mathematics

Review

Review of the Latest Progress in Controllability of Stochastic Linear Systems and Stochastic GE-Evolution Operator

Zhaoqiang Ge

School of Mathematics and Statistics, Xi'an Jiaotong University, No. 28, Xianning West Road, Xi'an 710049, China; gezqjd@mail.xjtu.edu.cn

Abstract: According to the spatial dimension, equation type, and time sequence, the latest progress in controllability of stochastic linear systems and some unsolved problems are introduced. Firstly, the exact controllability of stochastic linear systems in finite dimensional spaces is discussed. Secondly, the exact, exact null, approximate, approximate null, and partial approximate controllability of stochastic linear systems in infinite dimensional spaces are considered. Thirdly, the exact, exact null and impulse controllability of stochastic singular linear systems in finite dimensional spaces are investigated. Fourthly, the exact and approximate controllability of stochastic singular linear systems in infinite dimensional spaces are studied. At last, the controllability and observability for a type of time-varying stochastic singular linear systems are studied by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces, some necessary and sufficient conditions are obtained, the dual principle is proved to be true, an example is given to illustrate the validity of the theoretical results obtained in this part, and a problem to be solved is introduced. The main purpose of this paper is to facilitate readers to fully understand the latest research results concerning the controllability of stochastic linear systems and the problems that need to be further studied, and attract more scholars to engage in this research.

Keywords: controllability; observability; stochastic linear systems in finite and infinite dimensional spaces; stochastic singular linear systems in finite and infinite dimensional spaces; semigroup; evolution operator; GE-semigroup; GE-evolution operator; stochastic GE-evolution operator

Citation: Ge, Z. Review of the Latest Progress in Controllability of Stochastic Linear Systems and Stochastic GE-Evolution Operator. *Mathematics* **2021**, *9*, 3240. https://doi.org/10.3390/math9243240

Academic Editors: Mikhail Posypkin, Andrey Gorshenin and Vladimir Titarev

Received: 15 November 2021
Accepted: 11 December 2021
Published: 14 December 2021

1. Introduction

Since Kalman published the seminal paper [1], the controllability of stochastic systems has become a central problem in the study of mathematical control theory, a large number of academic papers have been published. For representative papers, see references [1–73]. However, even for the controllability of stochastic linear systems, there are still many important problems to be solved. In this paper, we discuss the latest development of controllability of stochastic linear systems and raise some unsolved issues. According to the spatial dimension, equation type and time sequence, the rest of the paper is organized as follows. In Section 2, the following contents are introduced concerning the controllability of stochastic linear systems in finite dimensional spaces: (i) The L^p-exact controllability and exact observability are discussed; (ii) The exact controllability by feedback controller is considered; (iii) The exact controllability of the stochastic linear systems with memory is investigated; (iv) Some theoretical results for these concepts are given and four important problems to be solved are put forward. In Section 3, the controllability of stochastic linear systems in infinite dimensional spaces is considered: (i) The null controllability is investigated by using C_0-semigroup in the sense of mild solution in Hilbert spaces; (ii) The approximate controllability and approximate null controllability are discussed by using C_0-semigroup in the sense of mild solution in Hilbert spaces; (iii) The partial approximate controllability is studied by using evolution operator in the sense of mild solution in Hilbert spaces; (iv) According to these theories, three problems that need to be studied are raised. In Section 4, the controllability of stochastic singular linear systems in finite dimensional spaces is dealt with: (i) The exact controllability is considered by using Gramian matrix; (ii) The exact null controllability is studied by using Gramian matrix; (iii) The impulse controllability and impulse observability are investigated in the sense of impulse solution; (iv) A problem that needs to be discussed is put forward. In Section 5, the controllability of stochastic singular linear systems in infinite dimensional spaces is studied: (i) The exact controllability for a type of time invariant systems is considered by using C_0-semigroup in the sense of strong solution in Hilbert spaces; (ii) The exact controllability and approximate controllability for a type of time invariant systems are investigated by using GE-semigroup in the sense of mild solution in Banach and Hilbert spaces, respectively; (iii) The exact controllability and approximate controllability for a type of time-varying systems are dealt with by using GE-evolution operator in the sense of mild solution in Hilbert spaces; (iv) The exact controllability and approximate controllability for a type of time invariant systems are considered by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces; (v) The exact controllability, approximate controllability, exact observability, and approximate observability for a type of time-varying systems are studied by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces. Some necessary and sufficient conditions concerning these concepts are obtained, the dual principle is proved to be true, an example is given to illustrate the validity of the theoretical results obtained in this part, and a problem to be solved is raised.

The main idea of this paper is to introduce the latest progress for the controllability of stochastic linear systems and the mathematical methods applied in this field, including GE-semigroup, GE-evolution operator, stochastic GE-evolution operator and so on. The main purpose of this paper is to facilitate readers to fully understand the latest research results concerning the controllability of stochastic linear systems and the problems that need to be further studied, and attract more scholars to engage in this research.

Notations. $(\Omega, F, \{F_t\}, P)$ is a complete probability space with filtration $\{F_t\}$ satisfying the usual condition (i.e., the filtration contains all $P-$null sets and is right continuous); all processes are $\{F_t\}-$adapted; $w(t)$ is a standard Wiener process defined on $(\Omega, F, \{F_t\}, P)$; E denotes the mathematical expectation; $\mathbb{R}^n$ is the $n-$dimensional real Euclidean space with the standard norm $\|\cdot\|_{\mathbb{R}^n}$, $\mathbb{R}^{n\times m}$ is the space of all $(n\times m)$ real matrices; $I_n \in \mathbb{R}^{n\times n}$ denotes the identical matrix; T denotes the transpose of a vector or a matrix; $H = \mathbb{R}^n, \mathbb{R}^{n\times m}$, etc, and $p \in [1,\infty)$; $L^p([0,\tau];H)$ denotes the set of all functions $f : [0,\tau] \to H$ satisfying $\|f(\cdot)\|_{L^p([0,\tau];H)} = (\int_0^\tau \|f(t)\|_H^p dt)^{1/p} < \infty$; $L^\infty([0,\tau];H)$ denotes the subset of $L^p([0,\tau];H)$ whose element is essentially bounded; $C([0,\tau];H)$ denotes the set of all functions $f : [0,\tau] \to H$, which are continuous on $[0,\tau]$ in the sense of $\|f(\cdot)\|_{C([0,\tau];H)} = \max_{t\in[0,\tau]}\|f(t)\|_H$; $L^p(\Omega, F_t, P, H)$ denotes the set of all random variables $\eta \in H$, such that η is F_t-measurable and $\|\eta\|_p = (E(\|\eta\|_H^p))^{1/p} < +\infty$; $L^p([0,\tau],\Omega,F_t,H)$ denotes the set of all processes $x(t) \in H$ such that $\|x(t)\|_p < +\infty, \forall t \in [0,\tau]$; $L^p([0,\tau],\Omega,H)$ denotes the set of all processes $x(t) \in L^p([0,\tau],\Omega,F_t,H)$ such that $E\int_0^\tau \|x(t)\|_H^p d\tau < +\infty$; $L^\infty([0,\tau],\Omega,H)$ is the subset of $L^2([0,\tau],\Omega,H)$ where each element $x(\cdot)$ is essentially bounded; Let A be a linear operator. $\mathrm{dom}(A), \ker(A)$ and $\mathrm{ran}(A)$ denote its domain, kernel and range, respectively; I denotes the identical operator. Other mathematical symbols involved in this paper will be properly explained in the discussion.

2. Exact Controllability of Finite Dimensional Stochastic Linear Systems

In this section, we discuss the latest development of exact controllability of finite dimensional stochastic linear systems.

2.1. L^p-Exact Controllability

In 2017, Wang et al. consider the controllability of the following stochastic linear differential equation in [59]:

$$dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum_{k=1}^{d}[C_k(t)x(t) + D_k(t)u(t)]dw_k(t), t \geq 0, \tag{1}$$

where A, C_k: $[0,\tau] \times \Omega \to \mathbb{R}^{n\times n}$ and B, D_k: $[0,\tau] \times \Omega \to \mathbb{R}^{n\times m}(k = 1, 2, \cdots, d)$ are suitable matrix-valued processes; $x(t)$ is the state process valued in $\mathbb{R}^n$ and $u(t)$ is the control process valued in $\mathbb{R}^m$; $\{w_k(t)$: $(k = 1, 2, \cdots, d)\}$ is a system of independent one-dimensional standard Wiener processes, $w(t) = (w_1(t), \cdots, w_d(t))$. We will denote system (1) by $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$, with $C(\cdot) = (C_1(\cdot), \cdots, C_d(\cdot))$ and $D(\cdot) = (D_1(\cdot), \cdots, D_d(\cdot))$.

For the convenience of narration, the following notations and concepts are introduced. $L^p_F(\Omega; L^q([0,\tau]; H))$ is the set of all processes $x(\cdot)$ valued in H, such that

$$\|x(\cdot)\|_{L^p_F(\Omega;L^q([0,\tau];H))} = [E(\int_0^\tau \|x(t)\|^q_H dt)^{p/q}]^{1/p} < \infty,$$

$$L^p_F(\Omega; L^p([0,\tau]; H)) = L^p_F([0,\tau]; H), p \in [1, \infty].$$

$L^p_F(\Omega; C([0,\tau]; H))$ is the set of all processes $x(\cdot)$ valued in H, such that for almost $\omega \in \Omega, t \to x(t,\omega)$ is continuous and

$$\|x(\cdot)\|_{L^p_F(\Omega;C([0,\tau];H))} = [E(\sup_{t\in[0,\tau]} \|x(t)\|^p_H)]^{1/p} < \infty.$$

In the similar manner, one can define $L^\infty_F(\Omega; L^\infty([0,\tau]; H))$ and $L^\infty_F(\Omega; C([0,\tau]; H))$.

Hypothesis 1. *The $\mathbb{R}^{n\times n}-$valued processes $A(\cdot), C_k(\cdot)$ satisfy*

$$A(\cdot), C_k(\cdot) \in L^\infty_F(\Omega; L^\infty([0,\tau]; \mathbb{R}^{n\times n}))(k = 1, \cdots, d).$$

Hypothesis 2. *For some $\mu \in (1, \infty]$ and $\sigma \in (2, \infty]$, the following hold:*

$$B(\cdot) \in L^\mu_F(\Omega; L^{\frac{2\sigma}{\sigma+2}}([0,\tau]; \mathbb{R}^{n\times m})), \mu \in (1, \infty], \sigma \in (2, \infty),$$

$$B(\cdot) \in L^\mu_F(\Omega; L^2([0,\tau]; \mathbb{R}^{n\times m})), \mu \in (1, \infty], \sigma = \infty,$$

$$D_1(\cdot), \cdots, D_d(\cdot) \in L^\mu_F(\Omega; L^\sigma([0,\tau]; \mathbb{R}^{n\times m})).$$

Now, we introduce the following definition.

Definition 1. *(i) A process $u(t)(t \in [0,\tau])$ is called a feasible control of system (1) if under $u(t)$, for any $x_0 \in \mathbb{R}^n$, system (1) admits a unique strong solution $x(t) \in L^1_F(\Omega; C([0,\tau]; \mathbb{R}^n))$ satisfying $x(0) = x_0$. The set of feasible controls is denoted by $U[0,\tau]$;*

(ii) A control $u(t) \in U[0,\tau]$ is said to be L^p-feasible for system (1) if

$$p \geq 1, B(\cdot)u(\cdot) \in L^p_F(\Omega; L^1([0,\tau]; \mathbb{R}^n)), D_k(\cdot)u(\cdot) \in L^p_F(\Omega; L^2([0,\tau]; \mathbb{R}^{n\times n}))$$

holds true. The set of L^p-feasible controls is denoted by $U^p[0,\tau]$;

(iii) System (1) is said to be L^p-exactly controllable by $U[0,\tau]$ on $[0,\tau]$, if for any $x_0 \in \mathbb{R}^n$ and $\xi \in L^p(\Omega, F_\tau, P, \mathbb{R}^n)$, there exists a $u(\cdot) \in U[0,\tau]$ such that the solution $x(\cdot) \in L^1_F(\Omega; C([0,\tau]; \mathbb{R}^n))$ of (1) with $x(0) = x_0$ satisfies $x(\tau) = \xi$.

2.1.1. The Case $D(\cdot) = 0$

In this case, we consider system $[A(\cdot), C(\cdot); B(\cdot), 0]$, i.e., the state equation is

$$dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum_{k=1}^{d} C_k(t)x(t)dw_k(t), t \geq 0. \tag{2}$$

Thus, the control $u(\cdot)$ does not appear in the diffusion. The L^p-exact controllability of system (2) was discussed and the following results were obtained in [59].

Theorem 1 ([59])**.** *Let Hypothesis 1 hold. Let*

$$B(t)B(t)^T \geq \delta I_n, t \in [0, \tau], a.s.,$$

for some $\delta > 0$. Then for any $p > 1$, system (2) is L^p-exactly controllable on $[0, \tau]$ by $U^{p-} = \cap_{q \in (0,p)} U^q[0, \tau]$.

Theorem 2 ([59])**.** *Let Hypothesis 1 hold. Suppose there exists a continuous differentiable function $f : [0, \tau] \to \mathbb{R}^n$, $\|f(t)\|_{\mathbb{R}^n} = 1$, for all $t \in [0, \tau]$ such that $f(t)^T B(t) = 0$. Additionally, let*

$$C_k(\cdot) \in L^\infty_F(\Omega; C([0, \tau]; \mathbb{R}^{n \times n})), 1 \leq k \leq d. \tag{3}$$

Then for any $p > 1$, system (2) is not L^p-exactly controllable on $[0, \tau]$ by $U^p[0, \tau]$.

Corollary 1 ([59])**.** *Let Hypothesis 1 and (3) hold. Let $B \in \mathbb{R}^{n \times m}$.*

(i) If for some $p > 1$, system $[A(\cdot), C(\cdot); B, 0]$ is L^p-exactly controllable on $[0, \tau]$ by $U^p[0, \tau]$, then

$$\mathrm{rank} B = n, \tag{4}$$

where rank$B$ *denotes the rank of B;*

(ii) If (4) holds, then for any $p > 1$, system $[A(\cdot), C(\cdot); B, 0]$ is L^p-exactly controllable on $[0, \tau]$ by $U^{p-}[0, \tau]$.

The above result shows that the gap between condition (4) and the L^p-exact controllability of system $[A(\cdot), C(\cdot); B, 0]$ (by $U^p[0, \tau]$, or $U^{p-}[0, \tau]$) is very small.

2.1.2. The Case rank$D(\cdot) = n$

In this case, we let $d = 1$, i.e., the Wiener process is one-dimensional. The case $d > 1$ can be discussed similarly. For system $[A(\cdot), C(\cdot); B(\cdot), D(\cdot)]$, we assume the following:

$$D(t)D(t)^T \geq \delta I_n, a.s., a.e. t \in [0, \tau]. \tag{5}$$

In this case, $[D(t)D(t)^T]^{-1}$ exists and uniformly bounded. We define

$$\tilde{A}(t) = A(t) - B(t)D(t)^T[D(t)D(t)^T]^{-1}C(t),$$

$$\tilde{B}(t) = B(t)\{I_n - D(t)^T[D(t)D(t)^T]^{-1}D(t)\}, \tilde{D}(t) = B(t)D(t)^T[D(t)D(t)^T]^{-1},$$

and introduce the following controlled system:

$$dx(t) = [\tilde{A}(t)x(t) + \tilde{B}(t)v(t) + \tilde{D}(t)z(t)]dt + z(t)dw(t), t \in [0, \tau], x(0) = x_0, \tag{6}$$

with $x(t)$ being the state and $(v(\cdot), z(\cdot))$ being the control. For system (6), we need the following set and definition:

$$\tilde{U}^p[0, \tau] = \{v(\tau) : \tilde{B}(\tau)v(\tau) \in L^p_F(\Omega; L^1([0, \tau]; \mathbb{R}^n)).$$

Definition 2. *System (6) is said to be exactly null-controllable by*

$$\tilde{U}^p[0,\tau]\times L^p_F(\Omega;L^2([0,\tau];\mathbb{R}^n))$$

on the $[0,\tau]$, if for any $x_0\in\mathbb{R}^n$, there exists a pair

$$(v(\cdot),z(\cdot))\in\tilde{U}^p[0,\tau]\times L^p_F(\Omega;L^2([0,\tau];\mathbb{R}^n)),$$

such that the solution $x(\cdot)$ to

$$dx(t)=[\tilde{A}(t)x(t)+\tilde{B}(t)v(t)+\tilde{D}(t)z(t)]dt+z(t)dw(t),t\in[0,\tau],$$
$$x(0)=x_0,x(\tau)=\xi, \tag{7}$$

under $(v(\tau),z(\tau))$ satisfies $x(\tau)=0$.

The following results were obtained in [59].

Theorem 3 ([59])**.** *Let Hypothesis 1 and (5) hold. Suppose*

$$\tilde{A}(t)\in L^\infty_F(\Omega;L^{1+\epsilon}([0,\tau];\mathbb{R}^{n\times n})),\tilde{D}(t)\in L^\infty_F(\Omega;L^2([0,\tau];\mathbb{R}^{n\times n})),$$

where $\epsilon>0$ is a given constant. Then system (1) is L^p-exactly controllable on $[0,\tau]$ by $U^p[0,\tau]$ if and only if system (6) is L^p-exactly controllable on $[0,\tau]$ by $\tilde{U}^p[0,\tau]\times L^p_F(\Omega;L^2([0,\tau];\mathbb{R}^n))$.

Theorem 4 ([59])**.** *Let Hypothesis 1 and (5) hold. Suppose*

$$\tilde{A}(t)\in L^\infty_F(\Omega;L^{1+\epsilon}([0,\tau];\mathbb{R}^{n\times n})),\tilde{B}(t)\in L_F^{\max\{2,p\}+\epsilon}(\Omega;L^{2+\xi}([0,\tau];\mathbb{R}^{n\times m})),$$
$$\tilde{D}(t)\in L^\infty_F(\Omega;L^{2+\epsilon}([0,\tau];\mathbb{R}^{n\times n})), \tag{8}$$

where $\epsilon>0$ is a given constant. Then the following are equivalent:
(i) System (6) is L^p-exactly controllable on $[0,\tau]$ by $\tilde{U}^p[0,\tau]\times L^p_F(\Omega;L^2([0,\tau];\mathbb{R}^n))$;
(ii) System (6) is exactly null-controllable on $[0,\tau]$ by $\tilde{U}^p[0,\tau]\times L^p_F(\Omega;L^2([0,\tau];\mathbb{R}^n))$;
(iii) Matrix G defined below is invertible:

$$G=E\int_0^\tau Y(t)\tilde{B}(t)\tilde{B}(t)^TY(t)^Tdt, \tag{9}$$

where $Y(\cdot)$ is the adapted solution to the following stochastic linear equation:

$$dY(t)=-Y(t)\tilde{A}(t)dt-Y(t)\tilde{D}(t)dw(t),t\geq 0,Y(0)=I_n.$$

Theorem 5 ([59])**.** *Let Hypothesis 1, (5), and (8) hold. Then system (1) is L^p-exactly controllable on $[0,\tau]$ by $U^p[0,\tau]$ if and only if G defined by (9) is invertible.*

In the above, we have discussed the two extreme cases: either $D(\cdot)=0$ or $\mathrm{rank}D(\cdot)=n$. The case in between remains open. Therefore, we have the following open problem.

Problem 1. *If $0<\mathrm{rank}D(\cdot)<n$, what are the conditions under which system (1) can be L^p-exactly controlled?*

2.1.3. Duality and Observability Inequality

In this subsection, we introduce the dual principle for system (1). The following result was obtained in [59].

