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Real Hypersurfaces in Complex Grassmannians of Rank Two

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Abstract: It is known that there does not exist any Hopf hypersurface in complex Grassmannians of rank two of complex dimension $2m$ with constant sectional curvature for $m \geq 3$. The purpose of this article is to extend the above result, and without the Hopf condition, we prove that there does not exist any locally conformally flat real hypersurface for $m \geq 3$.

Keywords: complex Grassmannians of rank two; real hypersurface; locally conformally flat; constant sectional curvature

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1. Introduction

Recall that a Riemannian manifold (M^n, g) of dimension $n \geq 4$ is called *locally conformally flat* if and only if the Weyl curvature tensor on M^n vanishes identically, of which the study has always been an important subject in Riemannian geometry, and especially from the point of view of submanifold theory. On the latter, the locally conformally flat hypersurfaces of dimension greater than three in space forms were classified completely by do Carmo et al. [1]. It is worth pointing out that it admits no real hypersurfaces even with harmonic Weyl curvature tensor when the ambient space is complex space form $\bar{N}^n(\bar{c})$ of constant holomorphic sectional curvature $\bar{c} \neq 0$ and complex dimension $n \geq 3$, which was proven by Ki et al. in [2]. Thus, it follows that there are no locally conformally flat real hypersurfaces in such complex space form $\bar{N}^n(\bar{c})$.

In recent decades, the study of Riemannian submanifolds has been extended to the ambient spaces, which are symmetric spaces other than real space forms and complex space forms. In particular, related to the study of real hypersurfaces in both complex two-plane Grassmannian $SU_{m+2}/S(U_2U_m)$ and complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, there are many interesting results that have been established in the last few decades; for details, see, e.g., [3–12] and the references therein.

The compact complex two-plane Grassmannian $SU_{m+2}/S(U_2U_m)$ consists of all the complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$, whereas the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$ of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$ is noncompact. By a unified notation, we denote by $\hat{M}^m(c)$ the one of the compact type (resp. noncompact type) for $c > 0$ (resp. $c < 0$), where $c = \max\|K\|/8$ is a scaling factor for its Riemannian metric g and sectional curvature K (see [11,12]). These are Hermitian symmetric spaces of rank two and complex dimension $2m$ equipped with a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J.

Let M be a connected and orientable real hypersurface in $\hat{M}^m(c)$ with N its normal vector field, whose induced metric is still denoted by g. Then, the Reeb vector field ξ on M is defined by $\xi = -JN$. Moreover, besides the almost contact metric structure (ϕ, ξ, η, g) induced from g and J, there exists a local almost contact metric three-structure $(\phi_a, \xi_a, \eta_a, g)$

induced from g and $\mathfrak{J}$, where $\xi_a = -J_a N$ for $a \in \{1,2,3\}$ and $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathfrak{J}$ (for details, see Section 2). In particular, we denote by $\mathfrak{D}^\perp$ the distribution on M spanned by $\{\xi_1, \xi_2, \xi_3\}$.

For the shape operator A, a real hypersurface in $\hat{M}^m(c)$ is said to be *Hopf* if it satisfies $A\xi = \alpha\xi$ for $\alpha = g(A\xi, \xi)$. The study of Hopf hypersurfaces in $\hat{M}^m(c)$ was initiated by Berndt and Suh in [13] for $c > 0$ and [14] for $c < 0$, respectively. More precisely, we have the following two well-known classification theorems.

Theorem 1 ([13])**.** *Let M be a connected real hypersurface in $SU_{m+2}/S(U_2U_m)$, $m \geq 3$. Then, both $\mathbb{R}\xi$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if M is an open part of one of the following spaces:*

- *(A) A tube around a totally geodesic $SU_{m+1}/S(U_2U_{m-1})$ in $SU_{m+2}/S(U_2U_m)$;*
- *(B) A tube around a totally geodesic $\mathbb{H}P_n = Sp_{n+1}/Sp_1Sp_n$ in $SU_{m+2}/S(U_2U_m)$, where $m = 2n$ is even.*

Theorem 2 ([14])**.** *Let M be a connected real hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 2$. Then, both $\mathbb{R}\xi$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if one of the following holds:*

- *(A) M is an open part of a tube around a totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$;*
- *(B) M is an open part of a tube around a totally geodesic $\mathbb{H}H_n = Sp_{1,n}/Sp_1Sp_n$ in $SU_{2,m}/S(U_2U_m)$, where $m = 2n$ is even;*
- *(C_1) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \in \mathfrak{J}N$;*
- *(C_2) M is an open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$;*
- *(D) The normal bundle of M consists of singular tangent vectors of type $JN \perp \mathfrak{J}N$. Moreover, M has at least four distinct principal curvatures, which are given by (c is a negative constant):*

$$\alpha = 2\sqrt{-c}, \quad \gamma = 0, \quad \lambda = \sqrt{-c},$$

with corresponding principal curvature spaces:

$$T_\alpha = \mathbb{R}\xi \oplus \mathfrak{D}^\perp, \quad T_\gamma = \mathfrak{J}\xi, \quad T_\lambda \subset \mathcal{H}.$$

If μ is another (possibly nonconstant) principal curvature function, then $T_\mu \subset \mathcal{H}$, $JT_\mu \subset T_\lambda$, and $\mathfrak{J}T_\mu \subset T_\lambda$.

Since then, a number of interesting results on Hopf hypersurfaces in $\hat{M}^m(c)$ have been obtained continuously. One of the best known results is that it admits no Hopf hypersurfaces with a parallel Ricci tensor in $\hat{M}^m(c)$, by which we easily see that there does not exist any Einstein–Hopf hypersurface (see [15,16]). From this, it is clear that all of these canonical real hypersurfaces of $\hat{M}^m(c)$ given in Theorem **??** and Theorem 2 are not of a constant sectional curvature.

Meanwhile, such non-existence results have also been established by geometers without the Hopf condition. When $c > 0$, Suh in [17] proved that there are no real hypersurfaces with a parallel shape operator, and later, this was generalized to the ones with a semi-parallel shape operator by Loo [18]. Moreover, for real hypersurfaces in $\hat{M}^m(c)$, an immediate consequence of the Codazzi equation states that the totally umbilicity is too strong to be satisfied (see also [11]). It should be pointed out that all the results mentioned above are related to real hypersurfaces in $\hat{M}^m(c)$ for $m \geq 3$.

