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Faster and Slower Soliton Phase Shift: Oceanic Waves Affected by Earth Rotation

Mostafa M. A. Khater ^{1,2,*} and Aliaa Mahfooz Alabdali ^{3,†} ¹ Department of Mathematics, Faculty of Science, Jiangsu University, Zhenjiang 212013, China² Department of Basic Science, Obour High Institute for Engineering and Technology, Cairo 11828, Egypt³ Faculty of Computing & Information Technology, King Abdulaziz University, P.O. Box 344, Rabigh 21911, Saudi Arabia; amalabdali@kau.edu.sa

* Correspondence: 1000005364@ujs.edu.cn or mostafa.khater2024@yahoo.com

† These authors did all this work equally.

Abstract: This research paper investigates the accuracy of a novel computational scheme (Khater II method) by applying this new technique to the fractional nonlinear Ostrovsky (FNO) equation. The accuracy of the obtained solutions was verified by employing the Adomian decomposition (AD) and El Kalla (EK) methods. The AD and EK methods are considered as two of the most accurate semi-analytical schemes. The FNO model is a modified version of the well-known Korteweg–de Vries (KdV) equation that considers the effects of rotational symmetry in space. However, in the KdV model, solutions to the KdV equations substitute this effect with radiating inertia gravity waves, and thus this impact is ignored. The analytical, semi-analytical, and accuracy between solutions are represented in some distinct plots. Additionally, the paper's novelty and its contributions are demonstrated by comparing the obtained solutions with previously published results.

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1. Introduction

Plasma physics is one of the most crucial fields of our time which is attracting a large number of researchers [1,2]. In modern plasma physics, it is essential to work on dense Langmuir turbulence [3–5]. This significance stems from their ability to provide a practical solution to Langmuir's condensation issue [6]. The Langmuir condensation paradox was characterized by an increase in the sum of long-wave disturbances [7]. Landau or radiation cannot dampen the vibration; similarly, the Coulomb relation cannot dampen the frequency fluctuations of oscillations at extreme intervals [8,9]. Plasma destruction allows the release of condensing disturbances, resolving the Langmuir turbulence discharge amplitude problem and improving the graphical representation of Langmuir turbulence [10–14].

The increasing interest in creating many theoretical and computational methods for studying these models exemplifies the strength of nonlinear partial differential equations (NLPDEs) such as the Khater method, generalized Khater method, Khater II method, the exp-function method, extended simplest equation method, Riccati expansion method, etc. [15–20]. Additionally, many fractional derivatives have been derived to convert the fractional nonlinear partial differential equations into ordinary differential equations with integer-order such as He's fractional derivative and the two-scale fractal derivative [21–23]. These models are used to describe a wide variety of complicated events in various disciplines, including biology, quantum physics, electrochemistry, mechanical engineering, and mechanical sciences [24–28]. When evaluating new features of the research paradigm, the use of specialized schemes is considered a significant advantage [29–31]. On the other

hand, several researchers failed to examine and provide a cohesive strategy applicable to all NLPDEs [32–34].

In this context, we studied the analytical solutions of the FNO equation which is given by [35–38]

$$\left(\mathcal{D}_t^\varepsilon \mathcal{V} - \beta \mathcal{V}_{xxx} + (\mathcal{V}^2)_x\right)_x = \gamma \mathcal{V}, \quad 0 < \varepsilon < 1, \quad (1)$$

where $\mathcal{V} = \mathcal{V}(x, t)$ is a simplified description of the nonlinear ocean wave dynamics that takes Earth rotation into account. β represents the in-compressible liquid surfaces. On the other hand, γ is the Earth's spinning effect.