Theorem 6 ([59]). *Let hypotheses 1 and 2 hold. Then system (1) is L^p-exactly controllable on $[0,\tau]$ by $U^{p,\mu,\sigma}[0,\tau]$ if and only if there exists a $\delta > 0$ such that the following, called an observability inequality holds:*

$$\|B(\cdot)^T y(\cdot) + \sum_k^d D_k(\cdot) z_k(\cdot)\|_{U^{p,\mu,\sigma}[0,\tau]^*} \geq \delta \|\eta\|_{L^q(\Omega,F_\tau,P,\mathbb{R}^n)}, \forall \eta \in L^q(\Omega, F_\tau, P, \mathbb{R}^n),$$

where

$$U^{p,\mu,\sigma}[0,\tau] = L_F^{\frac{\mu p}{\mu-p}}(\Omega; L^{\frac{2\sigma}{\sigma-2}}([0,\tau];\mathbb{R}^m)), p \in [1,\mu), \mu \in (1,\infty], \sigma \in (2,\infty),$$

$$U^{p,\mu,\sigma}[0,\tau] = L_F^{p}(\Omega; L^{\frac{2\sigma}{\sigma-2}}([0,\tau];\mathbb{R}^m)), p \in [1,\mu), \mu = \infty, \sigma \in (2,\infty),$$

$$U^{p,\mu,\sigma}[0,\tau] = L_F^{\frac{\mu p}{\mu-p}}(\Omega; L^2([0,\tau];\mathbb{R}^m)), p \in [1,\mu), \mu \in [1,\infty], \sigma = \infty,$$

$$U^{p,\mu,\sigma}[0,\tau] = L_F^{p}(\Omega; L^2([0,\tau];\mathbb{R}^m)), p \in [1,\mu), \mu = \sigma = \infty;$$

$U^{p,\mu,\sigma}[0,\tau]^*$ denotes the adjoint space of $U^{p,\mu,\sigma}[0,\tau]$; $(y(\cdot), z(\cdot))$ (with $z(\cdot) = (z_1(\cdot), \cdots, z_d(\cdot))$) is the unique adapted solution to the following system:

$$dy(t) = -[A(t)^T y(t) + \sum_{k=1}^d C_k(t)^T z_k(t)]dt + \sum_{k=1}^d z_k(t) dw_k(t), t \in [0,\tau], y(\tau) = \eta. \quad (10)$$

Now, we introduce the following definition which makes the name "observability inequality" aforementioned meaningful.

Definition 3. *Let Hypothesis 1 hold and $(y(t), z(t))$ be the adapted solution to system (10) with $\eta \in L^q(\Omega, F_\tau, P, \mathbb{R}^n)$. (i) For the pair (B(·),D(·)) with $B(\cdot), D_k(\cdot) \in L_F^1([0,\tau];\mathbb{R}^{n\times m})(k = 1,2,\cdots,d)$ and $D(\cdot) = (D_1(\cdot),\cdots,D_d(\cdot))$, the map*

$$\eta \to K^*\eta = B(\cdot)^T y(\cdot) + \sum_k^d D_k(\cdot)^T z_k(\cdot)$$

is called an $Y[0,\tau]-$observer of (10) if $K^*\eta \in Y[0,\tau], \forall \eta \in L^q(\Omega, F_\tau, P, \mathbb{R}^m)$, where $Y[0,\tau]$ is a subspace of $L_F^1([0,\tau];\mathbb{R}^m)$. System (10), together with the observer of (10) is denoted by $[A(\cdot)^T, C(\cdot)^T; B(\cdot)^T, D(\cdot)^T]$;

(ii) Subsystem $[A(\cdot)^T, C(\cdot)^T; B(\cdot)^T, D(\cdot)^T]$ is said to be L^q-exactly observable by $Y[0,\tau]$ observations if from the observation $K^* \in Y[0,\tau]$, the terminal value $\eta \in L^q(\Omega, F_\tau, P, \mathbb{R}^n)$ of $y(\cdot)$ at τ can be uniquely determined, i.e., the map $K^* : L^q(\Omega, F_\tau, P, \mathbb{R}^n) \to Y[0,\tau]$ admits a bounded inverse.

With the above definition, the following result was obtained in [59]:

Theorem 7 ([59]). *Let Hypotheses 1 and 2 hold true. Then, system (1) is L^p-exactly controllable on $[0,\tau]$ by $U^{p,\mu,\sigma}[0,\tau]$ if and only if system $[A(\cdot)^T, C(\cdot)^T; B(\cdot)^T, D(\cdot)^T]$ is L^p-exactly observable by $U^{p,\mu,\sigma}[0,\tau]^*$ observations.*

2.2. Exact Controllability by Feedback Controller

In 2018, Barbu and Tubaro consider the exact controllability by feedback controller of the following stochastic linear system in [60]:

$$dx(t) + A(t)x(t)dt = B(t)u(t)dt + \sum_{k=1}^d C_k x(t) dw_k(t), x(0) = x_0, \quad (11)$$

with the final target $x(\tau) = \xi$, where $A(\cdot), B(\cdot) \in C([0,\infty); \mathbb{R}^{n\times m})$; for some $\gamma > 0$, $B(t)B(t)^T \geq \gamma^2 I_n, \forall t \in [0,\infty); C_k \in \mathbb{R}^{n\times n}$;

$$x(\cdot) \in L^2([0,\tau], \Omega, \mathbb{R}^n), u(\cdot) \in L^2([0,\tau], \Omega, \mathbb{R}^m); x_0, \xi \in \mathbb{R}^n.$$

The problem we address here is the following.

Problem 2. *Given $x_0, \xi \in \mathbb{R}^n$ find an F_t-adapted feedback controller $u = f(x)$ and $u \in L^2([0,\tau], \Omega, \mathbb{R}^m)$, such that the solution $x(t)$ to system (11) satisfies $x(0) = x_0, x(\tau) = \xi$.*

Let $F \in C([0,\tau]; \mathbb{R}^{n\times n})$ be the solution to equation

$$dF(t) = \sum_{k=1}^{d} C_k F(t) dw_k(t), t \geq 0, F(0) = I_n.$$

By the substitution $x(t) = F(t)z(t)$ one transforms via Ito's formula equation (see [60] for details) (11) into stochastic differential equation

$$\frac{dz(t)}{dt} + F(t)^{-1}A(t)F(t)z(t) = F(t)^{-1}B(t)u(t), z(0) = x_0. \tag{12}$$

In (12), we take as u the feedback controller

$$u(t) = -\tilde{\alpha}\text{sign}(F(t)^{-1}B(t))^T(z(t) - z_\tau)), t \geq 0, \tag{13}$$

where $\tilde{\alpha} \in L^2(\Omega, F_T, P, \mathbb{R}), z_\tau \in L^2(\Omega, F_T, P, \mathbb{R}^n)$ are given and $z_\tau = F(\tau)^{-1}\xi$; $\text{sign}: \mathbb{R}^n \to \mathbb{R}^n$ is the multivalued mapping $\text{sign} y = \frac{y}{\|y\|_{\mathbb{R}^n}}$ if $y \neq 0$, $\text{sign} y = \{\beta \in \mathbb{R}^n : \|\beta\|_{\mathbb{R}^n} \leq 1\}$ if $y = 0$. Arguing as in the proof of Proposition 3.1 in [60], it follows that (12) has unique absolutely continuous solution $z(t)$. We note that if $z(t)$ is an F_t-adapted solution to (12) and (13) then $x(t) = F(t)z(t)$ is the solution to closed loop system (11) with feedback control

$$u(t) = -\tilde{\alpha}\text{sign}((F(t)^{-1}B(t))^T F(t)^{-1}(x(t) - F(t)F(\tau)^{-1}x(\tau))).$$

The following results were obtained in [60].

Theorem 8 ([60]). *Let $\tau > 0, x_0 \in \mathbb{R}^n$ and $\xi \in L^2(\Omega, F_\tau, P, \mathbb{R}^n)$ be arbitrary but fixed. Then there is $\tilde{\alpha} \in L^2(\Omega, F_T, P, \mathbb{R})$, such that the controller (13) steers x_0 in z_τ, in time τ, with probability one.*

Remark 1. *It should be noted that, under the assumption of the Theorem 8, the solution $z(t)$ to (12) is not adapted. Therefore, the solution $x(t) = F(t)z(t)$ to system (11) is not F_t-adapted. Hence, further research is needed on Problem 2.*

Theorem 9 ([59]). *Consider system (11) where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, 1 \leq m \leq n$ is time independent and satisfy the Kalman rank condition* $\text{rank}[B, AB, \cdots, A^{n-1}B] = n$. *Assume also that $d = 1, C_1 = C$ and $C^2 = aC, C(\mathbb{R}^n) \subset B(\mathbb{R}^m)$ for some $a \in \mathbb{R}$. Let $\tau > 0$ and $x_0 \in \mathbb{R}^n$ be arbitrary but fixed. Then there is an F_t-adapted controller $u \in L^2([0,\tau], \Omega, \mathbb{R}^m)$ which steers x_0 in origin, in time τ, with probability one.*

Remark 2. *One might suspect that the controller u steering x_0 in origin can be found in feedback form but the problem is open.*

See [60] (p. 22) for example of this part.

2.3. Exact Controllability of Stochastic Differential Equation with Memory

In 2020, Wang and Zhou consider the exact controllability of the following controlled stochastic linear differential equation with a memory in [61].

$$dx(t) = [A(t)x(t)dt + B(t)u(t) + \int_0^t M(t,s)x(s)ds]dt$$
$$+ [C(t)x(t) + D(t)u(t)]dw(t), t \geq 0, \tag{14}$$

where $x(\cdot), u(\cdot)$ are the state variable, control variable which take values in $\mathbb{R}^n, \mathbb{R}^m$, respectively; for any $t, s \in [0, \tau]$ with $\tau \in [0, \infty)$, $A(t), M(t,s), C(t) \in \mathbb{R}^{n\times n}$, and $B(t), D(t) \in \mathbb{R}^{n\times m}$; $w(t)$ is 1-dimensional Wiener process. System (14) is denoted by $[A(\cdot), M(\cdot,\cdot), C(\cdot); B(\cdot), D(\cdot)]$.

The following is definition of controllability for system (14).

Definition 4. *For any $\tau_0, \tau(\tau_0 \leq \tau)$, the following system*

$$dx(t) = [A(t)x(t)dt + B(t)u(t) + \int_{\tau_0}^t M(t,s)x(s)ds]dt$$
$$+ [C(t)x(t) + D(t)u(t)]dw(t), t \geq 0, \tag{15}$$

$\tau_0 \in L^2(\Omega, F_{\tau_0}, P, \mathbb{R}^n)$, is called exactly controllable on $[\tau_0, \tau]$, if for any $\tau_0 \in L^2(\Omega, F_{\tau_0}, P, \mathbb{R}^n)$, $\tau \in L^2(\Omega, F_\tau, P, \mathbb{R}^n)$, there exists a control $u(\cdot) \in L^2([\tau_0, \tau], \Omega, \mathbb{R}^m)$, such that the solution $x(\cdot, \tau_0, x_{\tau_0}, u(\cdot))$ to system (15) with initial condition $x(\tau_0) = x_{\tau_0}$ satisfies $x(\tau, \tau_0, x_{\tau_0}, u(\cdot)) = x_\tau$ a.s.

Throughout this subsection, we introduce the following basic hypothesis:

$$A(\cdot), C(\cdot) \in L^\infty([0,\tau], \Omega, \mathbb{R}^{n\times n}), M(\cdot,\cdot) \in L^\infty([0,\tau]; L^\infty([0,\tau], \Omega, \mathbb{R}^{n\times n})),$$
$$B(\cdot), D(\cdot) \in L^\infty([0,\tau], \Omega, \mathbb{R}^{n\times m}).$$

2.3.1. Time Invariant Systems

In this subsection, we discuss system (14) with time invariant matrices: i.e.,

$$[A(\cdot), M(\cdot,\cdot), C(\cdot); B(\cdot), D(\cdot)] = [A, M, C; B, D].$$

To consider the exact controllability of system $[A, M, C; B, D]$, we adopt the partial controllability of controlled system as follows:

$$dx(t) = [A_0(t)x(t)dt + B_0(t)u(t)]dt + [A_1(t)x(t)dt + B_1(t)u(t)]dw(t), t \geq 0. \tag{16}$$

For fixed $\tau \geq 0$ and a matrix $Q \in \mathbb{R}^{l\times n}$, define $X_\tau = \{\xi \in L^2(\Omega, F_\tau, P, \mathbb{R}^l) : \xi(\omega) \in \mathrm{ran}(Q)\}$.

Definition 5. *Let a matrix $Q \in \mathbb{R}^{l\times n}$ be given. System (16) is called $Q-$partially controllable on $[0, \tau]$, if for any $x_0 \in \mathbb{R}^n, \xi \in X_\tau$, there exists a $u(\cdot) \in L^2([\tau_0, \tau], \Omega, \mathbb{R}^m)$, such that the solution $x(\cdot, x_0, u(\cdot))$ to system (16) with the initial condition $x(0) = x_0$ satisfies $Qx(\tau, x_0, u(\cdot)) = \xi$ a.s.*

Setting

$$\eta(\cdot) = \int_0^\cdot x(s)ds, y(\cdot) = \begin{bmatrix} x_1(\cdot) \\ \eta(\cdot) \end{bmatrix}, A_0 = \begin{bmatrix} A & M \\ I_n & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} B \\ 0 \end{bmatrix}, C_0 = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, D_0 = \begin{bmatrix} B \\ 0 \end{bmatrix},$$

we can rewrite system $[A, M, C; B, D]$ as follows:

$$dy(t) = [A_0(t)y(t)dt + B_0(t)u(t)]dt + [C_0 y(t)dt + D_0 u(t)]dw(t), t \geq 0. \quad (17)$$

The following results were obtained in [61].

Theorem 10 ([61])**.** *System $[A, M, C; B, D]$ is exactly controllable on $[0, \tau]$ with $x(0) = x_0$ if and only if system (17) is $[I_n, 0]-$partially controllable on $[0, \tau]$ with $y(0) = [x_0^T, 0^T]^T$.*

Theorem 11 ([61])**.** *If system $[A, M, C; B, D]$ is exactly controllable on $[0, \tau]$, then* rank$D = n$.

In what follows, we tend to present a rank criterion ensuring system $[A, M, C; B, D]'$s exact controllability. By Theorem 11, from now on, we suppose that rank$D = n$. Then, there exists an invertible $K \in \mathbb{R}^{m \times m}$, such that $DK = [I_n, 0]$. Set

$$u(\cdot) = K\begin{bmatrix} u_1(\cdot) \\ u_2(\cdot) \end{bmatrix} + Jy(\cdot), BK = [B_1, B_2],$$

where $B_1 \in R^{n \times n}$, $B_2 \in R^{n \times (m-n)}$, and $J \in R^{m \times 2n}$. Then, system (17) turns to

$$\begin{aligned} dy(t) = \{[A_0 + \begin{bmatrix} BJ - B_1([C, 0] + DJ) \\ 0 \end{bmatrix}]y(t) \\ + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}[u_1(t) + ([C, 0] + DJ)y(t)] + \begin{bmatrix} B_2 \\ 0 \end{bmatrix}u_2(t)\}dt \\ + \begin{bmatrix} I_n \\ 0 \end{bmatrix}[u_1(t) + ([C, 0] + DJ)y(t)]dw(t). \end{aligned} \quad (18)$$

Take

$$\tilde{A}_0 = A_0 + \begin{bmatrix} BJ - B_1([C, 0] + DJ) \\ 0 \end{bmatrix}, v(\cdot) = u(\cdot) + ([C, 0] + DJ)y(\cdot).$$

Then, system (17) or (18) can be rewritten as

$$\begin{aligned} d\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} = [\tilde{A}_0\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}v(t) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix}u_2(t)]dt \\ + \begin{bmatrix} I_n \\ 0 \end{bmatrix}v(t)dw(t), t \geq 0. \end{aligned} \quad (19)$$

In order to discuss the exact controllability of (19), we need to introduce the following stochastic linear differential equation

$$\begin{aligned} d\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} = [\tilde{A}_0\begin{bmatrix} x(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}v(t) + \begin{bmatrix} B_2 \\ 0 \end{bmatrix}u_2(t)]dt \\ + \begin{bmatrix} I_n \\ 0 \end{bmatrix}v(t)dw(t), t \in [0, \tau], x(\tau) = 0, \eta(0) = 0. \end{aligned} \quad (20)$$

Let

$$L = -([0, I_n]e^{-\tilde{A}_0^T \tau}[0, I_n]^T)^{-1}[0, I_n]e^{-\tilde{A}_0^T \tau}[I_n, 0]^T,$$

$$L_0 = [I_n, L^T], B_0 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}, \tilde{B}_0 = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The determinant of a square matrix F will be denoted by det$F$.
The following result was obtained in [61].

Theorem 12 ([61]). *Suppose that for any $u_2(\cdot) \in L^2([0,\tau],\Omega,\mathbb{R}^{m-n})$ system (20) admits a unique solution, and*

$$\det([0,I_n]e^{-\tilde{A}_0^T\tau}[0,I_n]^T) \neq 0, \forall t \in [0,\tau]$$

holds. Then system $[A,M,C;B,D]$ is exactly controllable if, and only if, the following rank condition holds:

$$\mathrm{rank}[L_0B_0, L_0\tilde{A}_0B_0, L_0\tilde{B}_0B_0, L_0\tilde{A}_0\bar{B}_0B_0, L_0\tilde{B}_0\tilde{A}_0B_0, \cdots] = n.$$

2.3.2. Time Varying System

In this part, we discuss time varying stochastic linear system with memory terms, and tend to to provide some criteria. In Section 2.3.1, we can present some criteria ensuring system $[A,M,C;B,D]$'s exact controllability. However, for time variant systems even for systems without memory terms, it is difficult to list those criteria. However, for some special systems, we still can make a try.

Case I. $M(t,s) = M_1(t)M_2(s), 0 \le s \le t \le \tau$, and

$$M_1(\cdot), M_2(\cdot) \in L^\infty([0,\tau],\Omega,\mathbb{R}^{n\times n}).$$

In this case, we can set

$$\eta(\cdot) = \int_0^\cdot M_2(s)x(s)ds, y(\cdot) = \begin{bmatrix} x_1(\cdot) \\ \eta(\cdot) \end{bmatrix}, A_0(\cdot) = \begin{bmatrix} A(\cdot) & M_1(\cdot) \\ M_2(\cdot) & 0 \end{bmatrix},$$

$$B_0(\cdot) = \begin{bmatrix} B(\cdot) \\ 0 \end{bmatrix}, C_0(\cdot) = \begin{bmatrix} C(\cdot) & 0 \\ 0 & 0 \end{bmatrix}, D_0(\cdot) = \begin{bmatrix} B(\cdot) \\ 0 \end{bmatrix}.$$

Hence, time varying system $[A(\cdot),M(\cdot,\cdot),C(\cdot);B(\cdot),D(\cdot)]$'s exact controllability turns to the $[I_n,0]-$partial controllability of the following linear system without memory term:

$$dy(t) = [A_0(t)y(t)dt + B_0(t)u(t)]dt + [C_0y(t) + D_0u(t)]dw(t), t \ge 0. \tag{21}$$

The following result provides an equivalent condition ensuring system (21)'s $[I_n,0]-$ partial controllability (see [61] (Theorem 3.1)).

Theorem 13 ([61]). *Assume that $M(t,s) = M_1(t)M_2(s), 0 \le s \le t \le \tau$. Then the following two statements are equivalent:*

(i) System (21) is $[I_n,0]-$partially controllable on [0, τ];

(ii) There exists a positive c such that the following observability inequality holds

$$\|\xi\|_{L^2(\Omega,F_\tau,P,\mathbb{R}^n)} \le c\|B_0(\cdot)^TY(\cdot) + D_0(\cdot)^TZ(\cdot)\|_{L^2([0,\tau],\Omega,\mathbb{R}^m)},$$

for all $\xi \in L^2([0,\tau],\Omega,\mathbb{R}^n)$, and $(Y(\cdot),Z(\cdot))$ solve the following equation:

$$dY(t) = [A_0(t)^TY(t)dt + C_0(t)^TZ(t)]dt + Z(t)dw(t), t \in [0,\tau], Y(\tau) = [I_n,0]^T\xi.$$

Remark 3. *Theorem 13 can be used to determine some stochastic system's exact controllability (see [61] (Example 3.2)).*

Case II. $M(t,s) = M(t-s), 0 \le s \le t \le \tau$, and $M(\tau) \in L^\infty([0,\tau],\Omega,\mathbb{R}^{n\times n})$.

In this case, for the stochastic system $[A(\cdot),M(\cdot,\cdot),C(\cdot);B(\cdot),D(\cdot)]$, we can present the following sufficient condition (see [61] (Proposition 3.4)).

Theorem 14 ([61]). *Assume that $M(t,s) = M(t-s), 0 \le s \le t \le \tau$, and*

$$M(\tau) \in L^\infty([0,\tau],\Omega,\mathbb{R}^{n\times n}).$$

If system $[A(\cdot), M(\cdot,\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is exactly controllable on $[\tau_0, \tau]$, for some $\tau_0 \in (0,\tau)$, then system $[A(\cdot), M(\cdot,\cdot), C(\cdot); B(\cdot), D(\cdot)]$ is exactly controllable on $[0,\tau]$.

The applicable example of this part can be found in [61] (p. 9).

According to the above discussion, further research is needed on the following problems.