Motivated by the above statements, the next important problem associated with real hypersurfaces in $\hat{M}^m(c)$ becomes natural and interesting:

Problem. *Does there exist any locally conformally flat real hypersurface in $\hat{M}^m(c)$ for $m \geq 3$?*

In this paper, we focus on studying the problem above, and as the main result, the following non-existence theorem is proven.

Theorem 3. *There does not exist any locally conformally flat real hypersurface in* $\hat{M}^m(c)$ *for* $m \geq 3$.

Remark 1. *It was recently proven in [19] that there does not exist any locally conformally flat real hypersurface in both the complex quadric* $Q^n = SO_{n+2}/SO_nSO_2$ *and its dual space* $Q^{n*} = SO_{n,2}/SO_nSO_2$ *for* $n \geq 3$*, which are viewed as another kind of Hermitian symmetric spaces with rank two.*

As a direct consequence of Theorem 3, we have the following result.

Corollary 1. *There does not exist any real hypersurface with constant sectional curvature in* $\hat{M}^m(c)$ *for* $m \geq 3$.

Finally, it should be noted that the new method used to prove Theorem 3 is now called the *Tsinghua principle* due to H. Li, L. Vrancken and X. Wang (cf. [20]), by which one can combine the Codazzi equation with the Ricci identity in a new way to obtain some nice linear equations involving the components of the second fundamental form. Recently, this remarkable principle has been widely applied and proven to be very useful for various purposes; see, e.g., [19–25].

2. Preliminaries

In this section, we begin with some basic geometric properties of the complex Grassmannians of rank two $\hat{M}^m(c)$ besides those stated in the introduction section. Then, we recall the theory of real hypersurfaces in $\hat{M}^m(c)$. Furthermore, we state some fundamental equations for such hypersurfaces, which are needed for the proof of Theorem 3. More details can be found in [11,13,14,18,26,27].

2.1. The Complex Grassmannians of Rank Two $\hat{M}^m(c)$

The complex Grassmannians of rank two $\hat{M}^m(c)$ are both equipped with a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$. Here, J_a is a local almost Hermitian structure, and it holds:

$$J_aJ_{a+1} = J_{a+2} = -J_{a+1}J_a, \ \ a \in \{1,2,3\}, \tag{1}$$

where the index is taken modulo three. Denote by $\hat{\nabla}$ the Levi–Civita connection corresponding to the Riemannian metric g on $\hat{M}^m(c)$. Then, the local one-forms q_1, q_2, and q_3 can be defined by:

$$\hat{\nabla}_X J_a := q_{a+2}(X)J_{a+1} - q_{a+1}(X)J_{a+2} \tag{2}$$

for any $X \in T\hat{M}^m(c)$. Here, the index is taken modulo three. The relation in (2) means that $\mathfrak{J}$ is parallel, corresponding to $\hat{\nabla}$. Moreover, it is known that the Kähler structure J and the quaternionic Kähler structure $\mathfrak{J}$ are related by the following:

$$JJ_a = J_aJ, \ \ (JJ_a)^2 = \mathrm{Id}, \ \ \mathrm{Tr}(JJ_a) = 0, \tag{3}$$

where the tensor field JJ_a defined locally is self-adjoint and Id denotes the identity transformation.

In terms of the structures mentioned above and tangent vector fields X, Y, Z on $\hat{M}^m(c)$, the Riemannian curvature tensor $\hat{R}$ of $\hat{M}^m(c)$ can be locally expressed as:

$$\begin{aligned}\hat{R}(X,Y) = c\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ\} \\ &+ c\sum_{a=1}^{3}\{g(J_aY,Z)J_aX - g(J_aX,Z)J_aY - 2g(J_aX,Y)J_aZ \\ &+ g(JJ_aY,Z)JJ_aX - g(JJ_aX,Z)JJ_aY\}.\end{aligned} \tag{4}$$

2.2. Real Hypersurfaces in $\hat{M}^m(c)$

Let M be a connected and oriented real hypersurface isometrically immersed in $\hat{M}^m(c)$ ($m \geq 3$) with a unit normal vector field N along M. We also denote by g the induced metric on M. For any tangent vector field X of M, we can decompose $JX \in T\hat{M}^m(c)$ in terms of its tangent and normal parts as:

$$JX = \phi X + \eta(X)N, \;\; JN = -\xi, \;\; \eta(X) = g(X,\xi), \tag{5}$$

where ϕ denotes a tensor field of type $(1,1)$ on M and η is the one-form over M, corresponding to the Reeb vector field ξ. Then, the almost contact metric structure (ϕ,ξ,η,g) induced from g and J satisfies the following relations:

$$\begin{cases}\phi^2X = -X + \eta(X)\xi, \;\; \phi\xi = 0, \\ g(\phi X,Y) = -g(X,\phi Y), \;\; \eta(\xi) = 1, \\ g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y).\end{cases} \tag{6}$$

Choose $\{J_1, J_2, J_3\}$ to be a canonical local basis of $\mathfrak{J}$, and this induces a local almost contact metric three-structure (ϕ_a,ξ_a,η_a,g) on M by, $a \in \{1,2,3\}$,

$$J_aX = \phi_aX + \eta_a(X)N, \;\; J_aN = -\xi_a, \;\; \eta_a(X) = g(X,\xi_a). \tag{7}$$

Furthermore, it follows that:

$$\begin{cases}\phi_a^2X = -X + \eta_a(X)\xi_a, \;\; \phi_a\xi_a = 0, \\ g(\phi_aX,Y) = -g(X,\phi_aY), \;\; \eta_a(\xi_a) = 1, \\ g(\phi_aX,\phi_aY) = g(X,Y) - \eta_a(X)\eta_a(Y).\end{cases} \tag{8}$$

From the relation in (1), we further obtain that:

$$\begin{cases}\phi_a\xi_{a+1} = \xi_{a+2} = -\phi_{a+1}\xi_a, \\ \phi_a\phi_{a+1}X = \phi_{a+2}X + \eta_{a+1}(X)\xi_a, \\ \phi_{a+1}\phi_aX = -\phi_{a+2}X + \eta_a(X)\xi_{a+1},\end{cases} \tag{9}$$

where the index is taken modulo three.

Moreover, in terms of (3), these two almost contact metric structures (ϕ,ξ,η,g) and (ϕ_a,ξ_a,η_a,g) are related by:

$$\phi_a\phi X - \eta(X)\xi_a = \phi\phi_aX - \eta_a(X)\xi, \;\; \phi\xi_a = \phi_a\xi. \tag{10}$$

Then, there exists a local symmetric $(1,1)$-tensor field θ_a on M defined by:

$$\theta_aX := \phi_a\phi X - \eta(X)\xi_a. \tag{11}$$

More identities of θ_a have been established in [11,18].