Definition 1. The conformable derivative is defined as follows [39–41]:

Suppose a continuous function $\mathcal{V}$ in the domain $[0, \infty)$ and co-domain $\mathbb{R}$. Thus, the conformable derivative of $\mathcal{V}$ with order ε which satisfies $\varepsilon \in (0, 1]$, is given by

$$\mathcal{D}^\varepsilon \mathcal{V}(t) = \lim_{\alpha \rightarrow 0} \frac{\mathcal{V}(t + \alpha t^{1-\varepsilon}) - \mathcal{V}(t)}{\alpha}, \quad \forall t > 0. \quad (2)$$

Additionally, the $\mathcal{V}^{(\varepsilon)}(0)$ is defined by

$$\mathcal{V}^{(\varepsilon)}(0) = \lim_{t \rightarrow 0^+} \mathcal{V}^{(\varepsilon)}(t), \quad (3)$$

if and only if $\mathcal{V}$ is ε -differentiable in $(0, \alpha)$, and $\lim_{t \rightarrow 0} \mathcal{V}^{(\varepsilon)}(t)$ exists.

Lemma 1. Let $\varepsilon \in (0, 1]$ and $\mathcal{V}$ and $\mathcal{S}$ be ε -differentiable functions $\forall t > 0$. Thus, the conformable derivative properties are formulated by [39]:

- (a) $\mathcal{D}^\varepsilon (\mathcal{L}_1 \mathcal{V} + \mathcal{L}_2 \mathcal{S}) = \mathcal{L}_1 \mathcal{D}^\varepsilon \mathcal{V} + \mathcal{L}_2 \mathcal{D}^\varepsilon \mathcal{S}, \forall \mathcal{L}_1, \mathcal{L}_2 \in \mathbb{R};$
- (b) $\mathcal{D}^\varepsilon (t^\gamma) = \gamma t^{\gamma-\varepsilon}, \forall \gamma \in \mathbb{R};$
- (c) $\mathcal{D}^\varepsilon (\mathcal{V} \mathcal{S}) = \mathcal{V} \mathcal{D}^\varepsilon \mathcal{S} + \mathcal{S} \mathcal{D}^\varepsilon \mathcal{V};$
- (d) $\mathcal{D}^\varepsilon \left(\frac{\mathcal{V}}{\mathcal{S}}\right) = \frac{\mathcal{S} \mathcal{D}^\varepsilon \mathcal{V} - \mathcal{V} \mathcal{D}^\varepsilon \mathcal{S}}{\mathcal{S}^2},$ provided $\mathcal{S} \neq 0$;
- (e) $\mathcal{D}^\varepsilon (\mathcal{C}) = 0,$ where $\mathcal{C}$ is a constant.

Lemma 2. Suppose that $\mathcal{V}$ is a ε -differentiable and differentiable function, then $\mathcal{D}^\varepsilon \mathcal{V}(t)$ is formulated by [40]

$$\mathcal{D}^\varepsilon \mathcal{V}(t) = t^{1-\varepsilon} \frac{\partial \mathcal{V}(t)}{\partial t}. \quad (4)$$

Theorem 1. Let $\mathcal{V} : (0, \infty) \rightarrow \mathcal{R}$ be a differentiable and ε -differentiable function. Let $\mathcal{S}$ be a differentiable function defined in the range of ε [41]. Consequently:

$$\mathcal{D}_t^\varepsilon (\mathcal{V} \circ \mathcal{S})(t) = t^{1-\varepsilon} \mathcal{S}(t)^{\varepsilon-1} \times \mathcal{S}'(t) \mathcal{D}_t^\varepsilon (\mathcal{V}(\Phi))|_{\{q=\mathcal{S}(t)\}}. \quad (5)$$

Implementing the following wave transformation (the fractional complex transformation [42,43]) along with the conformable fractional properties $\mathcal{V}(x, t) = \mathcal{U}(\mathfrak{z}), \mathfrak{z} = x + c \frac{t^\varepsilon}{\varepsilon}$, where c is an arbitrary constants, then integrating the obtained ODE twice with respect to $\mathfrak{z}$ and zero constants of integration converts Equation (1) into the following ODE:

$$c\mathcal{U} - \beta\mathcal{U}'' + \mathcal{U}^2 = 0. \quad (6)$$

Balancing the terms in Equation (6) ($\mathcal{U}''$ & $\mathcal{U}^2$) leads to $N = 2$. Applying the Khater II method [44] to Equation (6) leads to the following general solution:

