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Magnetic Jacobi Fields in 3-Dimensional Cosymplectic Manifolds

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Abstract: We classify the magnetic Jacobi fields in cosymplectic manifolds of dimension 3, enriching the results in the study of magnetic Jacobi fields derived from uniform magnetic fields. In particular, we give examples of Jacobi magnetic fields in the Euclidean space $\mathbb{E}^3$ and we conclude with the description of magnetic Jacobi fields in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

Keywords: magnetic Jacobi field; cosymplectic manifold; magnetic curve

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1. Introduction

It is well known that in Riemannian geometry the study of *geodesics* on manifolds plays an essential role. Hence, it is quite obvious that a family of curves defined on a Riemannian manifold (M, g) with respect to its geometric structure will reveal more information on the geometry of the manifold from the point of view of these curves' properties. A natural generalization of the geodesics is the *magnetic curves*. In a physical interpretation, the trajectory γ of a charged particle moving in a magnetic background (M, g, F) under the influence of the magnetic field F is called a magnetic curve. In particular, when the magnetic field vanishes and the charged particle moves only under the influence of gravity, its trajectory describes a geodesic of the underlying manifold.

The study of magnetic curves originates in the study of motion of a charged particle under the action of a static magnetic field (time-independent magnetic field) in Euclidean 3-space. In $\mathbb{E}^3$ a static magnetic field is a divergence-free vector field V and it defines the Lorentz force $L\gamma' = V \times \gamma'$, where $\times$ denotes the usual cross-product. The Lorentz equation (called also the Newton equation) is given by $\gamma'' = V \times \gamma'$. Next, in order not to take into consideration the orientation of $\mathbb{E}^3$, the vector fields can be identified with 2-forms, and the divergence-free condition for vector fields becomes equivalent to closedness for 2-forms. Hence, we can generalize this result in the following manner.

On a (complete) Riemannian manifold (M, g) of arbitrary dimension n, a closed 2-form F defines a magnetic field. For the magnetic background (M, g, F), one can define the Lorentz force ϕ as a skew-symmetric $(1,1)-$type tensor field corresponding to F via the metric g as: $g(\phi(X), Y) = F(X, Y)$, $\forall X, Y \in \mathfrak{X}(M)$. The Lorentz equation becomes $\nabla_{\gamma'}\gamma' = \phi\gamma'$, where ∇ denotes the Levi-Civita connection on M. The solutions γ of the Lorentz equation are called *magnetic curves*. As we pointed out before, for a trivial magnetic field $F = 0$, the trajectories γ are the geodesics.

An important class of magnetic fields are the *uniform magnetic fields* on (complete) Riemannian manifolds, i.e. when the corresponding Lorentz force is parallel, $\nabla\phi = 0$. Two well known examples of uniform magnetic fields are scalar multiples of the Kähler form on a *Kähler manifold* and scalar multiples of the volume form on a Riemannian surface, which are generically called *Kähler magnetic fields*, and were intensively studied, see e.g., [1] and references therein. Knowing the magnetic curves corresponding to uniform magnetic fields on a (complete) Riemannian manifold (M, g), a natural problem initiated was the study of *magnetic Jacobi fields*. More precisely, a vector field W along a magnetic curve γ is called a Jacobi magnetic field derived from the uniform magnetic field F if it satisfies the following *magnetic Jacobi equation*:

$$\frac{D^2}{ds^2}W - R(\gamma', W)\gamma' - \phi(\frac{D}{ds}W) = 0,$$

where R denotes the curvature tensor on M. Again, let us notice that when the magnetic field is trivial, $F = 0$, as magnetic curves correspond to geodesics, the magnetic Jacobi fields are just the usual Jacobi fields. Moreover, in the same manner as Jacobi fields, also the magnetic Jacobi fields are obtained by a variation of trajectories. In the study of magnetic Jacobi fields on Kähler manifolds, let us mention the papers of Adachi [1,2], and Shi [3] for the particular case of magnetic Jacobi fields for surface magnetic fields. On a Riemannian manifold, the first researchers who investigated the magnetic Jacobi fields along magnetic curves for uniform magnetic fields were probably Gouda [4] and Paternain and Paternain [5].

At this point, we would like to recall a nice motivation in order to extend the study of magnetic Jacobi fields also on almost contact metric manifolds. The motivation originates again from the three-dimensional case. Let us denote by (M^3, g) an oriented three-dimensional Riemannian manifold endowed with the volume form dv_g. The 2-forms may be identified with the vector fields via the Hodge $\star$ operator. Let us denote by F a magnetic field on M, by V its corresponding divergence-free vector field, and ω the dual 1-form of V with respect to the metric g. If V is unitary, then (ϕ, V, ω) is an almost contact structure on M compatible with the metric g. Hence, the magnetic background (M^3, g, F) may be regarded as an almost contact metric manifold endowed with closed fundamental 2-form [6].