If $\xi \notin \mathfrak{D}^\perp$, where $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1,\xi_2,\xi_3\}$ as stated in Section 1, there exists a unit vector field $X_0 \in \mathfrak{D}$ such that $\eta(X_0)X_0$ is the projection of ξ onto $\mathfrak{D}$ satisfying $-1 \leq \eta(X_0) \leq 1$ and $\eta(X_0) \neq 0$. Then, a distribution $\mathcal{F}^\perp$ can be defined by:

$$\mathcal{F}^\perp := \mathrm{Span}\{\xi_1,\xi_2,\xi_3,X_0,\phi_1X_0,\phi_2X_0,\phi_3X_0\}, \tag{12}$$

whereas $\mathcal{F}$ is the orthonormal complement of $\mathcal{F}^{\perp}$ such that $TM = \mathcal{F} \oplus \mathcal{F}^{\perp}$. We can check that $\{\xi_1, \xi_2, \xi_3, X_0, \phi_1 X_0, \phi_2 X_0, \phi_3 X_0\}$ are orthonormal and $\dim\mathcal{F} = 4m - 8$. In particular, $\mathcal{F}$ is invariant under ϕ, ϕ_a, and θ_a. From Lemma 3.2 (e) of [11], we know that $\theta_a^2 X = X - g(X, \phi\xi_a)\phi\xi_a$ for all $X \in TM$, and hence, it follows that $\theta_{a|\mathcal{F}}$ has two eigenvalues $\varepsilon = \pm 1$. If we denote by $\mathcal{F}_a(\varepsilon)$ the eigenspace corresponding to eigenvalue ε of $\theta_{a|\mathcal{F}}$, it holds that $\dim\mathcal{F}_a(1) = \dim\mathcal{F}_a(-1)$ is even, and we derive that $\phi\mathcal{F}_a(\varepsilon) = \phi_a\mathcal{F}_a(\varepsilon) = \theta_a\mathcal{F}_a(\varepsilon) = \mathcal{F}_a(\varepsilon)$, $\phi_b\mathcal{F}_a(\varepsilon) = \theta_b\mathcal{F}_a(\varepsilon) = \mathcal{F}_a(-\varepsilon)$ $(a \neq b)$.

If $\xi \in \mathfrak{D}^{\perp}$, we notice that $\dim\mathfrak{D} = 4m - 4$ for the orthonormal complement $\mathfrak{D}$ of $\mathfrak{D}^{\perp}$. Moreover, $\mathfrak{D}$ is invariant under ϕ, ϕ_a, and θ_a. In this case, Lemma 3.2 (e) of [11] also implies that $\theta_{a|\mathfrak{D}}$ has two eigenvalues $\varepsilon = \pm 1$, of which the corresponding eigenspace is denoted by $\mathfrak{D}_a(\varepsilon)$. Thus, we see that $\dim\mathfrak{D}_a(1) = \dim\mathfrak{D}_a(-1)$ is even and $\phi\mathfrak{D}_a(\varepsilon) = \phi_a\mathfrak{D}_a(\varepsilon) = \theta_a\mathfrak{D}_a(\varepsilon) = \mathfrak{D}_a(\varepsilon)$, $\phi_b\mathfrak{D}_a(\varepsilon) = \theta_b\mathfrak{D}_a(\varepsilon) = \mathfrak{D}_a(-\varepsilon)$ $(a \neq b)$.

Similar to the method in [18], by direct calculation, we can verify these facts mentioned above, which involve both of the cases $\xi \notin \mathfrak{D}^{\perp}$ and $\xi \in \mathfrak{D}^{\perp}$.

On the other hand, if we denote by ∇ and A the induced connection of $\hat{\nabla}$ on M and the shape operator of M, the formulas of Gauss and Weingarten are given by, respectively,

$$\hat{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \hat{\nabla}_X N = -AX \tag{13}$$

for all $X, Y \in TM$. Using $(JJ_a)^{\top} = \theta_a X$ and $(JJ_a)^{\perp} = \eta_a(\phi X)N$ for $X \in TM$, where $\cdot^{\top}$ and $\cdot^{\perp}$ denote the tangential part and normal part, respectively, we obtain from the expression of the curvature tensor $\hat{R}$ in (4) the equations of Gauss and Codazzi as follows (see [11]):

$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ c\sum_{a=1}^{3}\{g(\phi_a Y, Z)\phi_a X - g(\phi_a X, Z)\phi_a Y - 2g(\phi_a X, Y)\phi_a Z\} \\ &+ c\sum_{a=1}^{3}\{g(\theta_a Y, Z)\theta_a X - g(\theta_a X, Z)\theta_a Y\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{14}$$

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = c\{&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \\ &+ c\sum_{a=1}^{3}\{\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X, Y)\xi_a\} \\ &+ c\sum_{a=1}^{3}\{\eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X\}. \end{aligned} \tag{15}$$

By contracting Y and Z in (14), we have the following expression of the Ricci tensor of M:

$$\begin{aligned} \mathrm{Ric}(X,Y) = c\{&(4m+7)g(X,Y) - 3\eta(X)\eta(Y)\} \\ &+ c\sum_{a=1}^{3}\{\eta_a(\phi X)\eta_a(\phi Y) - 3\eta_a(X)\eta_a(Y) \\ &\qquad - \eta_a(\xi)g(\phi X, \phi_a Y) - \eta_a(\xi)\eta_a(Y)\eta(X)\} \\ &+ Hg(AX, Y) - g(A^2 X, Y), \end{aligned} \tag{16}$$

where $H = \mathrm{Tr}A$ denotes the mean curvature of the real hypersurface M in $\hat{M}^m(c)$.

In particular, the shape operator A and the Riemannian curvature tensor R are related by the Ricci identity:

$$(\nabla^2 A)(W, X, Y) - (\nabla^2 A)(X, W, Y) = R(W, X)AY - AR(W, X)Y. \tag{17}$$

In order to apply the Tsinghua principle, we shall give the following lemma on the covariant derivatives of tensors $\{\xi, \eta, \phi, \xi_a, \eta_a, \phi_a, \theta_a\}$ without proof, which can be proven by direct calculation. Furthermore, most of them were presented in [11].