Problem 3. *Find a $u(\cdot) \in L^2([0,\tau], \Omega, \mathbb{R}^m)$ in general case such that the system (14) is exactly controllable.*

Problem 4. *How to discuss the L^p-exact controllability for system (14)?*

3. Controllability of Infinite Dimensional Stochastic Linear Systems

In this section, we discuss the latest development of controllability of infinite dimensional stochastic linear systems.

In 2001, Sirbu and Tessitore discussed the null controllability of the following general infinite dimensional linear stochastic differential equation in [62]:

$$dx(t) = [Ax(t) + Bu(t)]dt + \sum_{k=1}^{\infty} C_k x(t) dw_{1,k}(t) + \sum_{j=1}^{\infty} D_j u(t) dw_{2,j}(t), x(0) = x_0, \tag{22}$$

where $x(\cdot)$ is the state process valued in H, $u(\cdot)$ is the control process valued in H, $A : \mathrm{dom}(A) \subseteq H \to H$ is the infinitesimal generator of a C_0-semigroup in H (the Hilbert space with product $< \cdot, \cdot >$), $B \in B(H)$ (the space of all bounded linear operators on H); $C_k, D_k \in B(H)$ for each $i \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} \|C_k\|^2_{B(H)} < +\infty, \sum_{k=1}^{\infty} \|D_k\|^2_{B(H)} < +\infty;$$

the countable set $\{w_{1,k}, w_{2,j}, k, j \in \mathbb{N}\}$ consists of independent standard Wiener processes defined on the stochastic basis $(\Omega, F, \{F_t\}, P)$.

Given any Hilbert space H, We denote by $C^2([0,\tau], \Omega, F_t, H)$ the space of all $\xi \in L^2([0,\tau], \Omega, F_t, H)$ such that ξ has a modification in $C([0,\tau]; L^2(\Omega, F, P, H))$, where

$L^2(\Omega, F, P, H) = \{x : x$ is $F-$adapted process valued in H with norm

$$(E(\|x\|^2_H))^{1/2} < +\infty\}.$$

As it is well known (see for instance [62]) for any initial data $x_0 \in L^2(\Omega, F_0, P, H)$ and any control $u \in L^2([0,\tau], \Omega, F_t, H)$ there exists a unique mild solution $x \in C^2([0,\tau], \Omega, F_t, H)$ of (22). When needed, we will denote the mild solution of (22) by $x(\cdot, x_0, u)$ (the definition of mild solution is in the ordinary sense).

Definition 6. *For $\tau > 0$, the state system (22) is $\tau-$null controllable if for each $x_0 \in L^2(\Omega, F_0, P, H)$ there exists $u \in L^2([0,\tau], \Omega, F_t, H)$ such that the solution $x(\tau, x_0, u) = 0$, $P-$almost surely. Moreover, the system is null controllable if it is $\tau-$null controllable for each $\tau > 0$.*

We recall a classical result on linear quadratic games for Equation (22). By $\Sigma^+(H)$ we denote the space of all self-adjoint, non-negative, bounded linear operators on H. Moreover, if $J \subset \mathbb{R}^+$ is an interval (bounded or unbounded), we denote by $C_s(J; \Sigma^+(H))$ the space of all maps $Q : J \to \Sigma^+(H)$, such that $Q(\cdot)v$ is continuous in H for every $v \in H$.

Definition 7. *We say that $Y \in C_s((0,\infty); \Sigma^+(H))$ is a mild solution of the Riccati equation*

$$\frac{dY(t)}{dt} = A^*Y(t) + Y(t)A - Y(t)B[I + \sum_{j=1}^{\infty} D_j^* Y(t) D_j]^{-1} B^* Y(t)$$

$$+\sum_{j=1}^{\infty} C_j^* Y(t) C_j + S, Y(0) = +\infty \quad (23)$$

if

(i) For each $\delta \in (0, +\infty), Y(\cdot + \delta)$ *is a mild solution of*

$$\frac{dY(t)}{dt} = A^* Y(t) + Y(t)A - Y(t)B[I + \sum_{j=1}^{\infty} D_j^* Y(t) D_j]^{-1} B^* Y(t)$$

$$+\sum_{j=1}^{\infty} C_j^* Y(t) C_j + S, Y(0) = Y(\delta) \in \Sigma^+(H);$$

(ii) $\lim_{(t,z)\to(0,v)} < Y(t)z, z >= +\infty$ *for all* $v \in H, v \neq 0$.

The following result was obtained in [62]:

Theorem 15 ([62])**.** *The following conditions are equivalent:*
(i) The Riccati Equation (23) has a mild solution;
(ii) The state system (22) is null controllable.

We assume that $F_t = \sigma\ w_{1,k}(s), w_{2,k}(s), s \in [0,t], k \in \mathbb{N}$ and introduce the following backward stochastic differential equation:

$$dp(t) = -[A^* p(t) + + \sum_{k=1}^{\infty} C_k^* q_{1,k}(t)]dt + \sum_{k=1}^{\infty} q_{1,k}(t) dw_{1,k}(t)$$

$$+\sum_{j=1}^{\infty} q_{2,k}(t) dw_{2,j}(t), p(\tau) = p_\tau.$$

The following duality approach was obtained in [62]:

Theorem 16 ([62])**.** *The following statements are equivalent:*
(i) System (1) is $\tau-$*null controllable;*
(ii) There exists a constant $C_\tau > 0$*, such that for all* $p_\tau \in L^2(\Omega, F_\tau, P, H)$ *the following observability relation holds:*

$$\|p(0)\|^2_{L^2(\Omega,F_0,P,H)} \leq C_\tau E \int_0^\tau \|B^* p(t) + \sum_{k=1}^{\infty} D_k^* q_{2,k}(t)\|^2_{L^2(\Omega,F_t,P,H)} dt.$$

Remark 4. *We can give the similar characterization for the exact controllability on the interval* $[0, \tau]$*. This is equivalent to the stronger observability inequality*

$$\|p(\tau)\|^2_{L^2(\Omega,F_\tau,P,H)} \leq C_\tau E \int_0^\tau \|B^* p(t) + \sum_{k=1}^{\infty} D_k^* q_{2,k}(t)\|^2_{L^2(\Omega,F_t,P,H)} dt.$$

See [62] (p. 392) for the applicable example.

Problem 5. *How about the controllability of the following system?*

$$dx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum_{k=1}^{\infty} C_k(t) x(t) dw_{1,k}(t)$$

$$+\sum_{j=1}^{\infty} D_j(t) u(t) dw_{2,j}(t), x(0) = x_0,$$

where $A(t) : \mathrm{dom}(A(t)) \subseteq H \to H$ is the generator of an evolution operator in the Hilbert space H, $B(t) : \mathrm{dom}(B(t)) \subset U \to H$ is unbounded, U is a Hilbert space; $C_k(t) \in P([0,\tau], B(H))$, $D_k(t) \in P([0,\tau], B(U,H))$, for each $i \in \mathbb{N}$, $P([0,\tau], B(U,H)) = \{C(\cdot) \in B(U,H) : C(\cdot)z$ is continuous for every $z \in U$ and $\sup_{0\le t\le\tau} \|C(t)\|_{B(U,H)} < +\infty\}$; and

$$\sum_{k=1}^{\infty} \sup_{0\le t\le\tau} \|C_k(t)\|^2_{B(H)} < +\infty, \sum_{k=1}^{\infty} \sup_{0\le t\le\tau} \|D_k(t)\|^2_{B(U,H)} < +\infty,$$

$B(U,H)$ denotes the set of all bounded linear operators from U to H; the countable set

$$\{w_{1,k}, w_{2,j}, k, j \in \mathbb{N}\}$$

consists of independent standard Wiener processes defined on the stochastic basis $(\Omega, F, \{F_t\}, P)$.

In 2015, Shen et al. studied the exact null controllability, approximate controllability and approximate null controllability of the following linear stochastic system in [63]:

$$dx(t) = [Ax(t) + Bu(t)]dt + Cx(t)dw(t), x(0) = x_0, \tag{24}$$

where $x(t)$ is the state process valued in H, $u(t)$ is the control process valued in U, $x(0) = x_0 \in L^2(\Omega, F_0, P, H)$, $w(t)$ is a standard Wiener process valued in W, and $A : D(A) \subseteq H \to H$ is the infinitesimal generator of a C_0-semigroup on H; $B \in B(U,H)$, $C \in B(H, B(W,H))$; H, U, W are separable Hilbert spaces. System (24) admits a unique mild solution $x(t, x_0, u) \in L^2_F(\Omega; C([0,\tau]; H))$.

We introduce the following backward stochastic system as our adjoint system to obtain sufficient conditions.

$$dy(t) = -[A^*y(t) + C^*z(t)]dt + z(t)dw(t), y(\tau) = \eta, \tag{25}$$

where A^*, C^* denote the adjoint operators of A, C, respectively.

For any $\eta \in H$, system (25) admits a unique mild solution $(y(t), z(t))$. In (25) $y(t)$ can be interpreted as an evolution process of the fair price, whereas $z(t)$ as the related consumption and portfolio process.

Remark 5. *When C is unbounded, the situation will be more complex.*

The closure of a set S will be denoted by $\overline{S}$.

Definition 8. *For $\tau > 0$, system (24) is null controllable at τ if for each $x_0 \in L^2(\Omega, F_0, P, H)$, there exists $u \in U$ such that $x(\tau, x_0, u) = 0, P - a.s.$*

System (24) is approximately controllable at τ if for each $x_0 \in L^2(\Omega, F_0, P, H)$, there exists $u \in U$ such that $\overline{\{x(\tau, x_0, u), u \in U\}} = L^2(\Omega, F_\tau, P, H), P - a.s.$

System (24) is approximately null controllable at τ if for each $x_0 \in L^2(\Omega, F_0, P, H)$, there exists $u \in U$ such that $x(\tau, x_0, u)$ can be arbitrarily close to 0, $P - a.s.$

The following results were obtained in [63].

Theorem 17 ([63]). *System (24) is null controllable if, and only if, there exists a positive constant c, such that*

$$\|y(0)\|^2_{L^2(\Omega,F_0,P,H)} \le c \int_0^\tau \|B^*y(s)\|^2_{L^2(\Omega,F_\tau,P,H)} ds.$$

Theorem 18 ([63]). *Let $(y(t), z(t))$ denote the solution of (25).*

(i) System (24) is approximate controllable at τ if and only if for every $(y(t), z(t))$ such that $B^*y(t) = 0$ we have $(y(t), z(t)) = 0, t \in [0,\tau], P - a.s.$;

(ii) System (24) is approximate null controllable at τ if, and only if, for every $y(t)$ such that $B^*y(t) = 0$ we have $y(0) = 0, t \in [0, \tau], P - a.s.$;

The illustrative example can be found in [63] (p. 601).

Problem 6. *If A, B, C are $A(t), B(t), C(t)$, respectively, and $A(t) : \mathrm{dom}(A(t)) \subseteq H \to H$ is the generator of an evolution operator; $B(t), C(t)$ are unbounded in (24), how about the controllability of this system?*

In 2019, Dou and Lu studied the partial approximate controllability for the following system in [64]:

$$\begin{aligned} dy(t) - A(t)y(t)dt &= (A_1(t)y(t) + Bu(t))dt \\ &+ A_2(t)y(t)dw(t), t \in (0, \tau], y(0) = y_0, \end{aligned} \tag{26}$$

here $A(t)$ is a linear operator on H, which generates strongly continuous evolution operator; $A_1(t), A_2(t) \in L^\infty([0, \tau]; B(H)), B \in B(U, H); U, H$ are separable Hilbert spaces; $u \in L^2([0, \tau], F_t, P, U), y_0 \in H, w(t)$ is a one-dimensional standard Wiener process. In (26), y is the state process valued in H and u is the control process valued in U. In what follows, $y(\cdot, y_0, u)$ denotes the mild solution to (26).

In order to discuss the partial approximate controllability of (26), we introduce the following equations and concepts.

$$dz(t) - A(t)^*z(t)dt = -(A_1^*z(t) + A_2^*Z(t))dt + Z(t)dw(t), t \in (0, \tau], z(\tau) = z_\tau, \tag{27}$$

where the final datum $z_\tau \in L^2(\Omega, F_\tau, P, H)$.

In what follows, we denoted by (z,Z) the mild solution to (27) (the definition of mild solution is in the ordinary sense).

Definition 9. *We say that (27) fulfills the unique continuation property (UCP) with respect to B^* if $z = Z = 0$ in H for a.e. $(t, \omega) \in [0, \tau] \times \Omega$, provided that $B^*z = 0$ in U for a.e. $(t, \omega) \in [0, \tau] \times \Omega$.*

$$\tilde{z}(t) + A(t)^*\tilde{z}(t) = -A_1(t)^*\tilde{z}(t), t \in [t_0, \tau], \tilde{z}(\tau) = \tilde{z}_\tau, \tag{28}$$

where the final data$\tilde{z}_\tau \in H$ and $t_0 \in [0, \tau]$.

Definition 10. *We say that (28) fulfills UCP if $\tilde{z} = 0$ in H for a.e. $t \in [t_0, \tau]$, provided that $B^*\tilde{z} = 0$ for a.e. $t \in [t_0, \tau]$.*

Hypothesis 3. *Solutions to (28) fulfill the UPC for any $t_0 \in [0, \tau]$.*

Denoted by $h_k(x)$ the kth Hermite polynomial (see [64]). For $k \in \mathbb{N} \cup \{0\}$, let

$$H_k = \mathrm{span}\{h_k(\int_0^\tau l(t)dw(t)) : l \in L^2([0, \tau], \mathbb{R}), \|l\|_{L^2([0,\tau],\mathbb{R})} = 1\}.$$

We have that $H_0 = \mathbb{R}, H_k$ and H_r are orthogonal subspaces of $L^2(\Omega, F_\tau, P, \mathbb{R})$ for $k \neq r$ and

$$L^2(\Omega, F_\tau, P, \mathbb{R}) = \oplus_{k=0}^\infty H_k.$$

For $k \in \mathbb{N} \cup \{0\}$, denote by $H_k(H)$ the closed subspace of $L^2(\Omega, F_\tau, P, H)$ generated by H valued random variable of the form $\sum_{j=1}^r l_j v_j (r \in \mathbb{N}), l_j \in H_k$, and $v_j \in H$. Let $\{e_j\}_{j=1}^\infty$ be an orthonormal basis of H. It is easy to see that

$$H_k(H) = \{\sum_{j=1}^\infty l_j e_j : \{l_j\}_{j=1}^\infty \subset H_k, E\sum_{j=1}^\infty |l_j|^2 < +\infty\}.$$

$H_0(H) = H, H_k(H)$ and $H_r(H)$ are orthogonal subspaces of $L^2(\Omega, F_\tau, P, H)$ for $k \neq r$ and

$$L^2(\Omega, F_\tau, P, H) = \oplus_{k=0}^{\infty} H_k(H).$$

Write

$$L_m^2(\Omega, F_\tau, P, H) = \oplus_{k=0}^{m} H_k(H).$$

Clearly $L_m^2(\Omega, F_\tau, P, H)$ is a closed subspace of $L^2(\Omega, F_\tau, P, H)$. Denote by Γ_m the orthogonal projection from $L^2(\Omega, F_\tau, P, H)$ to $L_m^2(\Omega, F_\tau, P, H)$.

Definition 11. *System (26) is said to be $m-$approximately controllable if for any $\epsilon > 0, y_0 \in H$ and $y_1 \in L_m^2(\Omega, F_\tau, P, H)$, there is a control $u \in L^2([0,\tau], \Omega, U)$, such that the corresponding mild solution fulfills that $\|\Gamma_m(y(\tau, y_0, u) - y_1)\|_{L^2(\Omega, F_\tau, P, H)} < \epsilon$.*

The system (26) is said to be partially approximately controllable if it is $m-$approximately controllable for all $m \in \mathbb{N}$.

To study the above controllability problem, we need the following notion.

Definition 12. *Equation (27) is said to fulfill the m-unique continuation property (m-UCP) if $z = Z = 0$ in H for a.e. $(t, \omega) \in [0,\tau] \times \Omega$, provided that $z_\tau \in L_m^2(\Omega, F_\tau, P, H)$ and $B^* z = 0$ in U for a.e. $(t, \omega) \in [0,\tau] \times \Omega$.*

Equation (27) is said to fulfill the partial UCP if it fulfills m-UCP for all $m \in \mathbb{N}$.

The following results were obtained in [64].

Theorem 19 ([64]). *(i) System (26) is m-approximately controllable if and only if (27) fulfills the m-UCP;*

(ii) System (26) is partially approximately controllable if and only if (27) fulfills the partial UCP.

Theorem 20 ([64]). *Suppose that Hypothesis 3 holds. Then system (26) is partially approximate controllable.*

Problem 7. *If B is $B(t)$, and $A_1(t), B(t), A_2(t)$ are unbounded in (26), how about the controllability of this system?*

4. Controllability of Finite Dimensional Stochastic Singular Linear Systems

Stochastic singular linear systems are also called stochastic implicit systems, stochastic differential algebraic systems, stochastic descriptor systems, stochastic degenerate systems, and stochastic generalized systems, etc. Controllability is the important concept for stochastic singular linear systems. So far, however, few results have been obtained. In this section, we discuss the latest development of controllability of finite dimensional stochastic singular linear systems.

In 2013, Gashi and Pantelous studied the exact controllability of the following stochastic singular linear system in [65,66].

$$Ldx(t) = [Mx(t) + Bu(t)]dt + [Cx(t) + Du(t)]dw(t), x(0 = x_0), \tag{29}$$

where $L, M, C \in \mathbb{R}^{n\times n}$, $\det L = 0$; $B, D \in \mathbb{R}^{n\times m}$, $x(t)$ is the state process valued in $\mathbb{R}^n$, $u(t)$ is the state process valued in $\mathbb{R}^m$, $w(t)$ is a one-dimensional standard Wiener process, (L, M) is regular, i.e., matrix pencil $\det(sL - M)$ is not identically zero ($s \in \mathbb{R}$). Let us begin by stating the definition of exact controllability.

Definition 13. *System (29) is called exactly controllable at time τ if for any $x_0 \in \mathbb{R}^n$ and $\xi \in L^2(\Omega, F_\tau, P, \mathbb{R}^n)$, there exists at least one admissible control $u(\cdot) \in L^2([0,\tau], \Omega, \mathbb{R}^m)$, such* *that the corresponding trajectory $x(\cdot)$ satisfies the initial condition $x(0) = x_0$ and the terminal condition $x(\tau) = \xi$, a.s.*

The following result was obtained in [65,66].

Theorem 21 ([65,66])**.** *(i) A necessary condition for exact controllability of (29) is*

$$\text{rank}\tilde{K}_1 = n - \sigma; \tag{30}$$

(ii) Let the condition (30) hold. A necessary and sufficient condition for exact controllability of (29) is

$$\text{rank}G_\tau = n - \sigma.$$

Here, G_τ is the Gramian matrix defined as

$$G_\tau = E \int_0^\tau \Phi(t) \tilde{K}_{12} \tilde{K}_{12}^T \Phi(t)^T dt,$$

where $\Phi(t)$ is the unique solution to the matrix stochastic differential equation

$$d\Phi(t) = -\Phi(t)[\tilde{N}dt + \tilde{K}_{11}dw(t)], \Phi(0) = I.$$

For the detail see [65] (Theorem 4) and [65] (Theorem 2).

In 2015, Gashi and Pantelous studied the exact controllability of the stochastic singular linear system (29) on the basis of [65,66] in [67], in which L is skew-symmetric and M is symmetric. The following result was obtained in [67].

Theorem 22 ([67])**.** *(i) A necessary condition for exact controllability of (29) is*

$$\text{rank}\tilde{K}_1 = n - q - 2p; \tag{31}$$

(ii) Let the condition (31) hold. A necessary and sufficient condition for exact controllability of (29) is

$$\text{rank}G_\tau = n - q - 2p.$$

Here, G_τ is the Gramian matrix defined as

$$G_\tau = E \int_0^\tau \Phi(t) \tilde{K}_{12} \tilde{K}_{12}^T \Phi(t)^T dt,$$

where $\Phi(t)$ is the unique solution to the matrix stochastic differential equation

$$d\Phi(t) = -\Phi(t)[\tilde{N}dt + \tilde{K}_{11}dw(t)], \Phi(0) = I.$$

For the detail see [67] (Theorem 5).

See [67] (p. 9) for practical example.

In 2021, Ge and Ge considered the exact null controllability of stochastic singular linear system (29).