$$\mathcal{U}(3) = \sum_{i=1}^n \left(a_i f(3)^i + b_i \phi(3) f(3)^{i-1} \right) + a_0 = a_2 f(3)^2 + a_1 f(3) + a_0 + b_2 f(3) \phi(3) + b_1 \phi(3), \quad (7)$$

where a_0, a_1, a_2, b_1, b_2 are arbitrary constants to be subsequently calculated while $f'(3) = -\delta - f(3)^2$, $\phi'(3) = -f(3)\phi(3)$, $\phi(3)^2 = \alpha \left(\frac{f(3)^2}{\delta} + 1 \right)$, where $\alpha = 1$, and $\delta \neq 0$. Gathering the coefficients of $f(3)$, $\phi(3)$, $f(3)\phi(3)$, $f(3)^2\phi(3)$, $f(3)^3\phi(3)$ and equating them with zero, leads to a system of algebraic equations.

The rest of the paper's sections are ordered as following; Section 2 gives the novel analytical solutions of the FNO model with the Khater II method. Section 3 investigates the accuracy of the obtained analytical solutions by employing AD and EK schemes. Furthermore, these solutions are demonstrated in some graphs in two- and three-dimensional as well as contour figures. Section 4 illustrates the paper's contributions and the solutions' novelty. Section 5 gives the conclusion of the whole study.

2. Analytical Solutions

Solving the obtained system of algebraic equations by Mathematics 12 leads to the following values of the aforementioned constants.

Case I:

$$a_0 = 2c, a_1 = 0, a_2 = \frac{3c}{\delta}, b_1 = 0, b_2 = \frac{3c}{\sqrt{\alpha}\sqrt{\delta}}, \beta = \frac{c}{\delta}.$$

Case II:

$$a_0 = -\frac{1}{2}(3c), a_1 = 0, a_2 = -\frac{3c}{2\delta}, b_1 = 0, b_2 = 0, \beta = -\frac{c}{4\delta}.$$

Case III:

$$a_0 = \frac{c}{2}, a_1 = 0, a_2 = \frac{3c}{2\delta}, b_1 = 0, b_2 = 0, \beta = \frac{c}{4\delta}.$$

Thus, the traveling wave solutions of the FNO model is given by

$$\mathcal{U}_{I,1}(x, t) = c \left(3 \tan \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right) \left(\tan \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right) - \frac{\sec \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right)}{\sqrt{\alpha}} \right) + 2 \right), \quad (8)$$

$$\mathcal{U}_{I,2}(x, t) = c \left(3 \cot \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right) \left(\frac{\csc \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right)}{\sqrt{\alpha}} + \cot \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right) \right) + 2 \right), \quad (9)$$

$$\mathcal{U}_{II,1}(x, t) = \frac{1}{2}(-3)c \sec^2 \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right), \quad (10)$$

$$\mathcal{U}_{II,2}(x, t) = \frac{1}{2}(-3)c \csc^2 \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right), \quad (11)$$

$$\mathcal{U}_{III,1}(x, t) = \frac{3}{2}c \tan^2 \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right), \quad (12)$$

$$\mathcal{U}_{\text{III},2}(x,t) = \frac{1}{2} \left(3c \cot^2 \left(\sqrt{\delta} \left(\frac{ct^\epsilon}{\epsilon} + x \right) \right) + c \right). \quad (13)$$

3. Solutions' Accuracy

Applying the AD and EK methods to the FNO model based on the obtained analytical solutions (8), (10) and (12) when $\alpha = -1$, $c = 3$, $\delta = -4$. Thus, the semi-analytical solutions of the FNO model based on the AD method are given by

$$\mathcal{U}_{\text{Semi-Analy.}}(x,0) = \begin{cases} -\frac{96x^{10}}{5} + \frac{576x^9}{7} - 156x^8 + \frac{344x^7}{5} + \frac{848x^6}{5} - \frac{1404x^5}{5} + 72x^4 + 60x^3 - 36x^2 - 18x + 6, \\ 32x^6 - 120x^4 + 18x^2 - \frac{9}{2}, \\ 0. \end{cases} \quad (14)$$