Lemma 1. *Let M be a real hypersurface in $\hat{M}^m(c)$, $m \geq 3$, with the Levi–Civita connection ∇ of the induced metric g. Then, for the two contact metric structures (ϕ, ξ, η, g) and $(\phi_a, \xi_a, \eta_a, g)$, $a \in \{1, 2, 3\}$, we have:*

$$\begin{aligned}
\nabla_X \xi &= \phi A X, \\
(\nabla_X \eta) Y &= -g(AX, \phi Y), \\
(\nabla_X \phi) Y &= \eta(Y) AX - g(AX, Y)\xi, \\
\nabla_X \xi_a &= \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2}, \\
(\nabla_X \eta_a) Y &= -g(AX, \phi_a Y) + q_{a+2}(X)\eta_{a+1}(Y) - q_{a+1}(X)\eta_{a+2}(Y), \\
(\nabla_X \phi_a) Y &= \eta_a(Y) AX - g(AX, Y)\xi_a + q_{a+2}(X)\phi_{a+1} Y - q_{a+1}(X)\phi_{a+2} Y, \\
(\nabla_X \theta_a) Y &= \eta_a(\phi Y) AX - g(AX, Y)\phi\xi_a + q_{a+2}(X)\theta_{a+1} Y - q_{a+1}(X)\theta_{a+2} Y.
\end{aligned}$$

3. Key Lemmas and Important Classification Results

In this section, following the idea of Tsinghua principle, we first prove the next two lemmas, which are related to the general real hypersurfaces and the locally conformally flat real hypersurfaces, respectively, in $\hat{M}^m(c)$ for $m \geq 3$.

Lemma 2. *Let M be a real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then, in terms of these two almost contact metric structures (ϕ, ξ, η, g) and $(\phi_a, \xi_a, \eta_a, g)$ with the index $a \in \{1, 2, 3\}$, for any tangent vector fields $W, X, Y, Z \in TM$, we have:*

$$\underset{WXY}{\mathfrak{S}} \mathbf{I}(W, X, Y, Z) = -\underset{WXY}{\mathfrak{S}} \{g(R(W, X)Y, AZ) + g(R(W, X)Z, AY)\}, \tag{18}$$

where $\underset{WXY}{\mathfrak{S}}$ is the cyclic summation over W, X, Y and $\mathbf{I}(W, X, Y, Z)$ is given by:

$$\begin{aligned}
\mathbf{I}(W, X, Y, Z) :=\ & c\{g(AW, \phi Y)g(\phi X, Z) - g(AW, \phi X)g(\phi Y, Z) \\
& + 2g(AW, \phi Z)g(\phi X, Y) - 3g(AW, Y)\eta(X)\eta(Z) + 3g(AW, X)\eta(Y)\eta(Z)\} \\
& + c\sum_{a=1}^{3} \{g(AW, \phi_a Y)g(\phi_a X, Z) - g(AW, \phi_a X)g(\phi_a Y, Z) \\
& + 2g(AW, \phi_a Z)g(\phi_a X, Y) - 3g(AW, Y)\eta_a(X)\eta_a(Z) + 3g(AW, X)\eta_a(Y)\eta_a(Z)\} \\
& + c\sum_{a=1}^{3} \{g(\phi_a AW, \phi X) + \eta_a(AW)\eta(X) - g(AW, X)\eta_a(\xi)\}g(\theta_a Y, Z) \\
& - c\sum_{a=1}^{3} \{g(\phi_a AW, \phi Y) + \eta_a(AW)\eta(Y) - g(AW, Y)\eta_a(\xi)\}g(\theta_a X, Z) \\
& + c\sum_{a=1}^{3} \{g(AW, Y)\eta_a(\phi X)\eta_a(\phi Z) - g(AW, X)\eta_a(\phi Y)\eta_a(\phi Z)\}.
\end{aligned}$$

Proof. First of all, we put:

$$\mathfrak{B} := \underset{WXY}{\mathfrak{S}} \{g((\nabla^2 A)(W, X, Y), Z) - g((\nabla^2 A)(W, Y, X), Z)\}. \tag{19}$$

Then, this lemma shall be proven by calculating $\mathfrak{B}$ through two different ways. From the Codazzi equation in (15), we calculate:

$$\begin{aligned}
&g((\nabla^2 A)(W,X,Y),Z) - g((\nabla^2 A)(W,Y,X),Z)\\
=&c\{(\nabla_W \eta)(X)g(\phi Y,Z) + \eta(X)g((\nabla_W \phi)Y,Z) - (\nabla_W \eta)(Y)g(\phi X,Z)\\
&-\eta(Y)g((\nabla_W \phi)X,Z) - 2g((\nabla_W \phi)X,Y)\eta(Z) - 2g(\phi X,Y)g(\nabla_W \xi,Z)\}\\
&+c\sum_{a=1}^{3}\{(\nabla_W \eta_a)(X)g(\phi_a Y,Z) + \eta_a(X)g((\nabla_W \phi_a)Y,Z) - (\nabla_W \eta_a)(Y)g(\phi_a X,Z)\\
&-\eta_a(Y)g((\nabla_W \phi_a)X,Z) - 2g((\nabla_W \phi_a)X,Y)\eta_a(Z) - 2g(\phi_a X,Y)g(\nabla_W \xi_a,Z)\}\\
&+c\sum_{a=1}^{3}\{[(\nabla_W \eta_a)(\phi X) + g((\nabla_W \phi)X,\xi_a)]g(\theta_a Y,Z) + \eta_a(\phi X)g((\nabla_W \theta_a)Y,Z)\}\\
&-c\sum_{a=1}^{3}\{[(\nabla_W \eta_a)(\phi Y) + g((\nabla_W \phi)Y,\xi_a)]g(\theta_a X,Z) + \eta_a(\phi Y)g((\nabla_W \theta_a)X,Z)\}.
\end{aligned}$$

This, combined with the equations in Lemma 1, gives:

$$g((\nabla^2 A)(W,X,Y),Z) - g((\nabla^2 A)(W,Y,X),Z) = \mathbf{I}(W,X,Y,Z). \tag{20}$$

On the other hand, by using the Ricci identity (17), we obtain:

$$\begin{aligned}
\mathfrak{B} &= \underset{WXY}{\mathfrak{S}}\{g((\nabla^2 A)(W,X,Y),Z) - g((\nabla^2 A)(W,Y,X),Z)\}\\
&= \underset{WXY}{\mathfrak{S}}\{g((\nabla^2 A)(W,X,Y),Z) - g((\nabla^2 A)(X,W,Y),Z)\}\\
&= -\underset{WXY}{\mathfrak{S}}\{g(R(W,X)Y,AZ) + g(R(W,X)Z,AY)\}.
\end{aligned} \tag{21}$$

Hence, this assertion immediately follows from (20) and (21). □

Further, for a locally conformally flat real hypersurface M in $\hat{M}^m(c)$, by Lemma 2, we derive the following lemma, which is of great significance for the later proof.