Here, we assume that there are a pair of nonsingular deterministic and constant matrices $P_1, Q \in \mathbb{R}^{n \times n}$ such that the following condition is satisfied:

$$P_1 L Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, P_1 M Q = \begin{bmatrix} B_1 & 0 \\ 0 & I_{n_2} \end{bmatrix},$$
$$P_1 B = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, P_1 C Q = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, P_1 D = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \tag{32}$$

where $N \in \mathbb{R}^{n_2 \times n_2}$ denotes a nilpotent matrix with order h, i.e., $h = \min\{k : k \geq 1, N^k = 0\}$; $B_1, D_1 \in \mathbb{R}^{n_1 \times n_1}, C_1, G_1 \in \mathbb{R}^{n_1 \times m}, C_2 \in \mathbb{R}^{n_2 \times m}$, and $n_1 + n_2 = n$. Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Q^{-1}x$, system (29) is equivalent to

$$dx_1(t) = (B_1x_1(t) + C_1u(t))dt + (D_1x_1(t) + G_1u(t))dw(t), x_1(0) = x_{10}, \tag{33}$$

$$Ndx_2(t) = x_2(t)dt + C_2u(t)dt, x_2(0) = x_{20}. \tag{34}$$

Now, we consider the initial value problem (34). In the following, assume that the solution to (33) is the strong solution in the ordinary sense and (34) admits the stochastic Laplace transform (see [68]). Applying the stochastic Laplace transform to (34), we have

$$(sN - I_{n_2})X_2(s) = Nx_{20} + C_2U(s). \tag{35}$$

Definition 14. *(Impulse Solution) Suppose that $x_2(t)$ is the inverse stochastic Laplace transform of $X_2(s)$ obtained from (35). Then, $x_2(t)$ is the impulse solution to (34) in the sense of the stochastic Laplace transform, or simply, the impulse solution to (34). In this case, if $x_1(t)$ denotes the solution to (33), then $x(t) = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ is called the impulse solution of Equation (29).*

Let $\Phi(t)$ be the solution of system

$$d\Phi(t) = (B_1dt + D_1dw(t))\Phi(t), \Phi(0) = I_{n_1}, \tag{36}$$

Definition 15. *(Exact Null Controllability) System (33) and (34) is said to be exactly null controllable on $[0,\tau]$ if for any $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \in R^n$, there exists $u \in L^2([0,\tau], \Omega, R^m)$, such that (33) and (34) has a unique solution $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ satisfying the initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$ in addition to the terminal condition $\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} = 0$.*

It is obvious that if (33) and (34) is exactly null controllable, so is (33) and (34). In general, if $N \neq 0$, then (33) and (34) is not necessarily exactly null controllable. Consequently, we assume that $N = 0$ in the following.

The following result was obtained in [68].

Theorem 23 ([68]). *If $G_1 = 0$, then the necessary condition for (33) to be exactly null controllable on $[0,\tau]$ is that*

$$E(\int_0^\tau f^2(t)\Phi^{-1}(t)C_1(\Phi^{-1}(t)C_1)^{\mathrm{T}}dt) \tag{37}$$

is invertible for any real valued polynomial $f(t)$ not identical zero.

Let $\mathrm{rank}G_1 = n_1$; let $u(t) = M_1\begin{bmatrix} 0 \\ v(t) \end{bmatrix}, z(t) = D_1x_1(t)$, where M_1 denotes an $m \times m$ matrix, which satisfies $G_1M_1 = [I_{n_1} \quad 0]$, and $v(t)$ denotes an $(m - n_1)-$dimension vector.

For the above $u(t)$, system (33) and (34) is equivalent to

$$-dx_1(t) = (F_1x_1(t) + F_2z(t) + F_3v(t))dt - z(t)dw(t), x_1(0) = x_{10}, \tag{38}$$

$$x_2(t) = -C_2M_1\begin{bmatrix} 0 \\ v(t) \end{bmatrix}, t > 0, \tag{39}$$

where

$$F_1 = D_1 - B_1, F_2 = -I_{n_1}, F_3v(t) = -C_1M_1\begin{bmatrix} 0 \\ v(t) \end{bmatrix}.$$

Let $\Psi(t)$ denote the solution of system

$$d\Psi(t) = \Psi(t)(F_1dt + F_2dw(t)), \Psi(0) = I_{n_1}.$$

The following result was obtained in [68].

Theorem 24 ([68]). *System (38) and (39) is exactly null controllable on $[0, T]$ if, and only if,*

$$E(\int_0^\tau f^2(t)\Psi^{-1}(t)F_3(\Psi^{-1}(t)F_3)^{\mathrm{T}}dt)$$

is invertible for any real valued polynomial $f(t)$ not identical to zero.

The practical example can be found in [68] (supplementary file).

In 2021, Ge considered the impulse controllability and impulse observability of the following stochastic singular linear system in [69].

$$Adx(t) = Bx(t)dt + Cu(t)dt + Dx(t)dw(t), x(0) = x_0, \tag{40}$$

$$y(t) = Gx(t), \tag{41}$$

where $x(t) \in L^2([0,\tau],\Omega,\mathbb{R}^n)$ is the state vector, $u(t) \in L^2([0,\tau],\Omega,\mathbb{R}^m)$ is the control vector, $w(t)$ is one dimensional standard Wiener process, $x_0 \in L^2(\Omega, F_0, P, \mathbb{R}^n)$ is a given random variable, $y(t) \in L^2([0,\tau],\Omega,\mathbb{R}^l)$ is the measurement output.

For a stochastic singular system, impulse terms may exist in the solution. In a practical system, the impulse terms are generally undesirable because strong impulse behavior may impede the working of the system or even damage the system. Therefore, the impulse terms must be eliminated by imposing appropriate controls. In view of this fact, in this part, the concepts of impulse controllability and impulse observability for stochastic singular system (40) is considered.

In order to discusses the impulse controllability and impulse observability for stochastic singular system (40), let us introduce the class H_n of all processes $f(t) \in L^2([0,+\infty),\Omega,\mathbb{R}^n)$, such that

(i) $f(t)$ is mean square locally integrable;

(ii) There exist constants $a \geq 0$ and $M_0 > 0$ such that

$$(E\|f(t)\|^2_{\mathbb{R}^n})^{1/2} \leq M_0e^{at}, t \geq 0.$$

In the following, $C^k(J,\Omega,\mathbb{R}^n)$ denotes the set of all k times continuously differentiable stochastic processes $x(t) \in L^2(J,\Omega,\mathbb{R}^n)$, such that $x^{(i)}(t) \in L^2(J,\Omega,\mathbb{R}^n)(i = 0,1,\cdots,k)(J = [0,\tau]or[0,+\infty)$; we assume that there are a pair of non-singular matrices $P_1, Q \in \mathbb{R}^{n\times n}$, such that the following condition is satisfied

$$\begin{cases} P_1AQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, P_1BQ = \begin{bmatrix} B_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \\ P_1C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, P_1DQ = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, GQ = [G_1 \quad G_2], \end{cases} \tag{42}$$

where $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent, the index of nilpotency of N is denoted by h, i.e., $h = \min\{k : k \text{ is a positive integer}, k \geq 1, N^k = 0\}$, $B_1 \in \mathbb{R}^{n_1\times n_1}, C_1 \in \mathbb{R}^{n_1\times m}, C_2 \in \mathbb{R}^{n_2\times m}, D_1 \in \mathbb{R}^{n_1\times n_1}, G_1 \in \mathbb{R}^{l\times n_1}, G_2 \in \mathbb{R}^{l\times n_2}, n_1 + n_2 = n$. Let $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = Q^{-1}x(t)$, system (40) and (41) is equivalent to

$$dx_1(t) = (B_1x_1(t) + C_1u(t))dt + D_1x_1(t)dw(t), x_1(0) = x_{10}, \tag{43}$$

$$y_1(t) = G_1x_1(t), \tag{44}$$

$$Ndx_2(t) = x_2(t)dt + C_2u_2(t)dt, x_2(0) = x_{20}, \tag{45}$$

$$y_2(t) = G_2x_2(t). \tag{46}$$

Let $\Phi(t)$ be the solution of system

$$d\Phi(t) = (B_1dt + D_1dw(t))\Phi(t), \Phi(0) = I_{n_1},$$

the following results were obtained in [69]

Theorem 25 ([69])**.** *If $u \in L^2([0,\tau], \Omega, \mathbb{R}^m)$ is a bounded Borel measurable function, then subsystem (43) has a unique solution on $[0,\tau]$ with any $x_{10} \in L^2(\Omega, F_0, P, \mathbb{R}^{n_1})$, and the solution is given by the stochastic process*

$$x_1(t) = \Phi(t)x_{10} + \Phi(t)\int_0^t \Phi^{-1}(s)C_1u(s)ds. \tag{47}$$

Theorem 26 ([69])**.** *For any $x_{20} \in L^2(\Omega, F_0, P, \mathbb{R}^{n_2}), u \in C^{h-1}([0,+\infty), \Omega, \mathbb{R}^m)$ and $u^{(i)} \in H_m(i = 0,1,\cdots,h-1)$, subsystem (45) has a unique impulse solution, which is given by*

$$x_2(t) = -\sum_{i=1}^{h-1}\delta^{(i-1)}(t)[N^ix_{20} + \sum_{k=i}^{h-1}N^kC_2u^{(k-i)}(0)] - \sum_{i=0}^{h-1}N^iC_2u^{(i)}(t), \tag{48}$$

where $\delta(t)$ is the Dirac function, $\delta^{(i-1)}(t)$ is the $(i-1)$th derivative of $\delta(t)$.

Theorem 27 ([69])**.** *Assume that (40) and (41) is equivalent to (43)–(46),*

$$u \in C^{h-1}([0,+\infty), \Omega, \mathbb{R}^m)$$

is a bounded Borel measurable function, and $u^{(i)} \in H_m(i = 0,1,\cdots,h-1)$. Then, for any $x_0 \in L^2(\Omega, F_0, P, \mathbb{R}^n)$, system (40) has a unique impulse solution on $[0,\tau]$, which is given by

$$x(t) = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{49}$$

where $x_1(t)$ and $x_2(t)$ are given by (47) and (48), respectively.

Definition 16. *System (40) is called impulse controllable, if for any $x_0 \in L^2(\Omega, F_0, P, \mathbb{R}^n)$, there exists a bounded Borel measurable function $u \in C^h([0,+\infty), \Omega, \mathbb{R}^m)$ and $u^{(i)} \in H_m(i = 0,1,\cdots,h-1)$, such that the coefficient vectors of $\delta^{(i)}(t), i = 0,1,\cdots,h-2$, in the solution formula (49) are all zero.*

The following results were obtained in [69].

Theorem 28 ([69])**.** *System (40) is impulse controllable if, and only if, subsystem (45) is impulse controllable.*

Theorem 29 ([69])**.** *Subsystem (45) is impulse controllable if and only if for any*

$$x_{20} \in L^2(\Omega, F_0, P, \mathbb{R}^{n_2}),$$

there exists a bounded Borel measurable function $u \in C^{h-1}([0,+\infty), \Omega, \mathbb{R}^m)$ and $u^{(i)} \in H_m(i = 0,1,\cdots,h-1)$, such that

$$Nx_{20} + \sum_{i=0}^{h-2}N^{i+1}C_2u^{(i)}(0) = 0.$$

Theorem 30 ([69]). *System (40) is impulse controllable if, and only if,*

$$\mathrm{ran}(N) = \mathrm{ran}([NC_2 \quad \cdots \quad N^{h-1}C_2]),$$

where $\mathrm{ran}(N) = \{y : y = Nz, z \in L^2(\Omega, F_0, P, \mathbb{R}^{n_2})\}$, $\mathrm{ran}([NC_2 \quad \cdots \quad N^{h-1}C_2]) = \{y : \exists \alpha_k \in L^2(\Omega, F_0, P, \mathbb{R}^m), k = 1, 2, \cdots, h-1, y = \sum_{k=1}^{h-1} N^k C_2 \alpha_k\}$.

Now, we discuss the impulse observability of system (40) and (41). Without loss of generality, let $u(t) \equiv 0$.

Definition 17. *System (40) and (41) with subsystem (43)–(46) is called impulse observable if,* $y_2(t)|_{t=0} = 0$ *implies* $x_2(t)|_{t=0} = 0$.

Impulse observability guarantees the ability to uniquely determine the impulse behavior in solution from information of the impulse behavior in output, and focuses on the impulse terms that take infinite values in the solution.

The following results were obtained in [69].

Theorem 31 ([69]). *Subsystem (43) and (44) is always impulse observable.*

Theorem 32 ([69]). *System (40) and (41) is impulse observable if, and only if, one of the following conditions holds:*

(i) Subsystem (45) and (46) is impulse observable;

(ii)

$$\ker(\begin{bmatrix} G_2N \\ G_2N^2 \\ \vdots \\ G_2N^h \end{bmatrix}) = \ker(N).$$

where $\ker(N) = \{x : Nx = 0, x \in L^2(\Omega, F_0, P, R^{n_2})\}$,

$$\ker(\begin{bmatrix} G_2N \\ G_2N^2 \\ \vdots \\ G_2N^h \end{bmatrix}) = \{x : \begin{bmatrix} G_2N \\ G_2N^2 \\ \vdots \\ G_2N^h \end{bmatrix} x = 0, x \in L^2(\Omega, F_0, P, R^{n_2})\}.$$

For the impulse observability and impulse controllability, the so-called dual principle holds, which reveals the close relation between impulse observability and impulse controllability.

In order to introduce the dual principle for system (40) and (41), let us first introduce the dual system.

Definition 18. *The following system*

$$\begin{cases} A^T dz(t) = B^T z(t)dt + G^T v(t)dt + D^T z(t)dw(t), \\ w_0(t) = C^T z(t), \end{cases} \tag{50}$$

is called the dual system of the system (40) and (41).

The following dual principle was obtained in [69].

Theorem 33 ([69]). *Let (50) be the dual system of system (40) and (41). Then, system (40) and (41) is impulse observable (impulse controllable) if, and only if, its dual system (50) is impulse controllable (impulse observable).*

An illustrative example is given in [69] (p. 908).

Furthermore, in 2021, Ge discussed the exact observability for a kind of stochastic singular linear systems in the sense of impulse solution. Some necessary and sufficient conditions were obtained. See [70] (Theorems 3.1 and 3.3) for details.

Problem 8. *How to discuss the L^p-exact controllability for the following stochastic singular linear system?*

$$Ldx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum_{k=1}^{d}[C_k(t)x(t) + D_k(t)u(t)]dw_k(t), t \geq 0, x(0) = x_0.$$

where L as defined in (29); $A(t), B(t), C_k(t), D_k(t)$ as defined in (1).

5. Controllability of Infinite Dimensional Stochastic Singular Linear Systems

In this section the latest development of the controllability of infinite dimensional stochastic singular linear systems is discussed by using the methods of C_0-semigroup, GE-semigroup, GE-evolution operator, and stochastic GE-evolution operator, respectively. Some necessary and sufficient conditions concerning the controllability are introduced.

5.1. C_0-Semigroup Method for a Class of Time Invariant Systems in Hilbert Spaces

In 2015, Liaskos et al. studied the exact controllability of the following stochastic singular linear system by using the C_0-semigroup method in the sense of strong solution in Hilbert spaces in [71].

$$dLx(t) = [Mx(t) + Cu(t) + f(t)]dt + Bdw(t), t \in [0, \tau], x(0) = \xi. \tag{51}$$

In order to introduce the exact controllability, make the following assumptions and preparations.

Let H, U, K be separable and infinite dimensional Hilbert spaces, $x(t)$ be the state process valued in H, $u(t)$ be the control process valued in U, and $w(t)$ be a $U-$valued standard Wiener process in (51). The closure of an operator S will be denoted by $\overline{S}$. We use the notation $S^{\perp}$ for the orthogonal complement of a set S and for the restriction of the operator A to a linear subset S the symbol $A|_S$. For the coefficients L, M, C, f, B, ξ involved in (51), the following assumptions and definitions should be considered.

(A_1) (i) $L \in B(H), \ker(L) \neq \{0\}$. (ii) $\overline{\ker(L)} = \ker(L)$.

(A_2) (i) $M : \mathrm{dom}(M) \subseteq H \to H$ is a linear, densely defined and closed operator.

(ii) For the linear subspace $D = \{x \in \mathrm{dom}(M) : Mx + f(t) \in \mathrm{ran}(L)\}$, we assume that $D \cap \ker(L) = \{0\}$ and $P_1^{\perp}D$ is dense in $P_1^{\perp}H$, where $P_1, P_1^{\perp}$ are the projections onto $\ker(L)$ and $(\ker(L))^{\perp}$, respectively.

(A_3) (i) The operator pencil $\lambda L - M : \mathrm{dom}(M) \to H$ is of parabolic type, i.e., the restriction of the pencil $\lambda L - M : D \to \mathrm{ran}(L)$ is invertible with a bounded inverse $(\lambda L - M)^{-1}$, for all $\lambda > \omega$, where ω is a negative real constant. This regularity on the pencil also implies that $M(D) = \mathrm{ran}(L)$ and $M|_D : D \to \mathrm{ran}(L)$ is invertible with a bounded inverse M^{-1}.

(ii) The bounded pseudo-resolvent operators $R_1(\lambda) = (\lambda L - M)^{-1}L : H \to D$ and $R_2(\lambda) = L(\lambda L - M)^{-1} : \mathrm{ran}(L) \to L(D)$ satisfy $\|U(\lambda)\|_{B(H)} \leq \frac{c}{\lambda - \omega}$, for all $\lambda > \omega, 0 < c < 1$, where $U(\lambda)$ stands for both $R_1(\lambda), R_2(\lambda)$.

(A_4) $f \in L^1([0, \tau]; H) \cap L^2([0, \tau], \Omega, F_t, H)$, satisfying $f(t) \in L(D), P - a.s., a.e.$ in $[0, \tau]$.

(A_5) $B : U \to H$ is a linear operator with $\mathrm{ran}(B) \subseteq L(D)$, such that $B \in B(U, H)$.

(A_6) ξ is a $D-$valued random variable $P - a, s.$, with $\xi \in L^2(\Omega, F_0, P, H)$.

(A_7) $C \in B(K, H)$, with $\mathrm{ran}(C) \subseteq L(D)$, such that for any $u \in L^2([0, \tau], \Omega, K)$, the stochastic process $Cu(t), t \in [0, \tau]$ satisfies

$$E[\int_0^{\tau}\int_0^t \|(L^{\perp})^{-1}M_0 S_1(s-t)(L^{\perp})^{-1}(Cu(s) + f(s))\|_H dsdt] < \infty,$$

where $L^{\perp} = L|_{P_1^{\perp}H} : P_1^{\perp}H \to Q^{\perp}H, Q$ is the projection onto $\ker(L^*); M_0 = M(P_1^{\perp}|_D)^{-1}$, $S_1(t)$ is the C_0-semigroup in the closed subspace $P_1^{\perp}H$ generated by the operator $(L^{\perp})^{-1}M_0$.

Definition 19. *An $H-$valued stochastic process $x(t), t \in [0,\tau]$, is called a strong solution of the initial value problem (51), if*

(i) $x \in D, P-a.s., a.e.$ in $[0,\tau]$ and $x \in L^1([0,\tau];H), P-a.s.$

(ii) $Lx, Mx \in L^1([0,\tau];H), P-a.s.$

(iii) $Lx(t) = L\xi + \int_0^t [Mx(s) + Cu(t) + f(s)]ds + Bw(t), P-a.s., a.e.$ in $[0,\tau]$.

From the above, the controlled stochastic singular linear system (51) has a unique strong solution $x_u(t), t \in [0,\tau]$, which admits the form:

$$x_u(t) = \overline{(P_1^{\perp}|_D)^{-1}} S_1(t) P_1^{\perp}\xi + \int_0^t \overline{(P_1^{\perp}|_D)^{-1}} S_1(t-s)(L^{\perp})^{-1}[Cu(s) + f(s)]ds$$
$$+ \int_0^t \overline{(P_1^{\perp}|_D)^{-1}} S_1(t-s)(L^{\perp})^{-1} B dw(s), t \in [0,\tau]. \quad (52)$$

Definition 20. *Stochastic singular linear system (51) is called exactly controllable at time $\tau > 0$, if for any ξ which is $D-$valued random variable $P-a.s.$, with $\xi \in L^2(\Omega, F_0, PH)$ and for any ξ_τ which is also a $D-$valued random variable $P-a.s.$, with $\xi \in L^2(\Omega, F_\tau, P, H)$, there exists at least one control $u \in L^2([0,\tau],\Omega,K)$, such that the corresponding strong solution $x_u(t)$, which admits the form of (52), satisfies the initial condition $x_u(0) = \xi$ and the terminal condition $x_u(\tau) = \xi_\tau$.*

The following result was obtain in [71].