The semi-analytical solutions of the FNO model based on the EK method are given by

$$\mathcal{U}_{\text{Semi-Analy.}}(x,0) = \begin{cases} \begin{cases} -\frac{2048x^{22}}{1925} + \frac{1024x^{21}}{105} - \frac{422144x^{20}}{9975} + \frac{636416x^{19}}{5985} - \frac{28337536x^{18}}{187425} + \frac{185216x^{17}}{2975} + \frac{97792x^{16}}{525} \\ -\frac{1322624x^{15}}{3675} + \frac{2369408x^{14}}{15925} + \frac{708224x^{13}}{2275} - \frac{843792x^{12}}{1925} - \frac{1312x^{11}}{1925} + \frac{211184x^{10}}{525} - \frac{1168x^9}{5} \\ -\frac{6732x^8}{35} + \frac{1640x^7}{7} + \frac{112x^6}{5} - \frac{732x^5}{5} + 24x^4 + 60x^3 - 36x^2 - 18x + 6, \\ \frac{221184x^{14}}{2275} - \frac{12288x^{12}}{55} + \frac{19456x^{10}}{75} - \frac{6912x^8}{35} + \frac{544x^6}{5} - 48x^4 + 18x^2 - \frac{9}{2}, \\ 0. \end{cases} \end{cases} \quad (15)$$

The following Tables 1–6 show the values of analytical, semi-analytical, and absolute error through $\mathfrak{z} \in [0, 0.003]$.

Table 1. Accuracy between the analytical solution (8) and the semi-analytical solutions based on (14) (AD method).

Value of $\mathfrak{z}$	Analytical	Semi-Analytical	Absolute Error
0	6	6	0
0.0001	5.99819964	5.99819964	2.66454×10^{-15}
0.0002	5.99639856	5.99639856	3.81917×10^{-14}
0.0003	5.994596762	5.994596762	1.97176×10^{-13}
0.0004	5.992794244	5.992794244	6.12843×10^{-13}
0.0005	5.990991008	5.990991008	1.50635×10^{-12}
0.0006	5.989187053	5.989187053	3.12017×10^{-12}
0.0007	5.987382381	5.987382381	5.78648×10^{-12}
0.0008	5.985576991	5.985576991	9.87299×10^{-12}
0.0009	5.983770884	5.983770884	1.58265×10^{-11}
0.001	5.98196406	5.98196406	2.41354×10^{-11}
0.0011	5.98015652	5.98015652	3.53566×10^{-11}
0.0012	5.978348264	5.978348264	5.00977×10^{-11}
0.0013	5.976539292	5.976539292	6.90425×10^{-11}
0.0014	5.974729605	5.974729605	9.29203×10^{-11}
0.0015	5.972919203	5.972919203	1.22514×10^{-10}
0.0016	5.971108086	5.971108086	1.58691×10^{-10}
0.0017	5.969296256	5.969296255	2.02347×10^{-10}

Table 1. Cont.

Value of β	Analytical	Semi-Analytical	Absolute Error
0.0018	5.967483711	5.967483711	2.5447×10^{-10}
0.0019	5.965670453	5.965670452	3.16081×10^{-10}
0.002	5.963856482	5.963856481	3.88278×10^{-10}
0.0021	5.962041798	5.962041797	4.72212×10^{-10}
0.0022	5.960226401	5.960226401	5.69098×10^{-10}
0.0023	5.958410293	5.958410292	6.80211×10^{-10}
0.0024	5.956593473	5.956593472	8.06888×10^{-10}
0.0025	5.954775941	5.95477594	9.50533×10^{-10}
0.0026	5.952957699	5.952957698	1.11259×10^{-9}
0.0027	5.951138746	5.951138745	1.29459×10^{-9}
0.0028	5.949319083	5.949319081	1.49812×10^{-9}
0.0029	5.94749871	5.947498708	1.72481×10^{-9}
0.003	5.945677628	5.945677626	1.97638×10^{-9}

Table 2. Accuracy between the analytical solution (8) and the semi-analytical solutions based on (15) (EK method).