Lemma 3. *Let M be a locally conformally flat real hypersurface in $\hat{M}^m(c)$ for $m \geq 3$. Then, in terms of these two almost contact metric structures (ϕ,ξ,η,g) and (ϕ_a,ξ_a,η_a,g) with $a \in \{1,2,3\}$, for any tangent vector fields $W,X,Y,Z \in TM$, we have:*

$$\underset{WXY}{\mathfrak{S}}\mathbf{I}(W,X,Y,Z) = \underset{WXY}{\mathfrak{S}}\mathbf{II}(W,X,Y,Z) \tag{22}$$

where $\mathbf{I}(W,X,Y,Z)$ is defined as in Lemma 2 and:

$$\mathbf{II}(W,X,Y,Z) = \tfrac{1}{4m-3}\{\mathrm{Ric}(X,AY) - \mathrm{Ric}(Y,AX)\}g(W,Z). \tag{23}$$

Proof. We first recall that the curvature tensor of a locally conformally flat real hypersurface M in $\hat{M}^m(c)$, where the Weyl curvature tensor vanishes, is given by:

$$\begin{aligned}
g(R(X,Y)Z,W) =& \tfrac{1}{4m-3}\{\mathrm{Ric}(Y,Z)g(X,W) - \mathrm{Ric}(X,Z)g(Y,W)\\
&+ \mathrm{Ric}(X,W)g(Y,Z) - \mathrm{Ric}(Y,W)g(X,Z)\}\\
&+ \tfrac{r}{2(2m-1)(4m-3)}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\},
\end{aligned} \tag{24}$$

where r denotes the scalar curvature of M.

This, together with (21), yields:

$$\mathfrak{B} = \tfrac{1}{4m-3}\underset{WXY}{\mathfrak{S}}\{\mathrm{Ric}(X,AY) - \mathrm{Ric}(Y,AX)\}g(W,Z). \tag{25}$$

Thus, from (19) and (20), the assertion follows immediately, where by the expression of Ricci tensor in (16), we can rewrite $\mathbf{II}(W,X,Y,Z)$ as:

$$
\begin{aligned}
\mathbf{II}(W,X,Y,Z) &= \tfrac{3c}{4m-3}\{\eta(Y)\eta(AX)-\eta(X)\eta(AY)\}g(W,Z)\\
&+\tfrac{3c}{4m-3}\sum_{a=1}^{3}\{\eta_a(Y)\eta_a(AX)-\eta_a(X)\eta_a(AY)\}g(W,Z)\\
&-\tfrac{c}{4m-3}\sum_{a=1}^{3}\{\eta_a(\phi Y)\eta_a(\phi AX)-\eta_a(\phi X)\eta_a(\phi AY)\}g(W,Z)\\
&+\tfrac{c}{4m-3}\sum_{a=1}^{3}\eta_a(\xi)\{g(\phi Y,\phi_a AX)-g(\phi X,\phi_a AY)\}g(W,Z)\\
&-\tfrac{c}{4m-3}\sum_{a=1}^{3}\eta_a(\xi)\{\eta_a(AY)\eta(X)-\eta_a(AX)\eta(Y)\}g(W,Z).
\end{aligned}
\tag{26}
$$

□

Remark 2. *As the key to apply the Tsinghua principle successfully, we find it by calculation that all the terms, involving $q_a(X)$ for $a\in\{1,2,3\}$ and $X\in TM$, are canceled out by each other, and this greatly simplifies the calculation on the linear relationship described in $\mathbf{I}(W,X,Y,Z)$.*

At the end of this section, two important classification theorems of the real hypersurfaces with isometric Reeb flow in $\hat{M}^m(c)$ ($m\geq 3$) are stated for later use. Here, for a real hypersurface M in $\hat{M}^m(c)$, its Reeb flow is isometric if and only if it holds $\mathcal{L}_\xi g=0$ with $\mathcal{L}_\xi$ the Lie derivative $\mathcal{L}$ along the direction of ξ, which is also equivalent to $A\phi=\phi A$.

Theorem 4 ([28]). *Let M be a connected orientable real hypersurface in the complex two-plane Grassmannian $SU_{m+2}/S(U_2U_m)$, $m\geq 3$. Then, the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic $SU_{m+1}/S(U_2U_{m-1})$ in $SU_{m+2}/S(U_2U_m)$.*

Theorem 5 ([29]). *Let M be a connected orientable real hypersurface in the complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2U_m)$, $m\geq 3$. Then, the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2U_{m-1})$ in $SU_{2,m}/S(U_2U_m)$ or a horosphere whose center at infinity is singular.*

Remark 3. *It should be pointed out that these real hypersurfaces with isometric Reeb flow have at least three distinct constant principle curvatures (see [28,29] or [11]). This implies that it admits no totally umbilical real hypersurfaces in $\hat{M}^m(c)$ for $m\geq 3$. Otherwise, for such a real hypersurface in $\hat{M}^m(c)$ with $A\phi=\phi A$, it has isometric Reeb flow, and this is a contradiction.*

4. Proof of Main Theorem

Throughout this section, we always assume that M is a locally conformally flat real hypersurface in $\hat{M}^m(c)$, $m\geq 3$. For the Reeb vector field ξ on M, we prove that it belongs either to the distribution $\mathfrak{D}$ or to its orthonormal complement $\mathfrak{D}^\perp$.