Theorem 34 ([71])**.** *Suppose that $L^{\perp}S_1(t)v(t) - f(t) \in \mathrm{ran}(C), P-a.s., a.e.$ in $[0,\tau]$. Then there exists at least one $u \in L^2([0,\tau],\Omega,K)$, such that the corresponding strong solution $x_u(t)$, which admits the form of (52), satisfies the initial condition $x_u(0) = \xi$ and the terminal condition $x_u(\tau) = \xi_\tau$ and hence stochastic singular linear system (51) is exactly controllable.*

See [71] for the details of practical example.

In 2018, Liaskos et al. studied the exact controllability of the stochastic singular linear system (51) by using the C_0-semigroup method in the sense of strong solution in Hilbert spaces in [72].

Suppose that (A_1)–(A_6) hold true, and

$$E[\int_0^\tau \int_0^t \|M_0(L^{\perp})^{-1}S_2(s-t)(L^{\perp})^{-1}(Cu(s) + f(s))\|_H dsdt] < \infty.$$

Then, the controlled stochastic singular linear system (51) has a unique strong solution $x_u(t), t \in [0,\tau]$, which admits the form:

$$x_u(t) = \overline{(P_1^{\perp}|_D)^{-1}}(L^{\perp})^{-1}S_2(t)L\xi$$
$$+ \int_0^t \overline{(P_1^{\perp}|_D)^{-1}}(L^{\perp})^{-1}S_2(t-s)[Cu(s) + f(s)]ds$$
$$+ \int_0^t \overline{(P_1^{\perp}|_D)^{-1}}(L^{\perp})^{-1}S_2(t-s)Bdw(s), t \in [0,\tau], \quad (53)$$

where $S_2(t)$ is the C_0-semigroup generated by the operator $M_0(L^{\perp})^{-1}$.

The following result was obtained in [72]:

Theorem 35 ([72])**.** *Suppose that $S_2(t)v(t) - f(t) \in \mathrm{ran}(C), P-a.s., a.e.$ in $[0,\tau]$. Then, there exists at least one $u \in L^2([0,\tau],\Omega,K)$, such that the corresponding strong solution $x_u(t)$, which* *admits the form of (53), satisfies the initial condition $x_u(0) = \xi$ and the terminal condition $x_u(\tau) = \xi_\tau$ and hence stochastic singular linear system (51) is exactly controllable.*

5.2. GE-Semigroup Method for a Class of Time Invariant Systems

In this subsection, we discuss the controllability of the following time invariant stochastic singular linear system by using GE-semigroup in the sense of mild solution in Banach and Hilbert spaces, respectively,

$$Adx(t) = Bx(t)dt + Cv(t)dt + Ddw(t), x(0) = x_0, t \geq 0, \tag{54}$$

where $x(t)$ is the state process valued in H, $v(t)$ is the control process valued in U, $w(t)$ is the standard Wiener process on $Z, x_0 \in L^2(\Omega, F_0, P, H)$ is a given random variable, H, U, Z are Banach or Hilbert spaces; $A \in B(H), C \in B(U, H), D \in B(Z, H), B : \mathrm{dom}(B) \subseteq H \to H$ is a linear operator. This subsection is organized as follows. Firstly, the GE-semigroup is introduced and the mild solution of (54) is obtained; Secondly, the controllability of (54) is discussed in Banach spaces; Thirdly, the controllability of (54) is discussed in Hilbert spaces.

5.2.1. GE-Semigroup and Mild Solution of System (54)

In this part, the existence and uniqueness of the mild solution to system (54) are considered by GE-semigroup theory.

Definition 21 ([73–77])**.** *Suppose $\{U(t) : t \geq 0\}$ is one parameter family of bounded linear operators in Banach space H, and A is a bounded linear operator. If*

$$U(t+s) = U(t)AU(s), t, s \geq 0,$$

then $\{U(t) : t \geq 0\}$ is called a GE-semigroup induced by A.

If the GE-semigroup $U(t)$ satisfies

$$\lim_{t\to 0^+} \|U(t)x - U(0)x\|_H = 0,$$

for arbitrary $x \in H$, then it is called strongly continuous on H.

Lemma 1 ([73,74,76,77])**.** *If GE-semigroup $U(t)$ is strongly continuous on H, then there exist $M \geq 1$ and $\omega > 0$, such that*

$$\|U(t)\|_{L(H,H)} \leq Me^{\omega t}, t \geq 0,$$

i.e., $U(t)$ is exponentially bounded.

Definition 22 ([75–77])**.** *Suppose $U(t)$ is strongly continuous GE-semigroup induced by A. If*

$$Bx = \lim_{h\to 0^+} \frac{AU(h)A - AU(0)A}{h} x,$$

for every $x \in D_1$, where

$$D_1 = \{x : x \in \mathrm{dom}(B) \subseteq H, U(0)Ax = x, \exists \lim_{h\to 0^+} \frac{AU(h)A - AU(0)A}{h} x\},$$

then B is called a generator of GE-semigroup $U(t)$ induced by A.

Now, we consider the initial value problem (54).

Definition 23. *If B is a generator of GE-semigroup $U(t)$ induced by A, $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$, and $v(t) \in L^2([0,b], \Omega, U)$; $Cv(t), Ddw(t) \in A(L^2([0,b], \Omega, \overline{D_1}))$, the mild solution $x(t, x_0)$ to (54) is defined by*

$$x(t, x_0) = U(t)Ax_0 + \int_0^t U(t-\tau)Cv(\tau)d\tau + \int_0^t U(t-\tau)Ddw(\tau). \tag{55}$$

From the above knowledge, we have the following proposition.

Proposition 1 ([76,77])**.** *If B is the generator of GE-semigroup $U(t)$ induced by A, $v(t) \in L^2([0,b], \Omega, U)$, $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$; $Cv(t), Ddw(t) \in A(L^2([0,b], \Omega, \overline{D_1}))$, and $U(0)$ is a definite operator, then there exists unique mild solution $x(t, x_0)$ to (54), which is given by (55).*

In the following, we suppose that Proposition 1 holds true.

5.2.2. Controllability of System (54) in Banach Spaces

In this following, we discuss the exact (approximate) controllability of system (54) in Banach spaces. Some necessary and sufficient conditions are given.

Definition 24. *(a) Stochastic singular system (54) is said to be exactly controllable on $[0,b]$, if for all $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$, $x_b \in L^2(\Omega, F_b, P, \overline{D_1})$, there exists $v(t) \in L^2([0,b], \Omega, U)$, such that the mild solution $x(t, x_0)$ to (54) satisfies $x(T, x_0) = x_b$;*

(b) Stochastic singular system (54) is said to be approximately controllable on $[0,b]$, if for any state $x_b \in L^2(\Omega, F_b, P, \overline{D_1})$, any initial state $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$, and any $\epsilon > 0$, there exists a $v \in L^2([0,b], \Omega, U)$, such that the mild solution $x(t, x_0)$ satisfies

$$\|x(b, x_0) - x_b\|_{L^2(\Omega, F_{T_1}, P, \overline{D_1})} < \epsilon.$$

In order to discuss the controllability, we introduce the following concepts.

Banach space $\{v(t) \in U : Cv(t) \in A(\overline{D_1})\}$ is still denoted by U.

Controllability operator

$$C_0^b : L^2([0,b], \Omega, U) \to L^2(\Omega, F_b, P, \overline{D_1})$$

associated with system (54) is defined as

$$C_0^b v = \int_0^b U(b-\tau)Cv(\tau)d\tau.$$

It is obvious that operator C_0^b is a bounded linear operator, and its dual

$$C_0^{b*} : L^2(\Omega, F_b, P, (\overline{D_1})^*) \to L^2([0,b], \Omega, U^*)$$

is defined by

$$C_0^{b*} z^* = C^* U^*(b-\tau) E(z^* | F_\tau).$$

where $z^* \in L^2(\Omega, F_b, P, (\overline{D_1})^*)$.

The following results were obtained in [76].

Theorem 36 ([76])**.** *Stochastic singular system (54) is exactly controllable on $[0,b]$ if, and only if,* $\mathrm{ran}(C_0^b) = L^2(\Omega, F_b, P, \overline{D_1})$.

Theorem 37 ([76])**.** *Assume that H and U are reflexive Banach spaces. Stochastic singular system (54) is exactly controllable on $[0,b]$ if, and only if, one of the following conditions hold:*

(a) $\|C_0^{b*} z^*\|_{L^2([0,b], \Omega, U^*)} \geq \gamma \|z^*\|_{L^2(\Omega, F_b, P, (\overline{D_1})^*)}$ for some $\gamma > 0$ and all

$$z^* \in L^2(\Omega, F_T, P, (\overline{D_1})^*);$$

(b) $\ker(C_0^{b*}) = \{0\}$ *and* $\operatorname{ran}(C_0^{b*})$ *is closed.*

Theorem 38 ([76]). *Stochastic singular system (54) is approximately controllable on* $[0, b]$ *if, and only if,* $\overline{\operatorname{ran}(C_0^b)} = L^2(\Omega, F_b, P, \overline{D_1})$.

Theorem 39 ([76]). *Stochastic singular system (54) is approximate controllable on* $[0, b]$ *if, and only if,*

$$\ker(C_0^{b*}) = \{0\}.$$

See [76] (p. 908) for the illustrative example.

5.2.3. Controllability of System (54) in Hilbert Spaces

In this following, we discuss the exact (approximate) controllability of system (54) in Hilbert spaces. Some necessary and sufficient conditions are given. In order to discuss the controllability, we introduce the following operator.

Hilbert space $\{v(t) \in U : Cv(t) \in A(\overline{D_1})\}$ is still denoted by U.

Controllability Gramian operator $G_c^b : L^2(\Omega, F_b, P, \overline{D_1}) \to L^2(\Omega, F_b, P, \overline{D_1})$ in connection with stochastic descriptor linear system (54) is defined as

$$G_c^b z = \int_0^b S(b-t)CC^* S^*(b-t)E(z|F_t)dt.$$

The following results were obtained in [77].

Theorem 40 ([77]). *The necessary and sufficient condition for the stochastic singular linear system (54) to be exactly controllable on* $[0, b]$ *is that one of the following conditions is true:*
(a) $< G_c^b z, z >_{L^2(\Omega,F_b,P,\overline{D_1})} \geq \gamma \|z\|^2_{L^2(\Omega,F_b,P,\overline{D_1})}$ *for some* $\gamma > 0$ *and all* $z \in L^2(\Omega, F_b, P, \overline{D_1})$;
(b) $\lim_{\lambda \to 0^+} \|(\lambda I + G_c^b)^{-1} - (G_c^b)^{-1}\|_{B(L^2(\Omega,F_b,P,\overline{D_1})),L^2(\Omega,F_b,P,\overline{D_1}))} = 0$;
(c) $\lim_{\lambda \to 0^+} \|\lambda(\lambda I + G_c^b)^{-1}\|_{B(L^2(\Omega,F_b,P,\overline{D_1}),L^2(\Omega,F_b,P,\overline{D_1}))} = 0$;
(d) $\ker(C_0^{b*}) = \{0\}$ *and* $\operatorname{ran}(C_0^{b*})$ *is closed.*

Theorem 41 ([77]). *The necessary and sufficient condition for the stochastic singular linear system (54) to be approximately controllable on* $[0, b]$ *is that one of the following conditions is true:*
(a) $< G_c^b z, z >_{L^2(\Omega,F_b,P,\overline{D_1})} > 0$ *for all* $z \in L^2(\Omega, F_b, P, \overline{D_1}), z \neq 0$;
(b) $\lim_{\lambda \to 0^+} < \lambda(\lambda I + G_c^b)^{-1} x, z >_{L^2(\Omega,F_b,P,\overline{D_1})} = 0$ *for all* $x, z \in L^2(\Omega, F_b, P, \overline{D_1})$;
(c) $\lim_{\lambda \to 0^+} \|\lambda(\lambda I + G_c^b)^{-1} z\|_{L^2(\Omega,F_b,P,\overline{D_1})} = 0$ *for all* $z \in L^2(\Omega, F_b, P, \overline{D_1})$.

5.3. GE-Evolution Operator Method for a Class of Time-Varying Systems

In this subsection, we discuss the controllability of the following time varying stochastic singular linear system by using GE-evolution operator in Hilbert spaces,

$$Adx(t) = B(t)x(t)dt + C(t)v(t)dt + D(t)dw(t), x(0) = x_0, t \geq 0, \tag{56}$$

where $A \in B(H)$ is a deterministic and constant operator, $B(t) : \operatorname{dom}(B(t)) \subseteq H \to H$ is a linear operator (possibly unbounded), $B(t), C(t), D(t)$ are deterministic and time varying operators; $C(t) \in P([0, b], B(U, H)), D(t) \in P([0, b], B(Z, H))$; $x(t)$ is the state process valued in H, $v(t)$ is the control process in U, $w(t)$ is the stand Wiener process valued in $Z, x_0 \in L^2(\Omega, F_0, P, H)$ is a given random variable, H, U, Z are Hilbert spaces. This subsection is organized as follows. Firstly, the GE-evolution operator is introduced and the mild solution of (56) is obtained; Secondly, the controllability of (56) is discussed by GE-evolution operator in the sense of mild solution in Hilbert spaces.

5.3.1. GE-Evolution Operator and Mild Solution of System (56)

In the following, we discuss mild solution of time varying stochastic singular system (56) according to GE-evolution operator. First of all, we recall the GE-evolution operator, and then the mild solution of (56) is given.

Definition 25 ([78–80])**.** *Let $\Delta(b) = \{(t,s) : 0 \le s \le t \le b\}$. $U(t,s) : \Delta(b) \to B(H)$ is said to be a GE-evolution operator induced by A on $[0,b]$ if it has the following properties:*
(a) $U(t,s) = U(t,r)AU(r,s), 0 \le s \le r \le t \le b$;
(b) $U(s,s) = U_0, 0 \le s \le b$, where U_0 is a definite operator independent of s;
GE-evolution operator $U(t,s)$ is said to be strongly continuous on $[0,b]$ if it has the following property:
(c) $U(\cdot,s)$ is strongly continuous on $[s,b]$ and $U(t,\cdot)$ is strongly continuous on $[0,t]$;
GE-evolution operator $U(t,s)$ is said to be exponential bounded on $[0,b]$ if it has the following property:
(d) There exist $M \ge 1$ and $\omega > 0$, such that

$$\|U(t,s)\|_{B(H)} \le Me^{\omega(t-s)}, 0 \le s \le t \le b.$$

Definition 26 ([78–80])**.** *Assume that $U(t,s)$ is a strongly continuous and exponential bounded GE-evolution operator induced by A. If*

$$B(t)x = \lim_{h\to 0^+} \frac{AU(t+h,t)A - AU(t,t)A}{h} x, t \in [0,b],$$

for every $x \in D_0(t)$, where

$$D_0(t) = \{x : x \in \mathrm{dom}(B(t)) \subseteq H, U_0Ax = x,$$

$$\exists \lim_{h\to 0^+} \frac{AU(t+h,t)A - AU(t,t)A}{h} x, t \in [0,b]\},$$

then $B(t)$ is called a generator of GE-evolution operator $U(t,s)$.

In the following, we always assume that $B(t)$ is the generator of GE-evolution operator $U(t,s)$ induced by A and $D_0(t) = D_0$ is independent of t.

Now, we consider the initial value problem (56).

Definition 27. *If $x_0 \in L^2(\Omega, F_0, P, \overline{D_0})$, $v(t) \in L^2([0,T],\Omega,U)$; $C(t)v(t), D(t)dw(t) \in A(L^2([0,b],\Omega,\overline{D_0}))$, the mild solution $x(t,x_0)$ to (56) is defined by*

$$x(t,x_0) = U(t,0)Ax_0 + \int_0^t U(t,\tau)C(\tau)v(\tau)d\tau + \int_0^t U(t,\tau)D(\tau)dw(\tau). \tag{57}$$

Proposition 2 ([80])**.** *There exists unique mild solution $x(t,x_0)$ to (56), which is given by (57), if $v(t) \in L^2([0,b],\Omega,U)$, $x_0 \in L^2(\Omega,F_0,P,\overline{D_0})$; $C(t)v(t), D(t)dw(t) \in A(L^2([0,b],\Omega,\overline{D_0}))$, and $(U_0B(t))|_{D_0}$ satisfies the following assumptions:*
(P_1) For $t \in [0,b]$, $(\lambda I + (U_0B(t))|_{D_0})^{-1}$ exists for all λ with $Re\lambda \le 0$ and there is a constant $M > 0$, such that

$$\|(\lambda I + (U_0B(t))|_{D_0})^{-1}\|_{B(H)} \le \frac{M}{|\lambda|+1},$$

for all $Re\lambda \le 0, t \in [0,b]$.
(P_2) There exist constants $L > 0$ and $0 < \alpha \le 1$, such that

$$\|((U_0B(t))|_{D_0} - (U_0B(s))|_{D_0})((U_0B(\tau))|_{D_0})^{-1}\|_{B(H)} \le L|t-s|^{\alpha},$$

for $t, s, \tau \in [0,b]$.

In the following, we suppose that Proposition 2 holds true.

5.3.2. Controllability of System (56)

In this part, the exact controllability and approximate controllability of system (56) are discussed by using GE-evolution operator in the sense of mild solution in Hilbert spaces. In order to discuss the controllability, we introduce the following concepts.

Hilbert space $\{v(t) \in U : C(t)v(t) \in A(\overline{D_0})\}$ is still denoted by U.

Controllability operator $C_0^T : L^2([0,b],\Omega,U) \to L^2(\Omega,F_b,P,\overline{D_0})$ and Controllability Gramian $G_c^b : L^2(\Omega,F_b,P,\overline{D_0}) \to L^2(\Omega,F_b,P,\overline{D_0})$ associated with system (56) are defined as

$$C_0^b v = \int_0^b U(T,\tau)C(\tau)v(\tau)d\tau,$$

$$G_c^b z = \int_0^b U(b,\tau)C(\tau)C^*(\tau)U^*(b,\tau)E(z|F_\tau)d\tau,$$

respectively. It is obvious that operators C_0^b and G_c^b are bounded linear operators, and the dual

$$C_0^{b*} : L^2(\Omega,F_b,P,\overline{D_0}) \to L^2([0,b],\Omega,U)$$

of C_0^b is defined by $C_0^{b*}z = C^*(\tau)U^*(b,\tau)E(z|F_\tau)$, where $z \in L^2(\Omega,F_b,P,\overline{D_0})$ and

$$G_c^b = C_0^b C_0^{b*}.$$

Definition 28. *(a) Time varying stochastic singular system (56) is said to be exactly controllable on $[0,b]$, if for all $x_0 \in L^2(\Omega,F_0,P,\overline{D_0})$, $x_b \in L^2(\Omega,F_b,P,\overline{D_0})$, there exists $v(t) \in L^2([0,b],\Omega,U)$, such that the mild solution $x(t,x_0)$ to (56) satisfies $x(T,x_0) = x_b$;*

(b) Time varying stochastic singular system (56) is said to be approximately controllable on $[0,b]$, if for any state $x_b \in L^2(\Omega,F_b,P,\overline{D_0})$, any initial state $x_0 \in L^2(\Omega,F_0,P,\overline{D_0})$, and any $\epsilon > 0$, there exists a $v \in L^2([0,b],\Omega,U)$, such that the mild solution $x(t,x_0)$ to (56) satisfies

$$\|x(b,x_0) - x_b\|_{L^2(\Omega,F_b,P,\overline{D_0})} < \epsilon.$$

The following results were obtained in [80].

Theorem 42 ([80])**.** *The necessary and sufficient conditions for time-varying stochastic singular system (56) to be exactly controllable on $[0,b]$ are $\mathrm{ran}C_0^b = L^2(\Omega,F_b,P,\overline{D_0})$.*

Theorem 43 ([80])**.** *Time varying stochastic singular system (56) is exactly controllable on $[0,b]$ if, and only if, one of the following conditions is true:*
(a) $< G_c^b z, z >_{L^2(\Omega,F_b,P,\overline{D_0})} \geq \gamma \|z\|^2_{L^2(\Omega,F_b,P,\overline{D_0})}$ for some $\gamma > 0$ and all

$$z \in L^2(\Omega,F_b,P,\overline{D_0});$$

(b) $\lim_{\lambda\to 0^+} \|(\lambda I + G_c^T)^{-1} - (G_c^T)^{-1}\|_{L(L^2(\Omega,F_b,P,\overline{D_0}),L^2(\Omega,F_b,P,\overline{D_0}))} = 0$;
(c) $\lim_{\lambda\to 0^+} \|\lambda(\lambda I + G_c^T)^{-1}\|_{L(L^2(\Omega,F_b,P,\overline{D_0}),L^2(\Omega,F_b,P,\overline{D_0}))} = 0$;
(d) $\|C_0^{b*}z\|_{L^2([0,b],\Omega,U)} \geq \gamma\|z\|_{L^2(\Omega,F_b,P,\overline{D_0})}$ for some $\gamma > 0$ and all $z \in L^2(\Omega,F_b,P,\overline{D_0})$;
(e) $\ker(C_0^{b*}) = \{0\}$ and $\mathrm{ran}(C_0^{b*})$ is closed.