Value of β	Analytical	Semi-Analytical	Absolute Error
0	6	6	0
0.0001	5.99819964	5.99819964	7.10543×10^{-15}
0.0002	5.99639856	5.99639856	1.15463×10^{-13}
0.0003	5.994596762	5.994596762	5.86198×10^{-13}
0.0004	5.992794244	5.992794244	1.84031×10^{-12}
0.0005	5.990991008	5.990991008	4.50218×10^{-12}
0.0006	5.989187053	5.989187053	9.33031×10^{-12}
0.0007	5.987382381	5.987382381	1.72893×10^{-11}
0.0008	5.985576991	5.985576991	2.94902×10^{-11}
0.0009	5.983770884	5.983770884	4.72404×10^{-11}
0.001	5.98196406	5.98196406	7.20011×10^{-11}
0.0011	5.98015652	5.98015652	1.05417×10^{-10}
0.0012	5.978348264	5.978348264	1.49297×10^{-10}
0.0013	5.976539292	5.976539292	2.05637×10^{-10}
0.0014	5.974729605	5.974729605	2.76596×10^{-10}
0.0015	5.972919203	5.972919203	3.64495×10^{-10}
0.0016	5.971108086	5.971108086	4.71857×10^{-10}
0.0017	5.969296256	5.969296256	6.01343×10^{-10}
0.0018	5.967483711	5.967483711	7.55819×10^{-10}
0.0019	5.965670453	5.965670453	9.38302×10^{-10}
0.002	5.963856482	5.963856482	1.15199×10^{-9}
0.0021	5.962041798	5.962041798	1.40024×10^{-9}
0.0022	5.960226401	5.960226401	1.68662×10^{-9}
0.0023	5.958410293	5.958410293	2.01482×10^{-9}
0.0024	5.956593473	5.956593473	2.38874×10^{-9}
0.0025	5.954775941	5.954775941	2.81244×10^{-9}
0.0026	5.952957699	5.952957699	3.29015×10^{-9}
0.0027	5.951138746	5.951138746	3.82628×10^{-9}
0.0028	5.949319083	5.949319083	4.42541×10^{-9}
0.0029	5.94749871	5.94749871	5.09228×10^{-9}
0.003	5.945677628	5.945677628	5.83183×10^{-9}

Table 3. Accuracy between the analytical solution (10) and the semi-analytical solutions based on (14) (AD method).

Value of β	Analytical	Semi-Analytical	Absolute Error
0	−4.5	−4.5	0
0.0001	−4.49999982	−4.49999982	7.99361×10^{-15}
0.0002	−4.49999928	−4.49999928	1.14575×10^{-13}
0.0003	−4.49999838	−4.49999838	5.81757×10^{-13}
0.0004	−4.49999712	−4.49999712	1.84475×10^{-12}
0.0005	−4.4999955	−4.4999955	4.49862×10^{-12}
0.0006	−4.49999352	−4.49999352	9.33209×10^{-12}
0.0007	−4.49999118	−4.49999118	1.72866×10^{-11}
0.0008	−4.49998848	−4.49998848	2.9492×10^{-11}
0.0009	−4.49998542	−4.49998542	4.72387×10^{-11}
0.001	−4.499982	−4.499982	7.19993×10^{-11}
0.0011	−4.49997822	−4.49997822	1.05415×10^{-10}
0.0012	−4.49997408	−4.49997408	1.49301×10^{-10}
0.0013	−4.49996958	−4.49996958	2.0564×10^{-10}
0.0014	−4.49996472	−4.49996472	2.76596×10^{-10}
0.0015	−4.4999595	−4.499959501	3.64502×10^{-10}
0.0016	−4.49995392	−4.499953921	4.71861×10^{-10}
0.0017	−4.49994798	−4.499947981	6.01355×10^{-10}
0.0018	−4.499941681	−4.499941681	7.55829×10^{-10}
0.0019	−4.499935021	−4.499935022	9.38313×10^{-10}
0.002	−4.499928001	−4.499928002	1.152×10^{-9}
0.0021	−4.499920621	−4.499920622	1.40027×10^{-9}
0.0022	−4.499912881	−4.499912883	1.68665×10^{-9}
0.0023	−4.499904781	−4.499904783	2.01487×10^{-9}
0.0024	−4.499896322	−4.499896324	2.3888×10^{-9}
0.0025	−4.499887502	−4.499887505	2.81252×10^{-9}
0.0026	−4.499878322	−4.499878325	3.29025×10^{-9}
0.0027	−4.499868783	−4.499868786	3.82641×10^{-9}
0.0028	−4.499858883	−4.499858887	4.42556×10^{-9}
0.0029	−4.499848623	−4.499848628	5.09247×10^{-9}
0.003	−4.499838004	−4.49983801	5.83205×10^{-9}

