Obviously, we have

$$f * \varphi_{t'}(x') - f * \varphi_{t'}(y') = f * \Phi_{t'}(y'). \tag{49}$$

Let us first estimate $\|\Phi\|_{\mathcal{S}_{U,\tilde{U}}}$. From $x' \in y' + M_{t'+lJ}(\mathbb{B}^n)$, we deduce that

$$M_{t'}^{-1}(x' - y') \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n).$$

By this and the mean value theorem, we obtain

$$\begin{aligned} \|\Phi\|_{\mathcal{S}_{U,\tilde{U}}} &\le \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} \|\varphi(\cdot + h) - \varphi(\cdot)\|_{\mathcal{S}_{U,\tilde{U}}} \\ &= \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le U} (1 + |z|)^{\tilde{U}} |(\partial^\alpha \varphi)(z+h) - \partial^\alpha \varphi(z)| \\ &\le C \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le U+1} (1 + |z|)^{\tilde{U}} |(\partial^\alpha \varphi)(z+h)| \\ &\quad \times \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} |h|. \end{aligned} \tag{50}$$

From (2), we deduce

$$\|M_{t'}^{-1} M_{t'+lJ}\| \le a_5 2^{-a_6 lJ},$$

which implies

$$M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n) \subset a_5 2^{-a_6 lJ} \mathbb{B}^n.$$

By this and $h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)$, we have $|h| \le a_5 2^{-a_6 lJ}$. From this and (30), we deduce that

$$1 + |z| \le (1 + |z+h|)(1 + |h|) \le C(1 + |z+h|),\ \ z \in \mathbb{R}^n.$$

Applying this and $|h| \le a_5 2^{-a_6 lJ}$ in (50), we obtain

$$\begin{aligned} \|\Phi\|_{\mathcal{S}_{U,\tilde{U}}} &\le C \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} \sup_{z \in \mathbb{R}^n} \sup_{|\alpha| \le U+1} (1 + |z+h|)^{\tilde{U}} |(\partial^\alpha \varphi)(z+h)| \\ &\quad \times \sup_{h \in M_{t'}^{-1} M_{t'+lJ}(\mathbb{B}^n)} |h| \le C \|\varphi\|_{\mathcal{S}_{U+1,\tilde{U}}} a_5 2^{-a_6 lJ} \le C_4 2^{-a_6 lJ}, \end{aligned} \tag{51}$$

where a positive constant C_4 does not depend on L.

Moreover, notice that, for any $x' \in M_{t'+lJ}(\mathbb{B}^n) + y'$, there exists $z \in \mathbb{B}^n$ such that $x' = M_{t'+lJ}z + y'$. By (30), (2), and $t' \geq t_0$, we have

$$\begin{aligned}\left(1+\left|M_{t_0}^{-1}x'\right|\right) &\leq \left(1+\left|M_{t_0}^{-1}y'\right|\right)\left(1+\left\|M_{t_0}^{-1}M_{t'+lJ}\right\||z|\right) \\ &\leq \left(1+\left|M_{t_0}^{-1}y'\right|\right)\left(1+a_5 2^{-a_6(t'-t_0+lJ)}|z|\right) \leq 2a_5\left(1+\left|M_{t_0}^{-1}y'\right|\right).\end{aligned} \tag{52}$$

Thus, for any $x \in \Omega_{t_0}$, from (49), (52), (48), (51), Lemma 3, and (45), it follows that

$$\begin{aligned}2^L a_5^L F(x', t') &= 2^L a_5^L \left[|(f * \varphi_{t'})(x')|\,(1+|M_{t_0}^{-1}x'|)^{-L}(1+2^{t'+t_0})^{-L}\right] \\ &\geq [|f*\varphi_{t'}(y')| - |f*\Phi_{t'}(y')|]\left(1+\left|M_{t_0}^{-1}y'\right|\right)^{-L}\left(1+2^{t'+t_0}\right)^{-L} \\ &\geq F(y', t') - M_{U,\tilde{U}}^{(t_0,L)}f(x)\|\Phi\|_{\mathcal{S}_{U,\tilde{U}}} \\ &\geq M_{\varphi}^{(t_0,L)}f(x)/2 - C_4 2^{-a_6 lJ} C M_{U,\tilde{U}}^{0\,(t_0,L)}f(x) \\ &\geq M_{\varphi}^{(t_0,L)}f(x)/2 - C_4C_2C2^{-a_6 lJ}M_{\varphi}^{(t_0,L)}f(x).\end{aligned}$$