Theorem 44 ([80])**.** *The necessary and sufficient conditions for time varying stochastic singular system (56) to be approximately controllable on $[0,T]$ are that one of the following conditions is true:*
(a) $< G_c^b z, z >_{L^2(\Omega,F_b,P,\overline{D_0})} > 0$ for all $z \in L^2(\Omega,F_b,P,\overline{D_0}), z \neq 0$;
(b) $\lim_{\lambda\to 0^+} < \lambda(\lambda I + G_c^T)^{-1}x, z >_{L^2(\Omega,F_b,P,\overline{D_0})} = 0$ for all $x, z \in L^2(\Omega,F_b,P,\overline{D_0})$;
(c) $\lim_{\lambda\to 0^+} \|\lambda(\lambda I + G_c^T)^{-1}z\|_{L^2(\Omega,F_b,P,\overline{D_0})} = 0$ for all $z \in L^2(\Omega,F_b,P,\overline{D_0})$;
(d) $\ker(C_0^{b*}) = \{0\}$.

The details of applicable example can be found in [80].

5.4. Stochastic GE-Evolution Operator Method for a Class of Time Invariant Systems

In this subsection, we discuss the controllability of the following time varying stochastic singular linear system by using stochastic GE-evolution operator in Banach spaces,

$$Adx(t) = Bx(t)dt + Cv(t)dt + Dx(t)dw(t), t \geq 0, x(0) = x_0, \tag{58}$$

where $x(t)$ is the state process valued in H, $v(t)$ is the control process valued in U, $w(t)$ is the one-dimensional standard Wiener process, $x_0 \in L^2(\Omega, F_0, P, H)$ is a given random variable, H, U are Banach spaces; $A, D \in B(H), C \in B(U, H), B : \mathrm{dom}(B) \subseteq H \to H$ is a linear operator. The organization of this subsection is as follows. Firstly, the concept of stochastic GE-evolution operator is introduced, and the mild solution to system (58) is given by stochastic GE-evolution operator. Secondly, The exact controllability and approximate controllability of (58) are discussed by stochastic GE-evolution operator in the sense of mild solution in Banach spaces, respectively.

5.4.1. Stochastic GE-Evolution Operator and Mild Solution of System (58)

In the following, the stochastic GE-evolution operator is introduced, and the mild solution of system (58) is give by stochastic GE-evolution operator.

Definition 29 ([81]). *Let $\Delta_b = \{(t,s) : 0 \leq s \leq t \leq b\}$. A family of stochastic operators $\{S(t,s) : (t,s) \in \Delta_b\}$ on H is said to be a stochastic GE-evolution operator induced by A on $[0,b]$ if it has the following properties:*
(i) $S : \Delta_b \times \Omega \to B(H)$ is strongly measurable;
(ii) $S(t,s)$ is strongly F_t-measurable for $t \geq s$;
(iii) $S(s,s) = S_0, 0 \leq s \leq b$, and $S(t,r)AS(r,s) = S(t,s)$ for any $0 \leq s \leq r \leq t \leq b$, where $S_0 \in B(H)$ is a steady operator independent of s;
(iv) For any $\xi \in H, (t,s) \to S(t,s)\xi$ is mean square continuous from Δ_T into H.

In the following, we always suppose that B is a generator of GE-semigroup $U(t)$ induced by A.

Now, we consider the mild solution of stochastic singular linear system (58).

Definition 30. *If $v(t) \in L^2([0,b], \Omega, U), x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$, then the mild solution $x(t, x_0) \in L^2([0,b], \{F_t\}, \overline{D_1})$ to (58) is defined by*

$$x(t, x_0) = U(t)Ax_0 + \int_0^t U(t-\tau)Cv(\tau)d\tau + \int_0^t U(t-\tau)Dx(\tau, x_0)dw(\tau), \tag{59}$$

where $L^2([0,b], \{F_t\}, \overline{D_1})$ denotes the Banach space of all $\overline{D_1}-$valued processes x with norm

$$\|x\|_{L^2([0,b],\{F_t\},\overline{D_1})} = \sup_{t\in[0,b]}(E\|x(t)\|^2_{\overline{D_1}})^{1/2} < +\infty.$$

Lemma 2 ([81]). *If $v(t) \in L^2([0,b], \Omega, U), x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$;*

$$Cv(t) \in A(L^2([0,b], \Omega, \overline{D_1})),$$

then system (58) has a unique mild solution $x(t, x_0) \in L^2([0,b], \{F_t\}, \overline{D_1})$, which is given by (59).

Definition 31. *We say that stochastic GE-evolution operator $S(t,s)$ induced by A is related to the linear homogeneous equation*

$$Adx(t) = Bx(t)dt + Dx(t)dw(t), x(s) = x_0, 0 \leq s \leq t \leq b, \tag{60}$$

if $x(t) = S(t,s)Ax_0$ *is the mild solution to (60) with* $x(s) = S(s,s)Ax_0 = x_0$ *for arbitrary* $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$.

In the following, we suppose that there exists a stochastic GE-evolution operator $S(t,s)$ induced by A related to (60) and Lemma 2 holds true. Furthermore, we suppose that the following estimates hold for any $0 \le s \le t \le b$ and $\xi \in L^2(\Omega, F_s, P, \overline{D_1})$:

$$E\int_s^t \|S(r,s)\xi\|^2_{\overline{D_1}} dr \le c\|\xi\|^2_{L^2(\Omega,F_s,P,\overline{D_1})};$$

$$\sup_{r\in[s,t]} E\|S(r,s)\xi\|^2_{\overline{D_1}} \le c\|\xi\|^2_{L^2(\Omega,F_s,P,\overline{D_1})}.$$

We can obtain the following theorem.

Theorem 45 ([81])**.** *The mild solution* $x(t,x_0)$ *to (58) can be written in the form*

$$x(t,x_0) = S(t,0)Ax_0 + \int_0^t S(t,s)Cv(s)ds. \tag{61}$$

5.4.2. Controllability of System (58)

In the following, we discuss the exact and approximate controllability of stochastic singular linear system (58) by using stochastic GE-evolution operator theory, some criteria are obtained. In order to discuss the controllability, we introduce the following concepts.

Banach space $\{v(t) \in U : Cv(t) \in A(\overline{D_1})\}$ is still denoted by U.

Controllability operator $C_0^b : L^2([0,b], \Omega, U) \to L^2(\Omega, F_b, P, \overline{D_1})$ associated with system (58) is defined as

$$C_0^b v = \int_0^b S(T,\tau)Cv(\tau)d\tau.$$

It is obvious that operator C_0^b is a bounded linear operator, and the dual

$$C_0^{b*} : L^2(\Omega, F_b, P, \overline{D_0}) \to L^2([0,b], \Omega, U)$$

of C_0^b is defined by $C_0^{b*}z = C^*S^*(b,\tau)E(z|F_\tau)$, where $z \in L^2(\Omega, F_b, P, \overline{D_1})$.

Definition 32. *(a) Stochastic singular linear system (58) is called to be exactly controllable on* $[0,b]$*, if for all* $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$, $x_b \in L^2(\Omega, F_b, P, \overline{D_1})$*, there exists* $v(t) \in L^2([0,b], \Omega, U)$*, such that the mild solution* $x(t,x_0)$ *to stochastic singular linear system (58) which is given by (61) satisfies* $x(T,x_0) = x_b$*;*

(b) Stochastic singular linear system (58) is called to be approximately controllable on $[0,b]$*, if for any state* $x_b \in L^2(\Omega, F_b, P, \overline{D_1})$*, any initial state* $x_0 \in L^2(\Omega, F_0, P, \overline{D_1})$*, and any* $\epsilon > 0$*, existence* $v \in L^2([0,b], \Omega, U)$ *makes that the mild solution* $x(t,x_0)$ *which is given by (61) satisfies*

$$\|x(b,x_0) - x_b\|_{L^2(\Omega,F_b,P,\overline{D_1})} < \epsilon.$$

The following results were obtained in [81].

Theorem 46 ([81])**.** *Stochastic singular system (58) is exactly controllable on* $[0,b]$ *if, and only if,* $\mathrm{ran}(C_0^b) = L^2(\Omega, F_b, P, \overline{D_1})$.

Theorem 47 ([81])**.** *Assume that H and U are reflexive Banach spaces. Stochastic singular system (58) is exactly controllable on* $[0,b]$ *if and only if one of the following conditions holds:*
(a) $\|C_0^{b*}z^*\|_{L^2([0,b],\Omega,U^*)} \ge \gamma\|z^*\|_{L^2(\Omega,F_b,P,(\overline{D_1})^*)}$ *for some* $\gamma > 0$ *and all*

$$z^* \in L^2(\Omega, F_b, P, (\overline{D_1})^*);$$

(b) $\ker(C_0^{b*}) = \{0\}$ *and* $\mathrm{ran}(C_0^{b*})$ *is closed.*

Theorem 48 ([81]). *The necessary and sufficient condition for the stochastic singular linear system (58) to be approximately controllable on* $[0, b]$ *is* $\overline{\mathrm{ran}(C_0^b)} = L^2(\Omega, F_b, P, \overline{D_1})$.

Theorem 49 ([81]). *Stochastic singular systems (58) is approximate controllable on* $[0, b]$ *if, and only if, one of the following conditions holds:*
(a) $\|C_0^{b*}z^*\|_{L^2([0,b],\Omega,U^*)} > 0$ *for all* $z^* \in L^2(\Omega, F_b, P, (\overline{D_1})^*), z^* \neq 0$;
(b) $\ker(C_0^{b*}) = \{0\}$.

The practical example can be found in [81] if there is a need.

5.5. Stochastic GE-Evolution Operator Method for a Class of Time-Varying Systems

In this subsection, we study the controllability and observability of the following time varying stochastic singular linear system by using stochastic GE-evolution operator in Banach spaces,

$$O_1 dv(t) = O_2(t)v(t)dt + O_3(t)u(t)dt + O_4(t)v(t)dw(t), t \geq 0, v(0) = v_0,$$
$$x(t) = O_5(t)v(t), \tag{62}$$

where $v(t)$ is the state process valued in Y_1, $u(t)$ is the control process valued in Y_2, $w(t)$ is the one-dimensional standard Wiener process, $v_0 \in L^2(\Omega, F_0, P, Y_1)$ is a given random variable, $x(t)$ is the output process valued in Y_3, Y_1, Y_2, Y_3 are Banach spaces;

$$O_1 \in B(Y_1), O_3(t) \in P([0, T], B(Y_2, Y_1)), O_4(t) \in P([0, b], B(Y_1)),$$

$O_5(t) \in P([0, b], B(Y_1, Y_3)), O_2(t)$ is a linear operator from $\mathrm{dom}(O_2(t)) \subseteq Y_1$ to Y_1; $O_1, O_2(t)$, $O_3(t), O_4(t), O_5(t)$ are deterministic and constant operators; This subsection is organized as follows. Firstly, the mild solution of (62) is obtained by stochastic GE-evolution operator; Secondly, the exact controllability of (62) is discussed by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces; Thirdly, the approximate controllability of (62) is discussed by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces; Fourthly, the observability of (62) is studied, and the dual principle is given; At last, we give an example to illustrate the validity of the theoretical results obtained in this subsection.

5.5.1. Mild Solution of System (62)

In this part, we always suppose that $O_2(t)$ is a generator of GE-evolution operator $V(t, s)$ induced by O_1 and

$$D = \{v \in \mathrm{dom}O_2(t) \subseteq Y_1, V_0 O_1 v = v,$$
$$\exists \lim_{h \to 0^+} \frac{O_1 V(t + h, t)O_1 - O_1 V(t, t)O_1}{h} v, 0 \leq t \leq b\}$$

is independent of $t, 0 \leq t \leq b$.

Now, we consider the mild solution of time varying stochastic singular linear Equation (62).

Definition 33. *If* $u(t) \in L^2([0, b], \Omega, Y_2), v_0 \in L^2(\Omega, F_0, P, \overline{D})$, *then the mild solution* $v(t, v_0) \in L^2([0, b], \{F_t\}, \overline{D})$ *to time varying stochastic singular Equation (62) is defined by*

$$v(t, v_0) = V(t, 0)O_1 v_0 + \int_0^t V(t, \tau)O_3(\tau)u(\tau)d\tau + \int_0^t V(t, \tau)O_4(\tau)v(\tau, v_0)dw(\tau). \tag{63}$$

Lemma 3. *Time varying stochastic singular Equation (62) has a unique mild solution, which is given by (63), if* $u(t) \in L^2([0,b],\Omega,Y_2), v_0 \in L^2(\Omega,F_0,P,\overline{D})$;

$$O_3(t)u(t) \in O_1(L^2([0,b],\Omega,\overline{D})),$$

and $(V_0O_2(t))|_D$ *satisfies following assumptions:*

(P_1) For $t \in [0,b], (\lambda I + (V_0O_2(t))|_D)^{-1}$ *exists for all* λ *with* $Re\lambda \leq 0$ *and there is a constant M, such that*

$$\|(\lambda I + (V_0O_2(t))|_D)^{-1}\|_{B(Y_1)} \leq \frac{M}{|\lambda|+1},$$

for all $Re\lambda \leq 0, t \in [0,b]$, *where I denotes the identical operator on D,* $(V_0O_2(t))|_D$ *denotes the restriction of* $V_0O_2(t)$ *on D.*

(P_2) There exist constants L and $0 < \alpha \leq 1$, *such that*

$$\|((V_0O_2(t))|_D - (V_0O_2(s))|_D)((V_0O_2(\tau))|_D)^{-1}\|_{B(Y_1)} \leq L|t-s|^\alpha,$$

for $s,t,\tau \in [0,b]$.

Proof. First of all, according to Theorem 6.1 of [82] (see P.150 of [82]), we have that $V(t,s)|_{O_1(\overline{D})}$ is a unique evolution operator induced by O_1 with generator $O_2(t)$ on $O_1(\overline{D})$. Let Y_{11} denote the space of all $\overline{D}$ valued processes ξ, such that

$$|\xi|_{Y_{11}} = \sup_{t\in[0,b]}(E\|\xi(t)\|^2_{\overline{D}})^{1/2} < +\infty.$$

For any $\xi(t) \in Y_{11}$ define

$$P_1(\xi)(t) = V(t,0)O_1v_0 + \int_0^t V(t,s)O_3(s)u(s)ds$$

$$+\int_0^t V(t,s)O_4(s)\xi(s)dw(s), t \in [0,b],$$

and

$$P_2(\xi)(t) = \int_0^t V(t,s)O_4(s)\xi(s)dw(s), t \in [0,b].$$

Assume, see (d) of Definition 25, that $\|V(t,s)\|_{B(Y_1)} \leq M_1, 0 \leq s \leq t \leq b$, we have

$$|P_2(\xi)|_{Y_{11}} \leq \sup_{t\in[0,b]}(E\int_0^t \|V(t,s)O_4(s)\xi(s)\|^2_{\overline{D}}ds)^{1/2}$$

$$\leq M_1\|O_4(s)\|_{P([0,b],B(Y_1))}b^{1/2}|\xi|_{Y_{11}}, t \in [0,b].$$

Therefore, if b is sufficient small, P_1 is a contraction and it is easy to see that its unique fixed point can be identified as the mild solution to time varying stochastic singular Equation (62). The case of general b can be handled in a standard way. □

Theorem 50. *Suppose that stochastic GE-evolution operator* $G(t,s)$ *induced by* O_1 *is related to the linear homogeneous time varying stochastic singular equation*

$$O_1dv(t) = O_2(t)v(t)dt + O_4(t)v(t)dw(t), v(s) = v_0, 0 \leq s \leq t \leq b, \tag{64}$$

Lemma 3 holds true, and the following estimates hold for any $0 \leq s \leq t \leq b$ *and* $\xi \in L^2(\Omega,F_s,P,\overline{D})$:

$$E\int_s^t \|G(r,s)\xi\|^2_{\overline{D}}dr \leq c\|\xi\|^2_{L^2(\Omega,F_s,P,\overline{D})};$$

$$\sup_{r\in[s,t]}E\|G(r,s)\xi\|^2_{\overline{D_1}} \leq c\|\xi\|^2_{L^2(\Omega,F_s,P,\overline{D})}.$$

Then, the mild solution $v(t, v_0)$ to time varying stochastic singular Equation (62) can be written in the form

$$v(t, v_0) = G(t, 0)O_1 v_0 + \int_0^t G(t, s)O_3(s)u(s)ds. \tag{65}$$

Proof. Since $G(t, 0)O_1 v_0$ and $G(t, s)O_3(s)u(s)$ are mild solutions of time varying stochastic singular Equation (64) with $v(0) = v_0$ and $v(s) = G(s, s)O_3(s)u(s)$, respectively, we have that

$$G(t, 0)O_1 v_0 = V(t, 0)O_1 v_0 + \int_0^t V(t, \tau)O_4(\tau)G(\tau, 0)O_1 v_0 dw(\tau),$$

$$G(t, s)O_3(s)u(s) = V(t, s)O_1 G(s, s)O_3(s)u(s) + \int_s^t V(t, \tau)O_4(\tau)G(\tau, s)O_3(s)u(s)dw(\tau)$$

$$= V(t, s)O_3(s)u(s) + \int_s^t V(t, \tau)O_4(\tau)G(\tau, s)O_3(s)u(s)dw(\tau).$$

We have to prove that the process $v(t, v_0)$ in (65) is a solution to the integral Equation (63). By the representation of $v(\tau, v_0)$, we have

$$\int_0^t V(t, \tau)O_4(\tau)v(\tau, v_0)dw(\tau) = \int_0^t V(t, \tau)O_4(\tau)G(\tau, 0)O_1 v_0 dw(\tau)$$

$$+ \int_0^t V(t, \tau)O_4 D(\tau)(\int_0^\tau G(\tau, s)O_3(s)u(s)ds)dw(\tau)$$

$$= G(t, 0)O_1 v_0 - V(t, 0)O_1 v_0 + \int_0^t ds \int_s^t V(t, \tau)O_4(\tau)G(\tau, s)O_3(s)u(s)dw(\tau)$$

$$= G(t, 0)O_1 v_0 - V(t, 0)O_1 v_0 + \int_0^t [G(t, s)O_3(s)u(s) - V(t, s)O_1 G(s, s)O_3(s)u(s)]ds$$

$$= G(t, 0)O_1 v_0 - V(t, 0)O_1 v_0 + \int_0^t G(t, s)O_3(s)u(s)ds - \int_0^t V(t, s)O_3(s)u(s)ds,$$

where the stochastic Fubini theorem is given in Theorem 4.33 of [83]. Therefore,

$$v(t, v_0) = G(t, 0)O_1 v_0 + \int_0^t G(t, s)O_3(s)u(s)ds$$

$$= V(t, 0)O_1 v_0 + \int_0^t V(t, \tau)O_3(\tau)u(\tau)d\tau + \int_0^t V(t, \tau)O_4(\tau)v(\tau, v_0)dw(\tau),$$

which proves (63). □

In the following, we always assume that time varying stochastic singular Equation (62) has a unique mild solution in the form of (65).

In order to obtain the criteria of controllability, the following concepts are introduced.

Banach space $\{u(t) \in Y_2 : O_3(t)u(t) \in O_1(\overline{D})\}$ is still denoted by Y_2.

Controllability operator

$$Q_C^b : L^2([0, b], \Omega, Y_2) \to L^2(\Omega, F_b, P, \overline{D})$$

associated with time varying stochastic singular Equation (62) is defined as

$$Q_C^b u = \int_0^T G(T, \tau)O_3(\tau)u(\tau)d\tau.$$

It is obvious that operator Q_C^b is a bounded linear operator, and its dual

$$Q_C^{b*} : L^2(\Omega, F_b, P, (\overline{D})^*) \to L^2([0, b], \Omega, Y_2^*)$$

is defined by
$$Q_C^{b*}y^* = O_3^*(\tau)G^*(b,\tau)E(y^*|F_\tau).$$
where $y^* \in L^2(\Omega, F_b, P, (\overline{D})^*)$.