The almost contact metric manifolds include the particular class of quasi-Sasakian manifolds, which were defined by Blair in his Ph.D. thesis [7] (see also [8]) as normal almost contact metric manifolds M^{2n+1} with closed fundamental 2-form. According to [7], using the notion of the *rank* of a quasi-Sasakian manifold, which represents the rank of the 1-form η, i.e., η has *rank* $= 2p$ if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has *rank* $= 2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$, the *Sasakian manifolds* are quasi-Sasakian manifolds of rank $2n + 1$ and the cosymplectic manifolds are of rank 1.

On one hand, it is worth mentioning that a challenging problem was the study of *magnetic Jacobi fields in Sasakian manifolds* $(M, \varphi, \xi, \eta, g)$. In this case the Lorentz force is naturally obtained from the contact magnetic field $F = -q d\eta$, $q \in \mathbb{R}$, and hence $\phi = q\varphi$. We notice that the Lorentz force ϕ is no longer parallel, and thus the *magnetic field is not uniform*. The complete classification of magnetic Jacobi fields along contact magnetic curves in three-dimensional Sasakian space forms is given in [9], along with explicit examples of magnetic Jacobi fields on the unit 3-sphere $\mathbb{S}^3$, on the Heisenberg group $\mathtt{Nil}_3$, and on the model space of the SL-geometry $\mathtt{SL}_2\mathbb{R}$. These results were developed further for magnetic Jacobi fields in Sasakian space forms of dimension greater or equal to 5 [10].

On the other hand, a *cosymplectic manifold* possesses a closed fundamental 2-from, thus it defines a magnetic field. Moreover, since the field of endomorphisms φ is parallel, i.e., the Lorentz force is parallel, it follows that we deal with *uniform magnetic fields on cosymplectic manifolds*. The magnetic curves on cosymplectic manifolds of arbitrary dimension were completely classified in [11]. Moreover, in the same paper [11] it was proved that a reduction result showing that the study of the normal magnetic curves associated to a contact magnetic field on the cosymplectic manifold $M^{2n}(k) \times \mathbb{R}$ reduces to their study in $M^2(k) \times \mathbb{R}$. Special attention was paid to the product spaces $M^2(k) \times \mathbb{R}$, in order to unify the known results in the study of magnetic curves in $\mathbb{E}^3$ and in the product symmetric spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

The main objective of the present paper is the classification of magnetic Jacobi fields along magnetic curves derived from uniform magnetic fields on cosymplectic 3-dimensional manifolds.

The structure of the paper is as follows. In the next section we collect some fundamental results used in the sequel. Due to the strong connection of our study with the Euclidean 3-space, and since $\mathbb{E}^3$ can be endowed with a cosymplectic structure, we gradually introduce the reader in the study of magnetic Jacobi fields, presenting in Section 3 the results obtained in $\mathbb{E}^3$. Section 4 deals with the main definitions on cosymplectic manifolds and some first results in the study of magnetic Jacobi fields. For example, in Proposition 2 we show that the conservation law holds true for uniform magnetic fields in cosymplectic manifolds of arbitrary dimension. Next, Proposition 4 assures that the characteristic vector field ξ is a magnetic Jacobi field along any normal magnetic curve in a cosymplectic 3-manifold M^3, while Proposition 5 says that $\varphi\dot{\gamma}$ is a magnetic Jacobi field along a normal magnetic curve (which is not an integral curve of ξ) if and only if M^3 is a cosymplectic space form $M^3(c)$ with vanishing sectional curvature $c = 0$. Section 5 contains the main results, Theorems 2 and 3 obtained in the classification of magnetic Jacobi fields in 3-dimensional cosymplectic space forms according to the case when ξ is regarded as magnetic field and when $\dot{\gamma} \nparallel \xi$, respectively. Finally, in Section 6, Theorem 4 characterizes magnetic Jacobi fields in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

2. Magnetic Curves and Magnetic Jacobi Fields

There are two approaches in the study of magnetic curves along magnetic fields F on (complete) Riemannian manifolds (M, g). The first is the one mentioned in the Introduction, when a magnetic curve is regarded as a solution of the Lorentz equation:

$$\nabla_{\gamma'}\gamma' = \phi\gamma',$$

where ∇ denotes the Levi-Civita connection on M corresponding to the metric g on the Riemannian manifold M, and the Lorentz force ϕ is defined using the magnetic field F as $g(\phi\cdot, \cdot) = F$.

Notice that the magnetic curves satisfy the following conservation law, i.e., the magnetic curves evolve with constant speed. Unit speed magnetic curves are called normal magnetic curves.

On the other side, in a variational approach, the magnetic curves are solutions of a variational problem, i.e., they are the critical points of the Landau-Hall functional $\mathtt{LH}$ (on $C^\infty([a,b])$) $\mathtt{LH}(\gamma) = \mathtt{E}(\gamma) - q\int_a^b A(\gamma'(s))ds$, where $\mathtt{E}(\gamma) = \int_a^b \frac{1}{2}g(\gamma'(s), \gamma'(s))ds$ is the Dirichlet energy of γ and A is the potential 1-form generating the magnetic field F. A second variational formula for the integral $\mathtt{LH}$ leads to the concept of magnetic Jacobi field. We say that W is a *magnetic Jacobi field* along the magnetic curve γ on the magnetic background (M, g, F), if it satisfies the following second order differential equation:

$$\frac{D^2}{ds^2}W - R(\dot{\gamma}, W)\dot{\gamma} - \phi\left(\frac{D}{ds}W\right) - (\nabla_W\phi)\dot{\gamma} = 0, \tag{1}$$

where R denotes the Riemannian curvature tensor of M. See e.g., [1,12].