Table 4. Accuracy between the analytical solution (10) and the semi-analytical solutions based on (15) (EK method).

Value of β	Analytical	Semi-Analytical	Absolute Error
0	−4.5	−4.5	0
0.0001	−4.49999982	−4.49999982	0
0.0002	−4.49999928	−4.49999928	8.88178×10^{-16}
0.0003	−4.49999838	−4.49999838	8.88178×10^{-16}
0.0004	−4.49999712	−4.49999712	1.77636×10^{-15}
0.0005	−4.4999955	−4.4999955	8.88178×10^{-16}
0.0006	−4.49999352	−4.49999352	8.88178×10^{-16}
0.0007	−4.49999118	−4.49999118	0
0.0008	−4.49998848	−4.49998848	8.88178×10^{-16}
0.0009	−4.49998542	−4.49998542	0
0.001	−4.499982	−4.499982	8.88178×10^{-16}
0.0011	−4.49997822	−4.49997822	8.88178×10^{-16}
0.0012	−4.49997408	−4.49997408	1.77636×10^{-15}
0.0013	−4.49996958	−4.49996958	0
0.0014	−4.49996472	−4.49996472	8.88178×10^{-16}

Table 4. Cont.

Value of β	Analytical	Semi-Analytical	Absolute Error
0.0015	−4.4999595	−4.4999595	8.88178×10^{-16}
0.0016	−4.49995392	−4.49995392	0
0.0017	−4.49994798	−4.49994798	8.88178×10^{-16}
0.0018	−4.499941681	−4.499941681	0
0.0019	−4.499935021	−4.499935021	8.88178×10^{-16}
0.002	−4.499928001	−4.499928001	1.77636×10^{-15}
0.0021	−4.499920621	−4.499920621	8.88178×10^{-16}
0.0022	−4.499912881	−4.499912881	8.88178×10^{-16}
0.0023	−4.499904781	−4.499904781	0
0.0024	−4.499896322	−4.499896322	8.88178×10^{-16}
0.0025	−4.499887502	−4.499887502	1.77636×10^{-15}
0.0026	−4.499878322	−4.499878322	0
0.0027	−4.499868783	−4.499868783	8.88178×10^{-16}
0.0028	−4.499858883	−4.499858883	0
0.0029	−4.499848623	−4.499848623	8.88178×10^{-16}
0.003	−4.499838004	−4.499838004	8.88178×10^{-16}

Table 5. Accuracy between the analytical solution (12) and the semi-analytical solutions based on (14) (AD method).