5.5.2. Exact Controllability of System (62)

In this part, we discuss the exact controllability of time varying stochastic singular Equation (62) by stochastic GE-evolution operator theory, some criteria are obtained.

Definition 34. *Time varying stochastic singular Equation (62) is called to be exactly controllable on $[0,b]$, if for all $v_0 \in L^2(\Omega, F_0, P, \overline{D})$, $v_b \in L^2(\Omega, F_b, P, \overline{D})$, there exists $u(t) \in L^2([0,b], \Omega, Y_2)$, such that the mild solution $v(t, v_0)$ to time varying stochastic singular Equation (62) satisfies $v(b, v_0) = v_b$.*

From the Definition 34, we can obtain the following theorem immediately.

Theorem 51. *Time varying stochastic singular Equation (62) is exactly controllable on $[0,b]$ if, and only if,* $\operatorname{ran}(Q_C^b) = L^2(\Omega, F_b, P, \overline{D})$.

Theorem 52. *Assume that Y_1 and Y_2 are reflexive Banach spaces. Time varying stochastic singular Equation (62) is exactly controllable on $[0,b]$ if, and only if, one of the following conditions holds:*

(a) $\|Q_C^{b*}y^*\|_{L^2([0,b],\Omega,Y_2^*)} \geq \gamma\|y^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)}$ *for some $\gamma > 0$ and all*
$$y^* \in L^2(\Omega, F_b, P, (\overline{D})^*);$$
(b) $\ker(Q_C^{b*}) = \{0\}$ *and* $\operatorname{ran}(Q_C^{b*})$ *is closed.*

Proof. (a) $\Rightarrow$ (b) Notice that (a) implies that Q_C^{b*} is injective. To prove that Q_C^{b*} has closed range, assume that $Q_C^{b*}y_n^*$ is a Cauchy sequence in $L^2([0,b], \Omega, Y_2^*)$, then (a) implies that y_n^* is a Cauchy sequence in $L^2(\Omega, F_b, P, (\overline{D})^*)$. Since Q_C^{b*} is a bounded linear operator, if $\lim_{n\to+\infty} y_n^* = y^*$, then $\lim_{n\to+\infty} Q_C^{b*}y_n^* = Q_C^{b*}y^*$ and so Q_C^{b*} has closed range.
(b)$\Rightarrow$(a). (b) shows that Q_C^{b*} has an algebraic inverse with domain equal to $\operatorname{ran}(Q_C^{b*})$. Since $\operatorname{ran}(Q_C^{b*})$ is closed, it is a Banach space under the norm of $L^2([0,b], \Omega, Y_2^*)$, i.e.,
$$\|u^*\|_{\operatorname{ran}(Q_C^{b*})} = \|u^*\|_{L^2([0,b],\Omega,Y_2^*)}, u^* \in \operatorname{ran}(Q_C^{b*}).$$

By Corollary A.3.50 of [84], we have that $(Q_C^{b*})^{-1}$ is bounded on this range, i.e., there exists a $\gamma > 0$, such that
$$\|(Q_C^{b*})^{-1}u^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)} \leq \frac{1}{\gamma}\|u^*\|_{L^2([0,b],\Omega,Y_2^*)},$$
for every $u^* \in \operatorname{ran}(Q_C^{b*})$. Substituting $u^* = C_0^{T*}y^*$ proves (a).

It remains to show that (a) is equivalent to exact controllability of time varying stochastic singular Equation (62).

Necessity. Assume that time varying stochastic singular Equation (62) is exactly controllable. By Theorem 51, we have $\operatorname{ran}(Q_C^b) = L^2(\Omega, F_b, P, \overline{D})$.

If Q_C^b is a one to one operator, then $(Q_C^b)^{-1}$ exists on $L^2(\Omega, F_b, P, \overline{D})$. According to the continuity of operator Q_C^b we have that $(Q_C^b)^{-1}$ is a closed operator. From the closed graph theorem, we obtain that $(Q_C^b)^{-1}$ is a bounded linear operator on $L^2(\Omega, F_b, P, \overline{D})$, i.e.,
$$(Q_C^b)^{-1} \in B(L^2(\Omega, F_b, P, \overline{D}), L^2([0,b], \Omega, Y_2)).$$

Therefore

$$((Q_C^b)^{-1})^* \in B(L^2([0,b],\Omega,Y_2^*),L^2(\Omega,F_b,P,(\overline{D})^*)).$$

This implies that there exists $\gamma_b > 0$, such that

$$\|((Q_C^b)^{-1})^* v^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)} \le \gamma_b \|v^*\|_{L^2([0,b],\Omega,Y_2^*)}. \tag{66}$$

Assume $y^* \in L^2(\Omega,F_b,P,(\overline{D})^*)$, then

$$u^* = Q_C^{b*} y^* \in L^2([0,b],\Omega,Y_2^*).$$

Therefore, for all $y_0 \in L^2(\Omega,F_b,P,\overline{D})$, we find that

$$\begin{aligned} &< y_0, ((Q_C^b)^{-1})^* u^* >=< y_0, ((Q_C^b)^{-1})^* Q_C^{b*} y^* > \\ &=< (Q_C^T)^{-1} y_0, Q_C^{T*} y^* >=< y_0, y^* >, \end{aligned}$$

where $< y_0, y^* >= y^*(y_0)$. From (66), we obtain that

$$\begin{aligned} \|y^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)} &= \sup_{\|y_0\|_{L^2(\Omega,F_b,P,\overline{D})}=1} |< y_0, y^* >| \\ &\le \|((Q_C^b)^{-1})^* u^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)} \\ &\le \gamma_b \|u^*\|_{L^2([0,b],\Omega,Y_2^*)} = \gamma_b \|Q_C^{b*} y^*\|_{L^2([0,b],\Omega,Y_2^*)}, \end{aligned}$$

i.e.,

$$\begin{aligned} \|Q_C^{b*} y^*\|_{L^2([0,b],\Omega,Y_2^*)} &\ge \frac{1}{\gamma_b} \|y^*\|^2_{L^2(\Omega,F_b,P,(\overline{D})^*)} \\ &= \gamma \|y^*\|_{L^2(\Omega,F_b,P,(\overline{D})^*)}, \end{aligned}$$

where $\gamma = \frac{1}{\gamma_b}$. This implies that (a) holds.

If Q_C^b is not a one to one operator, then

$$\ker(Q_C^b) = \{u : u \in L^2([0,b],\Omega,Y_2), Q_C^b u = 0\} \neq \{0\}.$$

A factor space is defined as follows

$$Y_{21} = L^2([0,b],\Omega,Y_2)/\ker(Q_C^b) = \{u_1 : u_1 = \{u + u_2 : u_2 \in \ker(Q_C^b)\}\}.$$

For $u_1 \in Y_{21}$,

$$\|u_1\|_{Y_{21}} = \inf_{u_2 \in \ker(Q_C^b)} \|u + u_2\|_{L^2([0,b],\Omega,Y_2)}.$$

If we define operator

$$Q_1^b : Y_{21} \to L^2(\Omega,F_b,P,\overline{D}), Q_1^b u_1 = Q_C^b u,$$

then

$$Q_1^b \in B(Y_{21}, L^2(\Omega,F_b,P,\overline{D})),$$

and Q_1^b is a bijective operator. It can be seen from the above proof that

$$\|Q_1^{b*} y^*\|_{Y_{21}^*} \ge \gamma \|y^*\|^2_{L^2(\Omega,F_b,P,(\overline{D})^*)}.$$

According to the definition of Y_{21} and Q_1^b, we obtain

$$\|Q_1^{b*} y^*\|_{Y_{21}^*} = \|Q_C^{b*} y^*\|_{L^2([0,b],\Omega,Y_2^*)}.$$

This implies that (a) holds.
Sufficiency. Assume (a). It is need to prove that if $y \in L^2(\Omega, F_b, P, \overline{D})$, then $y \in \mathrm{ran}Q_C^b$. From

$$Q_C^b \in B(L^2([0,b],\Omega,Y_2), L^2(\Omega,F_b,P,\overline{D})),$$

we find that

$$Q_C^{b*} \in B(L^2(\Omega,F_b,P,(\overline{D})^*), L^2([0,b],\Omega,Y_2^*)).$$

For $y \in L^2(\Omega, F_b, P, \overline{D})$, we can define a functional f on $\mathrm{ran}Q_C^{b*}$ satisfying

$$f(Q_C^{b*}g^*) =< y, g^* >, g^* \in L^2(\Omega, F_b, P, (\overline{D})^*). \tag{67}$$

This implies that f is linear for $Q_C^{b*}g^*$. According to (a), if

$$\lim_{n\to\infty} Q_C^{b*}g_n^* = 0,$$

then

$$\lim_{n\to\infty} g_n^* = 0,$$

and

$$\lim_{n\to\infty} f(Q_C^{b*}g_n^*) = \lim_{n\to\infty} < y, g_n^* >= 0.$$

Therefore, f is continuous linear functional on

$$\mathrm{ran}(Q_C^{b*}) \subset L^2([0,b],\Omega,Y_2^*).$$

By Hahn–Banach theorem, we have that f can be extended as a continuous linear functional on $L^2([0,b],\Omega,Y_2^*)$. According to $Y_2^{**} = Y_2$, the existence of

$$u \in L^2([0,b],\Omega,Y_2) = L^2([0,b],\Omega,Y_2^{**})$$

makes

$$f(Q_C^{b*}g^*) =< u, Q_C^{b*}g^* >, g^* \in L^2(\Omega, F_b, P, (\overline{D})^*). \tag{68}$$

According to (67) and (68), we obtain that for every $g^* \in L^2(\Omega, F_b, P, (\overline{D})^*)$,

$$< y, g^* >=< Q_C^b u, g^* > .$$

Hence $y = Q_C^b u$, i.e.,

$$\mathrm{ran}(Q_C^b) = L^2(\Omega, F_b, P, \overline{D}).$$

From Theorem 51, time varying stochastic singular Equation (62) is exactly controllable. □

5.5.3. Approximate Controllability of System (62)

In this section, we discuss the approximate controllability of time varying stochastic singular Equation (62). Some necessary and sufficient conditions are obtained.

Definition 35. *Time varying stochastic singular Equation (62) is called to be approximately controllable on $[0,b]$, if for any state $v_b \in L^2(\Omega, F_b, P, \overline{D})$, any initial state $v_0 \in L^2(\Omega, F_0, P, \overline{D})$, and any $\epsilon > 0$, existence $u \in L^2([0,b],\Omega,Y_2)$ makes that the mild solution $v(t, v_0)$ to time varying stochastic singular Equation (62) satisfies*

$$\|v(b, v_0) - v_b\|_{L^2(\Omega,F_b,P,\overline{D})} < \epsilon.$$

It is obvious that the necessary and sufficient conditions for the time varying stochastic singular Equation (62) to be approximately controllable on $[0, b]$ are

$$\overline{\mathrm{ran}(Q_C^b)} = L^2(\Omega, F_b, P, \overline{D}). \tag{69}$$

Theorem 53. *Time varying stochastic singular Equation (62) is approximate controllable on $[0, b]$ if, and only if, one of the following conditions holds:*
(a) $\|Q_C^{b*} y^*\|_{L^2([0,b],\Omega,Y_2^*)} > 0$ *for all* $y^* \in L^2(\Omega, F_b, P, (\overline{D})^*), y^* \neq 0$;
(b) $\ker(Q_C^{b*}) = \{0\}$.

Proof. It is obvious that (a) is equivalent to (b). We only need to prove that (b) is equivalent to approximate controllability of time varying stochastic singular linear Equation (62).
If

$$\overline{\mathrm{ran}(Q_C^b)} = L^2(\Omega, F_b, P, \overline{D}), y^* \in \ker(Q_C^{b*}),$$

i.e., $Q_C^{b*} y^* = 0$, then

$$< u, Q_C^{b*} y^* >=< Q_C^b u, y^* >, u \in L^2([0, b], \Omega, Y_2).$$

Since $\overline{\mathrm{ran}(Q_C^b)} = L^2(\Omega, F_b, P, \overline{D})$, we have

$$< y, y^* >= 0, y \in L^2(\Omega, F_b, P, \overline{D}).$$

Therefore, $y^* = 0$, i.e., $\ker(Q_C^{b*}) = \{0\}$.
Conversely, if $\ker(Q_C^{b*}) = \{0\}$ but

$$\overline{\mathrm{ran}(Q_C^b)} \neq L^2(\Omega, F_b, P, \overline{D}),$$

then $\overline{\mathrm{ran}(Q_C^b)}$ is the proper subspace of $L^2(\Omega, F_b, P, \overline{D})$. According to Hahn–Banach theorem, there exists

$$y^* \in L^2(\Omega, F_b, P, (\overline{D})^*), y^* \neq 0,$$

such that

$$< Q_C^b u, y^* >= 0, u \in L^2([0, b], \Omega, Y_2).$$

Thus $< u, Q_C^{b*} y^* >= 0$, i.e., $Q_C^{b*} y^* = 0$. By $\ker(Q_C^{b*}) = \{0\}$, we find that $y^* = 0$. This contradicts $y^* \neq 0$. Therefore,

$$\overline{\mathrm{ran}(Q_C^b)} = L^2(\Omega, F_b, P, \overline{D}).$$

Hence (69) is true if, and only if, (b) holds, i.e., time varying stochastic singular Equation (62) is approximately controllable on $[0, b]$ if, and only if, (b) holds. □

5.5.4. Observability

Consider the following time varying stochastic singular equation

$$O_1 dv(t) = O_2(t)v(t)dt + O_4(t)v(t)dw(t), t \geq 0, v(0) = v_0, x(t) = O_5(t)v(t), \tag{70}$$

and its dual time varying stochastic singular equation

$$O_1^* dv^*(t) = O_2^*(t)v^*(t)dt + O_5^*(t)u^*(t)dt + O_4^*(t)v^*(t)dw(t), t \geq 0, v^*(0) = v_0^*. \tag{71}$$

For the time varying stochastic singular Equation (70), the following concepts are defined.
The observability operator of time varying stochastic singular Equation (70) on $[0, b]$ is the continuous linear operator

$$Q_O^T : L^2(\Omega, F_b, P, \overline{D}) \to L^2([0, b], \Omega, Y_3)$$

defined by $Q_O^b y = O_5(t)G(b,t)E(y|F_t)$, its dual operator

$$Q_O^{b*} : L^2([0,b],\Omega,Y_3^*) \to L^2(\Omega,F_b,P,(\overline{D})^*)$$

is defined by

$$Q_O^{b*} x^* = \int_0^b G^*(b,t)O_5^*(t)x^*(t)dt.$$

Definition 36. *Time varying stochastic singular Equation (70) is said to be exactly observable on $[0,b]$ if Q_O^b is injective and its inverse is bounded on* $\mathrm{ran}(Q_O^b)$.

In the case of Definition 36, the state v_0 can be uniquely and continuously constructed from the knowledge of the output $x(t)$ in $L^2([0,b],\Omega,Y_3)$.

Definition 37. *Time varying stochastic singular Equation (70) is said to be approximately observable on $[0,b]$ if Q_O^b is injective.*

In the case of Definition 37, the state v_0 can be uniquely constructed from the knowledge of the output $x(t)$ in $L^2([0,b],\Omega,Y_3)$.

We can obtain the following dual principle.

Theorem 54. *Assume that Y_1 and Y_3 are reflexive. Time varying stochastic singular Equation (70) is exactly (approximately) observable on $[0,b]$ if, and only if, its dual time varying stochastic singular Equation (71) is exactly (approximately) controllable on $[0,b]$.*

Proof. Here, we only prove the case of exact observability. Since

$$Q_O^{b*} x^* = \int_0^b G^*(b,t)O_5^*(t)x^*(t)dt$$

happens to be the controllability operator Q_C^b of time varying stochastic singular Equation (71), so $Q_C^{b*} = Q_O^b$.

If the time varying stochastic singular Equation (70) is exactly observable, then there exists $1/\gamma > 0$, such that

$$\|(Q_O^b)^{-1}x\|_{L^2(\Omega,F_b,P,\overline{D})} \leq \frac{1}{\gamma}\|x\|_{L^2([0,b],\Omega,Y_3)},$$

for all $x \in \mathrm{ran}(Q_O^b)$. This implies that

$$\gamma\|y\|_{L^2(\Omega,F_b,P,\overline{D})} = \gamma\|(Q_O^b)^{-1}Q_O^b y\|_{L^2(\Omega,F_b,P,\overline{D})}$$

$$\leq \|Q_O^b y\|_{L^2([0,b],\Omega,Y_3)} = \|Q_C^{b*} y\|_{L^2([0,b],\Omega,Y_3)},$$

where

$$y = (Q_O^b)^{-1}x, y \in L^2(\Omega,F_b,P,\overline{D}).$$

According to Theorem 52 (a), we have that (71) is exactly controllable.

Assume next that the time varying stochastic singular Equation (71) is exactly controllable. From Theorem 52 (b), we have that Q_O^b is injective and has closed range. According to closed graph theorem $(Q_O^b)^{-1}$ is bounded on $\mathrm{ran}Q_O^T$. □

Theorems 52 and Definitions 36 and 37 yield the following conditions for observability of time varying stochastic singular Equation (70).

Corollary 2. *Time varying stochastic singular Equation (70) is exactly observable on $[0,b]$ if, and only if, one of the following conditions holds for some $\gamma > 0$ and for all $y \in L^2(\Omega,F_b,P,\overline{D})$:*

(a) $\|Q_O^b y\|_{L^2([0,b],\Omega,Y_3)} \geq \gamma \|y\|_{L^2(\Omega,F_b,P,\overline{D})}$;
(b) $\ker(Q_O^b) = \{0\}$ *and* $\operatorname{ran}(Q_O^b)$ *is closed.*

Corollary 3. *Time varying stochastic singular Equation (70) is approximately observable on* $[0,b]$ *if, and only if,* $\ker(Q_O^b) = \{0\}$.

5.5.5. An Illustrative Example

In this part, we give an example to illustrate the effectiveness of the obtained results.

According to [72], in input–output economics, many models were established to describe the real economics. The economics Leontief dynamic input–output model can be extended as an ordinary differential equation of the form:

$$O_1 \frac{dv(t)}{dt} = O_2(t)v(t) + O_3(t)u(t), x(t) = O_5(t)v(t) \tag{72}$$

in Banach space Y_1, where $O_1 \in B(Y_1)$ and $O_2(t) : \operatorname{dom}(O_2(t)) \subseteq Y_1 \to Y_1$ is a linear and possibly unbounded operator, $O_3(t), O_5(t) \in P([0,b], B(Y_1))$, while $v(t)$ and $u(t)$ are state process and control process valued in Y_1, respectively, for $t \geq 0$. However, in reality, there are many unpredicted parameters and different types of uncertainty that have not been implemented in the mathematical modelling process of this equation. Nonetheless, according to [85,86], we can consider a stochastic version of the singular Equation (72) with the one-dimensional standard Wiener process $w(t)$ used to model the uncertainties of the form:

$$O_1 dv(t) = O_2(t)v(t)dt + O_3(t)u(t)dt + O_4(t)v(t)dw(t), x(t) = O_5(t)v(t), \tag{73}$$

where $O_4(t) \in P([0,b], B(Y_1))$. This stochastic version of the input-output model is a time varying stochastic singular equation in Banach space Y_1 of the form (62).

We consider the following unforced time varying stochastic singular equation, i.e., $u(t) = 0$ in time varying stochastic singular Equation (73):

$$O_1 dv(t) = O_2(t)v(t)dt + O_4(t)v(t)dw(t), x(t) = O_5(t)v(t). \tag{74}$$

Time varying stochastic singular Equation (74) is the form of time varying stochastic singular linear Equation (70). In what follows, we will verify the effectiveness of Corollary 3.