Proposition 1. *Let M be a locally conformally flat real hypersurface in $\hat{M}^m(c)$, $m\geq 3$. Then, the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$.*

Proof. We argue by contradiction. Assume that at some point $x\in M$, it satisfies that $\xi\notin\mathfrak{D}$ and $\xi\notin\mathfrak{D}^\perp$. Then, there exist a neighborhood U of x in M, on which we can write:

$$\xi=\eta(X_0)X_0+\eta(\xi_1)\xi_1 \tag{27}$$

for unit vector fields $X_0\in\mathfrak{D}$ and $\xi_1\in\mathfrak{D}^\perp$ such that:

$$\eta(X_0)\eta(\xi_1)\neq 0,\ \ [\eta(X_0)]^2+[\eta(\xi_1)]^2=1. \tag{28}$$

Put $e_1 = \xi_1$, $e_2 = \xi_2$, $e_3 = \xi_3$, $e_4 = X_0$, $e_5 = \phi_1 X_0$, $e_6 = \phi_2 X_0$, and $e_7 = \phi_3 X_0$. From Section 2, it follows that $\dim\mathcal{F}_1(1) = 2m-4 = \dim\mathcal{F}_1(-1)$ for $a=1$ and $m \geq 3$. Choosing some fixed unit vector field $e_8 \in \mathcal{F}_1(1)$, we can take $e_9 := \phi_1 e_8 \in \mathcal{F}_1(1)$, $e_{10} := \phi_2 e_8 \in \mathcal{F}_1(-1)$ and $e_{11} := \phi_3 e_8 \in \mathcal{F}_1(-1)$. Further, if we define $\mathcal{H} := \mathcal{F} \ominus \mathrm{Span}\{e_8, e_9, e_{10}, e_{11}\}$, it can be seen that $\theta_{1|\mathcal{H}}$ has also two eigenvalues $\varepsilon = \pm 1$, of which the corresponding eigenspace is denoted by $\mathcal{H}(\varepsilon)$. Then, $\dim\mathcal{H}(1) = 2m-6 = \dim\mathcal{H}(-1)$. Next, we proceed to choose a fixed unit vector field $e_{12} \in \mathcal{H}(1)$ and take $e_{13} := \phi_1 e_{12} \in \mathcal{H}(1)$, $e_{14} := \phi_2 e_{12} \in \mathcal{H}(-1)$, and $e_{15} := \phi_3 e_{12} \in \mathcal{H}(-1)$. Repeating this way, a local orthonormal frame field $\{e_i\}_{i=1}^{4m-1}$ along M can be chosen as:

$$\begin{cases} e_1 = \xi_1, \;\; e_2 = \xi_2, \;\; e_3 = \xi_3, \;\; e_4 = X_0, \\ e_5 = \phi_1 X_0, \;\; e_6 = \phi_2 X_0, \;\; e_7 = \phi_3 X_0, \\ e_{4p-8} \in \mathcal{F}_1(1), \;\; e_{4p-7} = \phi_1 e_{4p-8} \in \mathcal{F}_1(1), \\ e_{4p-6} = \phi_2 e_{4p-8} \in \mathcal{F}_1(-1), \;\; e_{4p-5} = \phi_3 e_{4p-8} \in \mathcal{F}_1(-1), \end{cases} \tag{29}$$

where $\dim\mathcal{F}_1(1) = 2m-4 = \dim\mathcal{F}_1(-1)$ and $4 \leq p \leq m+1$ for $m \geq 3$.

For the shape operator A of M, we set $Ae_i := \sum_{j=1}^{4m-1} a_{ij} e_j$ for $1 \leq i \leq 4m-1$. Obviously, $a_{ij} = a_{ji}$ for $1 \leq i, j \leq 4m-1$.

In order to apply Lemma 3, we first calculate $\mathbf{I}(W,X,Y,Z)$ and $\mathbf{II}(W,X,Y,Z)$ directly, by choosing appropriate $W = e_i$, $X = e_j$, $Y = e_k$, $Z = e_\ell$, $1 \leq i, j, k, \ell \leq 4m-1$. Moreover, making use of (22), we shall obtain a system of linear equations of the components a_{ij}, by means of which we will complete the proof of Proposition 1. For the convenience of calculation, the following agreement is presented:

$$\alpha = \tfrac{1}{4m-3}\eta(\xi_1), \;\; m \geq 3.$$

From (28), it is obvious that $-1 < \alpha < 1$ and $\alpha \neq 0$.

Firstly, we choose $W, X, Y, Z \in \{e_{4p-8}, e_{4p-7}, e_{4p-6}, e_{4p-5}\}$ for $4 \leq p \leq m+1$.

We begin with the calculation by taking in (22):

$$(W,X,Y,Z) = (e_{4p-8}, e_{4p-7}, e_{4p-6}, e_{4p-8}), \;\; (e_{4p-8}, e_{4p-7}, e_{4p-5}, e_{4p-7}),$$

respectively. Then, we have the equations:

$$a_{4p-8,4p-5} + (3-\alpha)a_{4p-7,4p-6} = 0, \;\; a_{4p-7,4p-6} + (3-\alpha)a_{4p-8,4p-5} = 0,$$

which implies that $a_{4p-8,4p-5} = a_{4p-7,4p-6} = 0$.

Similarly, in (22), we consider the following:

$$\begin{aligned} (W,X,Y,Z) = \;& (e_{4p-8}, e_{4p-7}, e_{4p-6}, e_{4p-7}), \;\; (e_{4p-8}, e_{4p-7}, e_{4p-5}, e_{4p-8}), \\ & (e_{4p-8}, e_{4p-7}, e_{4p-5}, e_{4p-5}), \;\; (e_{4p-7}, e_{4p-6}, e_{4p-5}, e_{4p-7}), \end{aligned}$$

and further obtain that $a_{4p-8,4p-6} = a_{4p-7,4p-5} = a_{4p-6,4p-5} = a_{4p-8,4p-7} = 0$.

By summarizing the conclusion above, for $4 \leq p \leq m+1$, we have:

$$\begin{cases} a_{4p-8,4p-7} = a_{4p-8,4p-6} = a_{4p-8,4p-5} = 0, \\ a_{4p-7,4p-6} = a_{4p-7,4p-5} = a_{4p-6,4p-5} = 0. \end{cases} \tag{30}$$

Next, if we take in (22), respectively,

$$\begin{aligned} (W,X,Y,Z) = \;& (e_{4p-8}, e_{4p-7}, e_{4p-6}, e_{4p-5}), \;\; (e_{4p-8}, e_{4p-7}, e_{4p-5}, e_{4p-6}), \\ & (e_{4p-7}, e_{4p-6}, e_{4p-5}, e_{4p-8}), \end{aligned}$$

then a system of equations can be obtained as:

$$\begin{cases} 2a_{4p-6,4p-6} = a_{4p-8,4p-8} + a_{4p-7,4p-7}, \\ 2a_{4p-5,4p-5} = a_{4p-8,4p-8} + a_{4p-7,4p-7}, \\ 2a_{4p-7,4p-7} = a_{4p-6,4p-6} + a_{4p-5,4p-5}. \end{cases}$$

This gives $a_{4p-8,4p-8} = a_{4p-7,4p-7} = a_{4p-6,4p-6} = a_{4p-5,4p-5}$.