Value of β	Analytical	Semi-Analytical	Absolute Error
0	0	0	0
0.0001	-1.8×10^{-7}	0	1.8×10^{-7}
0.0002	-7.2×10^{-7}	0	7.2×10^{-7}
0.0003	-1.62×10^{-6}	0	1.62×10^{-6}
0.0004	-2.88×10^{-6}	0	2.88×10^{-6}
0.0005	-4.5×10^{-6}	0	4.5×10^{-6}
0.0006	-6.47999×10^{-6}	0	6.47999×10^{-6}
0.0007	-8.81999×10^{-6}	0	8.81999×10^{-6}
0.0008	-1.152×10^{-5}	0	1.152×10^{-5}
0.0009	-1.458×10^{-5}	0	1.458×10^{-5}
0.001	-1.8×10^{-5}	0	1.8×10^{-5}
0.0011	-2.17799×10^{-5}	0	2.17799×10^{-5}
0.0012	-2.59199×10^{-5}	0	2.59199×10^{-5}
0.0013	-3.04199×10^{-5}	0	3.04199×10^{-5}
0.0014	-3.52798×10^{-5}	0	3.52798×10^{-5}
0.0015	-4.04998×10^{-5}	0	4.04998×10^{-5}
0.0016	-4.60797×10^{-5}	0	4.60797×10^{-5}
0.0017	-5.20196×10^{-5}	0	5.20196×10^{-5}
0.0018	-5.83195×10^{-5}	0	5.83195×10^{-5}
0.0019	-6.49794×10^{-5}	0	6.49794×10^{-5}
0.002	-7.19992×10^{-5}	0	7.19992×10^{-5}
0.0021	-7.93791×10^{-5}	0	7.93791×10^{-5}
0.0022	-8.71189×10^{-5}	0	8.71189×10^{-5}
0.0023	-9.52187×10^{-5}	0	9.52187×10^{-5}
0.0024	−0.000103678	0	0.000103678
0.0025	−0.000112498	0	0.000112498
0.0026	−0.000121678	0	0.000121678
0.0027	−0.000131217	0	0.000131217
0.0028	−0.000141117	0	0.000141117
0.0029	−0.000151377	0	0.000151377
0.003	−0.000161996	0	0.000161996

Table 6. Accuracy between the analytical solution (12) and the semi-analytical solutions based on (15) (EK method).

Value of β	Analytical	Semi-Analytical	Absolute Error
0	0	0	0
0.0001	-1.8×10^{-7}	0	1.8×10^{-7}
0.0002	-7.2×10^{-7}	0	7.2×10^{-7}
0.0003	-1.62×10^{-6}	0	1.62×10^{-6}
0.0004	-2.88×10^{-6}	0	2.88×10^{-6}
0.0005	-4.5×10^{-6}	0	4.5×10^{-6}
0.0006	-6.47999×10^{-6}	0	6.47999×10^{-6}
0.0007	-8.81999×10^{-6}	0	8.81999×10^{-6}
0.0008	-1.152×10^{-5}	0	1.152×10^{-5}
0.0009	-1.458×10^{-5}	0	1.458×10^{-5}
0.001	-1.8×10^{-5}	0	1.8×10^{-5}
0.0011	-2.17799×10^{-5}	0	2.17799×10^{-5}
0.0012	-2.59199×10^{-5}	0	2.59199×10^{-5}
0.0013	-3.04199×10^{-5}	0	3.04199×10^{-5}
0.0014	-3.52798×10^{-5}	0	3.52798×10^{-5}
0.0015	-4.04998×10^{-5}	0	4.04998×10^{-5}
0.0016	-4.60797×10^{-5}	0	4.60797×10^{-5}
0.0017	-5.20196×10^{-5}	0	5.20196×10^{-5}
0.0018	-5.83195×10^{-5}	0	5.83195×10^{-5}
0.0019	-6.49794×10^{-5}	0	6.49794×10^{-5}
0.002	-7.19992×10^{-5}	0	7.19992×10^{-5}
0.0021	-7.93791×10^{-5}	0	7.93791×10^{-5}
0.0022	-8.71189×10^{-5}	0	8.71189×10^{-5}
0.0023	-9.52187×10^{-5}	0	9.52187×10^{-5}
0.0024	-0.000103678	0	0.000103678
0.0025	-0.000112498	0	0.000112498
0.0026	-0.000121678	0	0.000121678
0.0027	-0.000131217	0	0.000131217
0.0028	-0.000141117	0	0.000141117
0.0029	-0.000151377	0	0.000151377
0.003	-0.000161996	0	0.000161996

4. Results and Discussion

This section investigates the constructed wave solutions of the FNO model that has been derived for describing the dynamical behavior of the oceanic waves affected by the Earth's rotation based on the newly formulated analytical scheme (Khater II method). This method is one of indirect analytical schemes which lead to a difference between its results and the other direct methods' results. This difference can be demonstrated by comparing our obtained results with those in [36–38]. In these published papers [36–38], many direct methods have been applied such as the generalized Jacobi elliptical functional method and modified Khater method, the Exp-function method, the hyperbolic tangent method and an exponential function approach. Comparing our results with the constructed solutions in these papers explains the novelty of our solutions where all our solutions are different and new from their solutions.