If for some concrete engineering practice, the following data are taken in time varying stochastic singular Equation (74):

$$O_1 = \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix}, O_2(t) = \begin{bmatrix} -(2t+1)U_1 & 0 \\ 0 & 5(t^2+1)U_2 \end{bmatrix},$$

$$O_4(t) = \begin{bmatrix} (2t)^{1/2}U_1 & 0 \\ 0 & 3t^2U_2 \end{bmatrix}, O_5(t) = \begin{bmatrix} 7(t+1)^2U_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where U_1, U_2 are identical operators in Banach spaces Y_{11}, Y_{12}, respectively. Time varying stochastic singular Equation (74) can be written as

$$\begin{aligned} \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dv_1(t) \\ dv_2(t) \end{bmatrix} &= \begin{bmatrix} -(2t+1)U_1 & 0 \\ 0 & 5(t^2+1)U_2 \end{bmatrix} \begin{bmatrix} v_1(t)dt \\ v_2(t)dt \end{bmatrix} \\ &\quad + \begin{bmatrix} (2t)^{1/2}U_1 & 0 \\ 0 & 3t^2U_2 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} dw(t), \\ x(t) &= \begin{bmatrix} 7(t+1)^2U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \end{aligned} \tag{75}$$

where $\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} \in Y_{11} \oplus Y_{12} = Y_1$. We can find that $\overline{D} = Y_{11}$. According to [87], we can obtain

$$G(t,s) = \begin{bmatrix} \exp[-\frac{3}{2}t^2 - t + \frac{3}{2}s^2 + s + \int_s^t (2r)^{1/2} w(r) ds] U_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is obvious that time varying stochastic singular Equation (75) satisfies the conditions of Lemma 3. If $\begin{bmatrix} y \\ 0 \end{bmatrix} \in L^2(\Omega, F_b, P, \overline{D})$, and

$$Q_O^b \begin{bmatrix} y \\ 0 \end{bmatrix} = O_5(t) G(b,t) E(\begin{bmatrix} y \\ 0 \end{bmatrix} | F_t) = 0, t \in [0,b],$$

then

$$O_5(b) G(b,b) E(\begin{bmatrix} y \\ 0 \end{bmatrix} | F_b) = 7(b+1)^2 \begin{bmatrix} y \\ 0 \end{bmatrix} = 0,$$

i.e., $y = 0$. This implies that $\ker(Q_O^b) = \{0\}$. Therefore time varying stochastic singular Equation (75) is approximately observable by Corollary 3.

In this section, we have discussed the controllability of some types of stochastic singular linear systems. However, the following problems still need to be studied.

Problem 9. *How about the controllability of the following system?*

$$Ldx(t) = [A(t)x(t) + B(t)u(t)]dt + \sum_{k=1}^{\infty} C_k(t) x(t) dw_{1,k}(t)$$
$$+ \sum_{j=1}^{\infty} D_j(t) u(t) dw_{2,j}(t), x(0) = x_0,$$

where $L \in B(H)$ and $\ker(L) \neq \{0\}$, $A(t) : \mathrm{dom}(A(t)) \subseteq H \to H$ is the generator of a GE-evolution operator induced by L in the Hilbert (or Banach) space H, $B(t) : \mathrm{dom}(B(t)) \subset U \to H$ is a linear operator, U is a Hilbert (or Banach) space; $C_k(t) \in P([0,b], B(H)), D_k(t) \in P([0,b], B(U,H))$, for each $i \in \mathbb{N}$; and in Hilbert spaces,

$$\sum_{k=1}^{\infty} \sup_{0 \le t \le b} \|C_k(t)\|^2_{B(H)} < +\infty, \sum_{k=1}^{\infty} \sup_{0 \le t \le b} \|D_k(t)\|^2_{B(U,H)} < +\infty;$$

in Banach spaces,

$$\sum_{k=1}^{\infty} \sup_{0 \le t \le b} \|C_k(t)\|_{B(H)} < +\infty, \sum_{k=1}^{\infty} \sup_{0 \le t \le b} \|D_k(t)\|_{B(U,H)} < +\infty;$$

the countable set $\{w_{1,k}, w_{2,j}, k, j \in \mathbb{N}\}$ consists of independent standard Wiener processes defined on the stochastic basis $(\Omega, F, \{F_t\}, P)$.

6. Conclusions

We have introduced the latest progress in controllability of stochastic linear systems and put forward some problems that need to be further studied, which includes stochastic linear systems in finite dimensional spaces, stochastic linear systems in infinite dimensional spaces, stochastic singular linear systems in finite dimensional spaces, and stochastic singular linear systems in infinite dimensional spaces. The controllability and observability for a type of time-varying stochastic singular linear systems have been studied by using stochastic GE-evolution operator in the sense of mild solution in Banach spaces, some necessary and sufficient conditions have been obtained, the dual principle has been proved to be true, an example has been given to illustrate the validity of the theoretical results obtained in this part. Readers can easily and comprehensively understand the latest progress concerning the controllability of stochastic linear systems and further problems to be solved. The next research direction is how to solve these problems.

Funding: This research was funded by National Natural Science Foundation of China grant numbers 12126401 and 11926402.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the anonymous reviewers and the Editor for their helpful comments.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Kalman, R.E. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mex.* **1960**, *5*, 102–119.
2. Connors, M.M. Controllability of discrete, linear, random dynamics systems. *SIAM Control Optim.* **1967**, *5*, 183–210. [CrossRef]
3. Aoki, M. On observability of stochastic discrete-time dynamic systems. *J. Frankl. Inst.* **1968**, *286*, 36–38. [CrossRef]
4. Fitts, J.M. On the global observability of nonlinear systems. In Proceedings of the 1st Sympsium on Nonlinear Estimation Theory and Applications, San Diego, CA, USA, 10–12 September 1970; p. 128.
5. Gershwin, S.B. A Controllability Theory for Nonlinear Dynamics Systems. Ph.D. Thesis, Harvard University, Cambridge, MA, USA, 1970.
6. Sunahara, Y.; Kishino, K.; Aihara, S. On stochastic observability of nonlinear discrete-time dynamical systems. *Int. J. Control* **1974**, *19*, 719–732. [CrossRef]
7. Sunahara, Y.; Kabluchi, T.; Asada, Y.; Aihara, S.; Kishino, K. On stochastic controllability for nonlinear systems. *IEEE Trans. Autom. Control* **1974**, *19*, 49–54. [CrossRef]
8. Sunahara, Y.; Aihara, S.; Kishino, K. On the stochastic observability and controllability for nonlinear systems. *Int. J. Control* **1975**, *22*, 65–82. [CrossRef]
9. Chen, H.F. On stochastic observability. *Sci. Sin.* **1977**, *20*, 305–324.
10. Sunahara, Y.; Aihara, S. On stochastic observability and controllability for nonlinear distributed parameter systems. *Inf. Control* **1977**, *34*, 348–371. [CrossRef]
11. Klamka, J.; Socha, L. Some remarks about stochastic controllability. *IEEE Trans. Autom. Control* **1977**, *22*, 880–881. [CrossRef]
12. Harris, S.E. Stochastic controllability of linear discrete systems with multicative noise. *Int. J. Control* **1978**, *27*, 213–227. [CrossRef]
13. Dubov, M.A.; Mordukhovich, B.S. Theory of controllability of linear stochastic systems. *Differ. Equ.* **1978**, *14*, 1609–1612.
14. Zabcjyk, J. *On Stochastic Controllability*; FDS Report, No 34; Universitat Bremen: Bremen, Germany, 1980.
15. Chen, H.F. On stochastic observability and controllability. *Automatica* **1980**, *16*, 179–190. [CrossRef]
16. Zabcjyk, J. Controllability of stochastic linear systems. *Syst. Control Lett.* **1981**, *1*, 25–31. [CrossRef]
17. Ehrhard, M.; Kliemann, W. Controllability of linear stochastic systems. *Syst. Control Lett.* **1982**, *2*, 145–153. [CrossRef]
18. Dubov, M.A.; Morduchovich, B.S. On controllability of infinite dimensional linear stochastic systems. In *Stochastic Control: Proceedings of the 2nd IFAC Symposium, Vilnius, Lithuanian SSR, USSR, 19–23 May 1986 (IFAC Symposia Series)*; Pergamon Press: Oxford, UK; New York, NY, USA, 1987; Volume 2, pp. 307–310.
19. Bensoussan, A. *Stochastic Control of Partially Observable Systems*; Cambridge University Press: London, UK, 1992.
20. Peng, S.G. Backward stochastic differential equation and exact controllability of stochastic control systems. *Prog. Nat. Sci.* **1994**, *4*, 274–283.
21. Bashirov, A.E.; Kerimov, K.R. On controllability conception for stochastic systems. *SIAM J. Control Optim.* **1997**, *35*, 384–398. [CrossRef]
22. Bashirov, A.E.; Mahmudov, N.I. On concepts of controllability for deterministic and stochastic systems. *SIAM J. Control Optim.* **1999**, *37*, 1808–1821. [CrossRef]
23. Bashirov, A.E.; Mahmudov, N.I. Controllability of linear deterministic and stochastic systems. In Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ, USA, 7–10 December 1999; Volume 4, pp. 3196–3202.
24. Mahmudov, N.I.; Denker, A. On controllability of linear stochastic systems. *Int. J. Control* **2000**, *73*, 144–151. [CrossRef]
25. Mahmudov, N.I. Controllability of linear stochastic systems. *IEEE Trans. Autom. Control* **2001**, *46*, 724–731. [CrossRef]
26. Mahmudov, N.I. Controllability of linear stochastic systems in Hilbert spaces. *J. Math. Anal. Appl.* **2001**, *259*, 64–82. [CrossRef]
27. Mahmudov, N.I. On controllability of semilinear stochastic systems in Hilbert spaces. *IMA J. Math. Control Inf.* **2002**, *19*, 363–376. [CrossRef]

28. Liu, Y.Z.; Peng, S.G. Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems. *Automatica* **2002**, *38*, 1417–1423. [CrossRef]
29. Mahmudov, N.I. Controllability and observability of linear stochastic systems in Hilbert spaces. *Prog. Probab.* **2003**, *53*, 151–167.
30. Mahmudov, N.I. Controllability of semilinear stochastic systems in Hilbert spaces. *J. Math. Anal. Appl.* **2003**, *288*, 197–211. [CrossRef]
31. Mahmudov, N.I. Approximate controllability of semilinear deterministic and stochastic evolution equation in abstract spaces. *SIAM J. Control Optim.* **2003**, *42*, 1604–1622. [CrossRef]
32. Mahmudov, N.I.; Zorlu, S. Controllability of non-linear stochastic systems. *Int. J. Control* **2003**, *76*, 95–104. [CrossRef]
33. Ugrinovskii, V.A. Observability of linear stochastic uncertain systems. *IEEE Trans. Autom. Control* **2003**, *48*, 2246–2269. [CrossRef]
34. Mahmudov, N.I.; Zorlu, S. Controllability of semilinear stochastic systems. *Int. J. Control* **2005**, *78*, 997–1004. [CrossRef]
35. Dauer, J.P.; Mahmudov, N.I.; Matar, M.M. Approximate controllability of backward stochastic evolution equations in Hilbert spaces. *J. Math. Anal. Appl.* **2006**, *323*, 42–56. [CrossRef]
36. Mahmudov, N.I.; Mckibben, M.A. Approximate controllability of second order neutral stochastic evolution equation. *Dyn. Contin. Discrite Impuls. Syst. Ser. B Appl. Algorithm* **2006**, *13*, 619–634.
37. Sakthivel, R.; Kim, J.H. On controllability of nonlinear stochastic systems. *Rep. Math. Phys.* **2006**, *58*, 433–443. [CrossRef]
38. Klamka, J. Stochastic controllability of linear stochastic systems with delay in control. *Bull. Pol. Acad. Sci. Tech. Sci.* **2007**, *55*, 23–29.
39. Gorac, D. Approximate controllability for linear stochastic differential equations in infinite dimensions. *Appl. Math. Optim.* **2009**, *60*, 105–132. [CrossRef]
40. Tang, S.J.; Zhang, X. Null controllability for forward and backward stochastic parabolic equations. *SIAM J. Control Optim.* **2009**, *48*, 2191–2216. [CrossRef]
41. Sakthivel, R.; Mahmudov, N.I.; Lee, S.G. Controllability of nonlinear impulsive stochastic systems. *Int. J. Control* **2009**, *82*, 801–807. [CrossRef]
42. Klamska, J. Stochastic controllability of systems with multiple delays in control. *Int. J. Appl. Math. Comput. Sci.* **2009**, *19*, 39–47. [CrossRef]
43. Bashirov, A.E.; Etikan, H.; Semi, N. Partial controllability of stochastic linear systems. *Int. J. Control* **2010**, *83*, 2564–2572. [CrossRef]
44. Sakthivel, R.; Nieto, J.J.; Mahmudov, N.I. Approximate controllability of nonlinear deterministic and stochastic systems with unbounded delay. *Taiwan J. Math.* **2010**, *14*, 1777–1797. [CrossRef]
45. Liu, F.; Peng, S.G. On controllability for stochastic control systems when the cofficient is time-varying. *J. Syst. Sci. Complex.* **2010**, *23*, 270–278. [CrossRef]
46. Liu, Q. Some results on the controllability of forward stochastic heat equations with control on the drift. *J. Funct. Anal.* **2011**, *260*, 832–851. [CrossRef]
47. Sathya, R.; Balachandram, K. Controllability of neutral impulsive Ito type stochastic integrodifferential systems. *Vietnam J. Math.* **2013**, *41*, 59–80. [CrossRef]
48. Liu, Q. Exact controllability for stochastic Schrodinger equations. *J. Differ. Equ.* **2013**, *255*, 2484–2504. [CrossRef]
49. Liu, X. Controllability of some coupled stochastic parabolic systems with fractional order spatial differential operators by one control in the drift. *SIAM J. Control Optim.* **2014**, *52*, 836–860. [CrossRef]
50. Liu, Q. Exact controllability for stochastic transport equations. *SIAM J. Control Optim.* **2014**, *255*, 397–419. [CrossRef]
51. Ning, H.; Qing, G. Approximate controllability of nonlinear stochastic partial differential systems with infinite delay. *Adv. Differ. Equ.* **2015**, *85*, 1–25. [CrossRef]
52. Gao, P.; Chen, M.; Li, Y. Observability estimates and null controllability for forward and backward linear stochastic Kuramoto-Sivashinsky equations. *SIAM J. Control Optim.* **2015**, *53*, 475–500. [CrossRef]
53. Das, S.; Pandey, D.; Sukavanam, N. Existence of solution and approximate controllability of a second order neutral stochastic differential equation with state dependent delay. *Acta Math. Sci.* **2016**, *36B*, 1509–1523. [CrossRef]
54. Fu, X.; Liu, X. Controllability and observability of some complex Ginzburg-Landau equations. *SIAM J. Control Optim.* **2017**, *55*, 1102–1127. [CrossRef]
55. Mokkedem, F.Z.; Fu, X. Approximate controllability for a semilinear stochastic evolution systems with infinite delay in L_p space. *Appl. Math. Optim.* **2017**, *75*, 253–283. [CrossRef]
56. Klamka, J.; Wyzwal, J.; Zawiski, R. On controllability of second order dynamical systems-a survey. *Bull. Pol. Acad. Sci. Tech. Sci.* **2017**, *65*, 279–295. [CrossRef]
57. Shukla, A.; Sukavanam, N.; Pandey, D.N. Controllability of semilinear stochastic control system with finite delay. *IMA J. Math. Control Inf.* **2018**, *35*, 427–449. [CrossRef]
58. Liu, L.Y.; Liu, X. Controllability and observability of some coupled stochastic parabolic system. *Math. Control Relat. Fields* **2018**, *8*, 829–854. [CrossRef]
59. Wang, Y.Q.; Yang, D.H.; Yong, J.M.; Yu, Z. Exact controllability of linear stochastic differential equations and related problems. *Math. Control Relat. Fields* **2017**, *7*, 305–345. [CrossRef]
60. Barbu, V.; Tubaro, L. Exact controllability of stochastic equations with multiplicative noise. *Syst. Control Lett.* **2018**, *122*, 19–23. [CrossRef]

61. Wang, Y.Q.; Zhou, X.X. Exact controllability of stochastic differential equations with memory. *Syst. Control Lett.* **2020**, *142*, 1–10. [CrossRef]
62. Sirbu, M.; Tessitore, G. Null controllability of an infinite dimensional SDE with stateand control-dependent noise. *Syst. Control Lett.* **2001**, *44*, 385–394. [CrossRef]
63. Shen, L.J.; Li, C.X.; Wang, L. Some remarks on the controllability of linear stochastic systems. In Proceedings of the 34th Chinese Control Conference, Hangzhou, China, 28–30 July 2015; pp. 598–601.
64. Dou, F.F.; Lu, Q. Partial approximate controllability for linear stochastic control systems. *SIAM J. Control Optim.* **2019**, *57*, 1209–1229. [CrossRef]
65. Gashi, B.; Pantelous, A.A. Linear backward stochastic differential equations of descriptor type: Regular systems. *Stoch. Anal. Appl.* **2013**, *31*, 142–166. [CrossRef]
66. Gashi, B.; Pantelous, A.A. Linear stochastic systems of descriptype: Theory and applications, safety, reliability, risk and life-cycle performance of structure and infrastructures. In Proceedings of the 11th International Conference on Structure Safety and Reliability, ICOSSAR 2013, New York, NY, USA, 16–20 June 2013; pp. 1047–1054.
67. Gashi, B.; Pantelous, A.A. Linear backward stochastic differential systems of descriptor type with structure and applications to engineering. *Probab. Eng. Mech.* **2015**, *40*, 1–11. [CrossRef]
68. Ge, Z.Q.; Ge, X.C. An exact null controllability of stochastic singular systems. *Sci. China Inf. Sci.* **2021**, *64*, 179202:1–179202:3. [CrossRef]
69. Ge, Z.Q. Impulse controllability and impulse observability of stochastic singular systems. *J. Syst. Sci. Complex.* **2021**, *34*, 899–911. [CrossRef]
70. Ge, Z.Q. Exact observability and stability of stochastic implicit systems. *Syst. Control Lett.* **2021**, *157*, 1–7. [CrossRef]
71. Liaskos, K.B.; Pantelous, A.A.; Stratis, I.G. Linear stochastic degenerate Sobolev equations and applications. *Int. J. Control* **2015**, *88*, 2538–2553. [CrossRef]
72. Liaskos, K.B.; Stratis, I.G.; Pantelous, A.A. Stochastic degenerate Sobolev equations: Well posedness and exact controllability. *Math. Meth. Appl. Sci.* **2018**, *41*, 1025–1032. [CrossRef]
73. Ge, Z.Q.; Zhu, G.T.; Feng, D.X. Exact controllability for singular distributed parameter system in Hilbert space. *Sci. China Ser. F Inf. Sci.* **2009**, *52*, 2045–2052. [CrossRef]
74. Ge, Z.Q.; Zhu, G.T.; Feng, D.X. Generalized operator semigroup and well-posedness of singular distributed parameter systems. *Sci. Sin. Math.* **2010**, *40*, 477–495.
75. Ge, Z.Q.; Feng, D.X. Well-posed problem of nonlinear singular distributed parameter systems and nonlinear GE-semigroup. *Sci. China Ser. F Inf. Sci.* **2013**, *56*, 128201:1–128201:14. [CrossRef]
76. Ge, Z.Q. Controllability and observability of stochastic singular systems in Banach spaces. *J. Syst. Sci. Complex.* **2021**. [CrossRef]
77. Ge, Z.Q. GE-semigroup method for controllability of stochastic descriptor linear systems. *Sci. China Inf. Sci.* **2021**. [CrossRef]
78. Ge, Z.Q.; Feng, D.X. Solvability of a time-varying singular distributed parameter system in Banach space. *Sci. Sin. Inf.* **2013**, *43*, 386–406. [CrossRef]
79. Ge, Z.Q.; Feng, D.X. Well-posed problem of nonlinear time varying singular distributed parameter systems. *Sci. Sin. Math.* **2014**, *44*, 1277–1298. [CrossRef]
80. Ge, Z.Q. GE-evolution operator method for controllability of time varying stochastic descriptor linear systems in Hilbert spaces. *IMA J. Math. Control. Inf.* **2021**. [CrossRef]
81. Ge, Z.Q. Controllability and observability of stochastic implicit systems and stochastic GE-evolution operator. *Numer. Algebra Control Optim.* **2021**. [CrossRef]
82. Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983; pp. 149–150.
83. Prato, G.D.; Zabczyk, J. *Stochastic Equation in Infinite Dimensions*; Cambridge University Press: Cambridge, UK, 2014; pp. 119–216.
84. Curtain, R.; Zwart, H.J. *An Introduction to Infinite Dimensional Linear Systems Theory*; Springer: New York, NY, USA, 1995; pp. 581–608.
85. Oksendal, B. *Stochastic Differential Equation: An Introduction with Application*; Springer: New York, NY, USA, 1998; pp. 101–217.
86. Mao, X. *Stochastic Differential Equation and Their Applications*; Horwood Publishing: Chichester, UK, 1997; pp. 114–217.
87. Hu, S.G.; Huang, C.M.; Wu, F.K. *Stochastic Differential Equation*; Science Press: Beijing, China, 2008; pp. 55–62.