We choose an integer $l \geq 1$ large enough such that $C_4C_2C2^{-a_6 lJ} \leq 1/4$. Therefore, for any $x \in \Omega_{t_0}$ and $x' \in M_{t'+lJ}(\mathbb{B}^n) + y'$, we further have

$$2^L a_5^L F(x', t') \geq M_{\varphi}^{(t_0,L)}f(x)/2 - C_4C_2C2^{-a_6 lJ}M_{\varphi}^{(t_0,L)}f(x) \geq M_{\varphi}^{(t_0,L)}f(x)/4. \tag{53}$$

Moreover, by $y' \in \theta(x, t')$ and Proposition 2, we have

$$\begin{aligned}M_{t'+lJ}(\mathbb{B}^n) + y' &\subseteq M_{t'+lJ}(\mathbb{B}^n) + M_{t'}(\mathbb{B}^n) + x \\ &\subseteq 2M_{t'}(\mathbb{B}^n) + x \subseteq \theta(x, t'-J).\end{aligned} \tag{54}$$

Thus, for any $x \in \Omega_{t_0}$ and $t \geq t_0$, by (53) and (54), we obtain

$$\begin{aligned}\left[M_{\varphi}^{(t_0,L)}f(x)\right]^q &\leq \frac{4^q 2^{Lq} a_5^{Lq}}{|M_{t'+lJ}(\mathbb{B}^n)|}\int_{y'+M_{t'+lJ}(\mathbb{B}^n)}[F(z,t')]^q dz \\ &\leq C4^q 2^{Lq} a_5^{Lq}\frac{2^{(l+1)J}}{|\theta(x, t'-J)|}\int_{\theta(x,t'-J)}\left[M_{\varphi}^{0\,(t_0,L)}f(z)\right]^q dz \\ &\leq C_3 M_{\Theta}\left(\left(M_{\varphi}^{0\,(t_0,L)}f\right)^q\right)(x),\end{aligned}$$

which shows the above claim (47).

Consequently, by (46), (47), and Proposition 3 with $p/q > 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left[M_{\varphi}^{(t_0,L)}f(x)\right]^p dx &\leq 2\int_{\Omega_{t_0}}\left[M_{\varphi}^{(t_0,L)}f(x)\right]^p dx \\ &\leq 2C_3{}^p\int_{\Omega_{t_0}}\left[M_{\Theta}\left(\left(M_{\varphi}^{0(t_0,L)}f\right)^q\right)(x)\right]^{p/q}dx \\ &\leq C_5\int_{\mathbb{R}^n}\left[M_{\varphi}^{0(t_0,L)}f(x)\right]^p dx,\end{aligned} \tag{55}$$

where the constant C_5 depends on $p/q > 1$, $L \geq 0$ and $p(\Theta)$ but is independent of $t_0 < 0$. This inequality is crucial as it gives a bound of the non-tangential by the radial maximal function in L^p. The rest of the proof is immediate.

For any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $t < 0$, by (2), we obtain

$$\left|M_t^{-1}y\right| = \left|M_t^{-1}M_0M_0^{-1}y\right| \leq \left\|M_t^{-1}M_0\right\|\left\|M_0^{-1}\right\||y|$$
$$\leq a_5 2^{a_6 t}\left\|M_0^{-1}\right\||y| \to 0 \text{ as } t \to -\infty.$$

Hence, we obtain that $M_\varphi^{(t_0,L)}f(x)$ converges pointwise and monotonically to $M_\varphi f(x)$ for all $x \in \mathbb{R}^n$ as $t_0 \to -\infty$, which together with (55) and the monotone convergence theorem, further implies that $M_\varphi f \in L^p$. Therefore, we can now choose $L = 0$, and again, by (55) and the monotone convergence theorem, we have $\|M_\varphi f\|_p^p \leq C_5\|M_\varphi^0 f\|_p^p$, where C_5 corresponds to $L = 0$ and is independent of $f \in \mathcal{S}'$. This finishes the proof of Theorem 1. □

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