The dynamical behavior of the oceanic waves affected by the Earth's rotation is explained through some graphs of the solutions in the contour, three, and two-dimensional Figures 1–3 when $\alpha = -1$, $c = 3$, $\delta = -4$, $\epsilon = 1$. Additionally, the stability of the obtained solutions can be checked through the Hamiltonian system's characterizations which lead

to demonstrate the stability property of Equation (8) when $t \in [-5, 5]$, $x \in [-5, 5]$ and $\alpha = 1$, $\delta = -4$ where the momentum of Equation (8) is given by

$$\begin{aligned} \mathcal{M} = & \frac{1}{8}c \left(400c + 6 \log(\cosh(10(c-1))) - 6 \log(\cosh(10(c+1))) + 3 \right. \\ & \times \left(\frac{1}{(\sinh(5(c+1)) + i \cosh(5(c+1)))^2} + \frac{1}{(\cosh(5-5c) + i \sinh(5-5c))^2} \right. \\ & \left. \left. + \frac{1}{(\cosh(5-5c) - i \sinh(5-5c))^2} + \frac{1}{(\sinh(5(c+1)) - i \cosh(5(c+1)))^2} \right) \right). \end{aligned} \quad (16)$$

Thus, we obtain:

$$\mathcal{M}_c \Big|_{c=2} = 185.000000 > 0, \quad (17)$$

where $\mathcal{M}_c = \frac{\partial \mathcal{M}}{\partial c}$. Consequently, this solution is a stable one. Handling the other obtained solutions through the same technique can explain their stability property.

Figure 1. Periodic solitary wave solutions of the FNO model in (a) 3D, (b) 2D, and (c) contour plots of Equation (8).

Figure 2. Periodic solitary wave solutions of the FNO model in (a) 3D, (b) 2D, and (c) contour plots of Equation (10).

Figure 3. Cont.

Figure 3. Periodic solitary wave solutions of the FNO model in (a) 3D, (b) 2D, and (c) contour plots of Equation (12).

Moreover, the accuracy of constructed solutions where the analytical scheme is a new scheme was verified by applying the AD and EK semi-analytical methods based on the constructed analytical solutions to evaluate the initial and boundary conditions of the FNO equation. The accuracy of the analytical and semi-analytical solutions is explained in the above—Tables 1–6 and is presented in Figures 4–9.

Figure 4. Cont.

Figure 4. Accuracy between the analytical (8) and semi-analytical solutions in Table 1. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Figure 5. Cont.

Figure 5. Accuracy between the analytical (8) and semi-analytical solutions in Table 2. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Figure 6. Accuracy between the analytical (10) and semi-analytical solutions in Table 3. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Figure 7. Accuracy between the analytical (10) and semi-analytical solutions in Table 4. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Figure 8. Cont.

Figure 8. Accuracy between the analytical (12) and semi-analytical solutions in Table 5. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Figure 9. Cont.

Figure 9. Accuracy between the analytical (12) and semi-analytical solutions in Table 6. (a) Analytical; (b) Semi-analytical; (c) Absolute Error.

Finally, the superiority of the constructed solutions was verified by comparing the absolute error for each solution to demonstrate which of the obtained solutions is more accurate. Based on the above—Tables 1–6—Equation (12) is more accurate than (8), (10). This accuracy is represented through Figures 10 and 11.

Figure 10. Absolute error based on Tables 1, 3, and 5 ((a) Khater II (b) AD methods' solutions accuracy).

Figure 11. Absolute error based on Tables 2, 4, and 6 ((a) Khater II (b) EK methods' solutions accuracy).

5. Conclusions

This research paper investigated the analytical and semi-analytical solutions of the FNO equation. A new analytical technique (Khater II method) was successfully applied to the investigated model, and many novel solutions were constructed. These solutions are represented by graphs to explain how the ocean wave's dynamics are affected by the Earth's rotation. Additionally, the AD and EK semi-analytical schemes were applied to evaluate the FNO model's semi-analytical solutions and check the accuracy of the obtained analytical solutions.

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