Moreover, we can take $(W, X, Y, Z) = (e_{4p_1-8}, e_{4p_1-7}, e_{4p_2-8}, e_{4p_2-7})$ in (22) such that $5 \leq p_2 = p_1 + 1 \leq m + 1$ if $m \geq 4$ to obtain the recurrence relation:

$$2a_{4p_2-8,4p_2-8} = a_{4p_1-8,4p_1-8} + a_{4p_1-7,4p_1-7}.$$

It follows that:

$$a_{ii} = a_{jj}, \ \ 8 \leq i < j \leq 4m - 1. \tag{31}$$

Consider $W, X, Y, Z \in \{e_1, \ldots, e_7, e_{4p-8}, e_{4p-7}, e_{4p-6}, e_{4p-5}\}$ for $4 \leq p \leq m + 1$.

Repeating the calculation by taking in (22) for $5 \leq k \leq 8$:

$$\begin{aligned} (W, X, Y, Z) = \ & (e_1, e_2, e_4, e_{4p-k}), \ \ (e_2, e_3, e_5, e_{4p-k}), \\ & (e_3, e_1, e_6, e_{4p-k}), \ \ (e_1, e_2, e_7, e_{4p-k}), \end{aligned}$$

respectively, we derive, for $4 \leq p \leq m + 1$,

$$a_{i,4p-8} = a_{i,4p-7} = a_{i,4p-6} = a_{i,4p-5} = 0, \ \ 4 \leq i \leq 7. \tag{32}$$

For $5 \leq k \leq 8$, taking in (22), respectively,

$$(W, X, Y, Z) = (e_4, e_5, e_1, e_{4p-k}), \ \ (e_5, e_6, e_2, e_{4p-k}), \ \ (e_6, e_7, e_3, e_{4p-k}),$$

from (28) and (32), we conclude, for $4 \leq p \leq m + 1$,

$$a_{i,4p-k} = 0, \ \ 1 \leq i \leq 3, \ \ 5 \leq k \leq 8. \tag{33}$$

Further, for $5 \leq k, \ell \leq 8$, we take $(W, X, Y, Z) = (e_1, e_2, e_{4p_1-k}, e_{4p_2-\ell})$ in (22) for $4 \leq p_1 < p_2 \leq m + 1$ if $m \geq 4$, and later, it holds between the cross terms:

$$a_{4p_1-k,4p_2-\ell} = 0, \ \ 4 \leq p_1 < p_2 \leq m + 1, \ \ 5 \leq k, \ell \leq 8. \tag{34}$$

Finally, we calculate by choosing $W, X, Y, Z \in \{e_1, \ldots, e_7, e_8, e_9, e_{10}, e_{11}\}$.

For $1 \leq i \leq 3$ and $4 \leq j \leq 7$, taking in (22) $(W, X, Y, Z) = (e_9, e_{10}, e_i, e_j)$ and:

$$\begin{aligned} (W, X, Y, Z) = \ & (e_9, e_{10}, e_1, e_1), \ \ (e_8, e_9, e_2, e_2), \ \ (e_8, e_9, e_6, e_6), \\ & (e_9, e_{10}, e_4, e_4), \ \ (e_8, e_9, e_4, e_4), \ \ (e_{10}, e_{11}, e_6, e_1), \\ & (e_9, e_{11}, e_1, e_1), \ \ (e_8, e_{10}, e_4, e_4), \ \ (e_{10}, e_{11}, e_7, e_1), \end{aligned}$$

respectively, from the combination of (28), (30), and (31), we obtain:

$$a_{ij} = 0, \ \ 1 \leq i < j \leq 7. \tag{35}$$

In particular, by taking in (22), respectively,

$$\begin{aligned} (W, X, Y, Z) = & (e_8, e_9, e_1, e_5), \ \ (e_8, e_9, e_2, e_6), \ \ (e_8, e_9, e_3, e_7), \ \ (e_4, e_5, e_8, e_9), \\ & (e_8, e_9, e_5, e_1), \ \ (e_8, e_9, e_6, e_2), \ \ (e_8, e_9, e_7, e_3), \end{aligned}$$

with the use of (28), (31), and (35), we have:

$$a_{ii} = a_{jj}, \ \ 1 \leq i < j \leq 8. \tag{36}$$

From the equations of (30)–(36), we easily see that M is totally umbilical, and by Remark 3, it is obviously a contradiction.

Hence, we complete the proof by this contradiction. □

In order to complete the proof of Theorem 3, by Proposition 1, we only need to consider the following two cases:

- **Case I**: The Reeb vector field ξ belongs to the distribution $\mathfrak{D}$;
- **Case II**: The Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$.

In the following, we first prove that **Case I** will not happen, as is shown in Proposition 2.

Proposition 2. *Let M be a locally conformally flat real hypersurface in $\hat{M}^m(c)$, $m \geq 3$. Then, the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$.*

Proof. Suppose that $\xi \in \mathfrak{D}$. Without loss of generality, we can also take $\xi = X_0$ for such unit vector field $X_0 \in \mathfrak{D}$. Choosing the local orthonormal frame field $\{e_i\}_{i=1}^{4m-1}$ the same as in (29), we repeat the calculations of Proposition 1 step by step and conclude that M is still totally umbilical, which combined with Remark 3 appears as a contradiction. By Proposition 1, we see that for a locally conformally flat real hypersurface M in $\hat{M}^m(c)$ ($m \geq 3$), its Reeb vector field ξ must belong to the distribution $\mathfrak{D}^{\perp}$. □

To complete the proof of Theorem 3, we are left just to consider **Case II**, which states that M is a locally conformally flat real hypersurface in $\hat{M}^m(c)$ ($m \geq 3$) with $\xi \in \mathfrak{D}^{\perp}$. In this case, by Section 2, we know that $\dim\mathfrak{D}^{\perp} = 4m-4$. Noting that $\dim\mathfrak{D}_1^{\perp}(1) = 2m-2 = \dim\mathfrak{D}_1^{\perp}(-1)$ for $a = 1$ and $m \geq 3$, we choose some fixed unit vector field $e_4 \in \mathfrak{D}_1^{\perp}(1)$ such that $e_5 := \phi_1 e_4 \in \mathfrak{D}_1^{\perp}(1)$, $e_6 := \phi_2 e_4 \in \mathfrak{D}_1^{\perp}(-1)$, and $e_7 := \phi_3 e_4 \in \mathfrak{D}_1^{\perp}(-1)$, respectively. If we define $\mathcal{P} := \mathfrak{D}^{\perp} \ominus \mathrm{Span}\{e_4, e_5, e_6, e_7\}$, the same as $\theta_{1|\mathfrak{D}^{\perp}}$, $\theta_{1|\mathcal{P}}$ also has two eigenvalues $\varepsilon = \pm 1$. We denote by $\mathcal{P}(\varepsilon)$ the corresponding eigenspace and further obtain that $\dim\mathcal{P}(1) = 2m-4 = \dim\mathcal{P}(-1)$ for $m \geq 3$. Later, another fixed unit vector field $e_8 \in \mathcal{P}(1)$ can be chosen such that $e_9 := \phi_1 e_8 \in \mathcal{P}(1)$, $e_{10} := \phi_2 e_8 \in \mathcal{P}(-1)$, and $e_{11} := \phi_3 e_8 \in \mathcal{P}(-1)$. Repeating this way, there exist a local orthogonal frame field $\{e_i\}_{i=1}^{4m-1}$ given by:

$$\begin{cases} e_1 = \xi_1, \;\; e_2 = \xi_2, \;\; e_3 = \xi_3, \\ e_{4q-8} \in \mathfrak{D}_1^{\perp}(1), \;\; e_{4q-7} = \phi_1 e_{4q-8} \in \mathfrak{D}_1^{\perp}(1), \\ e_{4q-6} = \phi_2 e_{4q-8} \in \mathfrak{D}_1^{\perp}(-1), \;\; e_{4q-5} = \phi_3 e_{4q-8} \in \mathfrak{D}_1^{\perp}(-1), \end{cases} \tag{37}$$

where $\dim\mathfrak{D}_1^{\perp}(1) = 2m-2 = \dim\mathfrak{D}_1^{\perp}(-1)$ and $3 \leq q \leq m+1$ for $m \geq 3$.

By choosing appropriate $W = e_i$, $X = e_j$, $Y = e_k$, $Z = e_\ell$, $1 \leq i, j, k, \ell \leq 4m-1$, we proceed to calculate $\mathbf{I}(W, X, Y, Z)$ and $\mathbf{II}(W, X, Y, Z)$ in (22).

Put $Ae_i = \sum_{j=1}^{4m-1} a_{ij} e_j$ for $1 \leq i \leq 4m-1$ with $a_{ij} = a_{ji}$ for $1 \leq i, j \leq 4m-1$. Similarly, by taking different values in (W, X, Y, Z), we still apply the relation in (22) of Lemma 3 to obtain a system of linear equations of the components a_{ij}.

We start with the calculation by taking in (22), for $6 \leq i \leq 8$ and $7 \leq j \leq 8$,

$$\begin{aligned} (W, X, Y, Z) = \;& (e_{4q-8}, e_{4q-7}, e_{4q-6}, e_{4q-i}), \;\; (e_{4q-8}, e_{4q-7}, e_{4q-5}, e_{4q-j}), \\ & (e_{4q-7}, e_{4q-6}, e_{4q-5}, e_{4q-7}), \end{aligned}$$

respectively, and it follows that:

$$\begin{cases} a_{4q-8,4q-7} = a_{4q-8,4q-6} = a_{4q-8,4q-5} = 0, \\ a_{4q-7,4q-6} = a_{4q-7,4q-5} = a_{4q-6,4q-5} = 0. \end{cases} \tag{38}$$

For $4 \leq q_2 = q_1 + 1 \leq m+1$, by further taking in (22), respectively,

$$
\begin{aligned}
(W,X,Y,Z) = \; & (e_{4q-8}, e_{4q-7}, e_{4q-6}, e_{4q-5}), \;\; (e_{4q-8}, e_{4q-7}, e_{4q-5}, e_{4q-6}), \\
& (e_{4q-7}, e_{4q-6}, e_{4q-5}, e_{4q-8}), \;\; (e_{4q_1-8}, e_{4q_1-7}, e_{4q_2-8}, e_{4q_2-7}),
\end{aligned}
$$

we conclude that:

$$a_{ii} = a_{jj}, \;\; 4 \le i < j \le 4m-1. \tag{39}$$

For $5 \le k, \ell \le 8$, taking $(W,X,Y,Z) = (e_1, e_2, e_{4q_1-k}, e_{4q_2-\ell})$ in (22), we obtain these cross terms, for $3 \le q_1 < q_2 \le m+1$ if $m \ge 4$,

$$a_{4q_1-k, 4q_2-\ell} = 0, \;\; 5 \le k, \ell \le 8. \tag{40}$$

Repeating the calculation and taking in (22), for $5 \le k \le 8$ and $5 \le \ell \le 6$,

$$
\begin{aligned}
(W,X,Y,Z) = \; & (e_{4q-8}, e_{4q-7}, e_1, e_{4q-k}), \;\; (e_{4q-7}, e_{4q-6}, e_2, e_{4q-5}), \\
& (e_{4q-5}, e_{4q-8}, e_2, e_{4q-6}), \;\; (e_{4q-6}, e_{4q-5}, e_2, e_{4q-\ell}), \\
& (e_{4q-7}, e_{4q-5}, e_3, e_{4q-6}), \;\; (e_{4q-8}, e_{4q-6}, e_3, e_{4q-5}), \\
& (e_{4q-6}, e_{4q-5}, e_3, e_{4q-\ell}),
\end{aligned}
$$

respectively, we obtain:

$$a_{i,4q-k} = 0, \;\; 1 \le i \le 3, \;\; 5 \le k \le 8. \tag{41}$$

Moreover, if in (22), we proceed to take, respectively,

$$
\begin{aligned}
(W,X,Y,Z) = & (e_5, e_6, e_1, e_1), \;\; (e_5, e_7, e_1, e_1), \;\; (e_4, e_5, e_2, e_2), \\
& (e_5, e_6, e_1, e_2), \;\; (e_5, e_6, e_2, e_1), \;\; (e_4, e_5, e_3, e_2),
\end{aligned}
$$

by virtue of (38) and (39), we obtain:

$$a_{12} = a_{13} = a_{23} = 0, \;\; a_{11} = a_{22} = a_{33} = a_{44}. \tag{42}$$

Summarizing all the results of (38)–(42), we immediately obtain:

$$a_{ij} = 0, \;\; a_{ii} = a_{jj}, \;\; 1 \le i < j \le 4m-1, \tag{43}$$

which implies that M is still totally umbilical, a contradiction to the statement in Remark 3. For this reason, exactly it admits no locally conformally flat real hypersurfaces in $\hat{M}^m(c)$, $m \ge 3$.

In conclusion, we have completed the proof of Theorem 3.

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