Analogously to magnetic curves, also Jacobi magnetic fields satisfy a conservation law, i.e., if $\nabla_W L$ is skew-adjoint with respect to g, then $g(\nabla_{\gamma'}W, \gamma')$ is constant along γ (see Lemma 1.2 of [1]). Moreover, we call a magnetic Jacobi field W along a trajectory γ *normal* if it satisfies $g(\nabla_{\gamma'}W, \gamma') = 0$. The normal magnetic Jacobi fields are obtained naturally by variations of normal trajectories, see [1].

A first example of a magnetic Jacobi field is the following. On a magnetic curve $\gamma(s)$ the velocity vector field $\gamma'(s)$ is a magnetic Jacobi field along $\gamma(s)$ (see [4]).

The major difficulty in explicitly solving the magnetic Jacobi equation consists in the presence of the last term in (1). When the covariant derivative of the Lorentz force has a particular concrete expression, we can think about solving the Equation (1). Let us point out some particular situations. For example:

- in Kähler manifolds (M, g, J) the (Kähler) magnetic fields are uniform, i.e., the Lorentz force $\phi = qJ$, $q \in \mathbb{R}$ is parallel, $\nabla\phi = 0$, and thus, the last term in (1) vanishes (see e.g., [2,3]).
- in Sasakian manifolds $\nabla\phi$ also has a relatively simple expression. If $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold, the Lorentz force is defined by $\phi = q\varphi$, where $(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$. The classification of nonuniform magnetic Jacobi fields is given in [9,10].
- in cosymplectic manifolds the magnetic field is again uniform. In the case when the magnetic field W is uniform, that is the Lorentz force $\phi = q\varphi$ is parallel, i.e., $\nabla\phi = 0$, we retrieve the equation of a magnetic Jacobi field given by Gouda [12]
$$\frac{D^2}{ds^2}W - R(\dot{\gamma}, W)\dot{\gamma} - \phi\left(\frac{D}{ds}W\right) = 0, \tag{2}$$
where R denotes the Riemannian curvature tensor of the cosymplectic manifold. In the sequel, this equation is of interest for us. Hence, we solve (2) in order to find all the magnetic Jacobi fields W in three-dimensional cosymplectic manifolds.

3. Magnetic Jacobi Fields in $\mathbb{E}^3$

In this section we describe first some already obtained results on magnetic curves in Euclidean 3-space, and second, we study the corresponding magnetic Jacobi fields.

3.1. Magnetic curves in $\mathbb{E}^3$

We consider the setting $(\mathbb{E}^3, \langle\ ,\ \rangle, \times)$, where $\langle\ ,\ \rangle$ denotes the scalar product and $\times$ is the usual cross product in the 3-dimensional space $\mathbb{R}^3$. As it was mentioned many times in some previous works on magnetic curves, the 3-dimensional case is very special. In a generic 3-dimensional Riemannian manifold (M^3, g) the 2-forms and the vector fields may be identified via the Hodge star operator $\star$ and the volume form dv_g of M^3. Hence, magnetic fields (corresponding to closed 2-forms) mean divergence free vector fields. Recall that the Killing vector fields are some important examples of divergence-free vector fields and they define the so-called *Killing magnetic fields*. Classically, one can define the cross product on M^3 as $g(X \times Y, Z) = dv_g(X, Y, Z)$, $\forall X, Y \in \mathfrak{X}(M^3)$. If we denote by V a Killing vector field on M^3, then $F_V = dv_g(V, \cdot, \cdot)$ represents the corresponding Killing magnetic field.

Let (x, y, z) be global coordinates on $\mathbb{E}^3$; a basis of Killing vector fields is given by three translational vector fields and three rotational vector fields with respect to the coordinates axes $\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right\}$.

Let $\gamma : I \to \mathbb{E}^3$ be a *normal magnetic curve*, namely a solution of the *magnetic equation* (the *Lorentz equation*):
$$\ddot{\gamma}(s) = V(s) \times \dot{\gamma}(s), \quad \text{where} \quad V(s) = V(\gamma(s)). \tag{3}$$

The *Lorentz force* has the expression:
$$\phi : \mathfrak{X}(\mathbb{E}^3) \to \mathfrak{X}(\mathbb{E}^3), \quad \phi X = V \times X, \ \ \forall X \in \mathfrak{X}(\mathbb{E}^3). \tag{4}$$

In order to solve the Lorentz Equation (3), the easiest case is to consider the Killing vector field $V_0 = \frac{\partial}{\partial z}$. Similar discussions can be made for the other two translational Killing vector fields, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively. Its corresponding magnetic curves were described for example in [13–15], and it was shown that they are helices with axis $\frac{\partial}{\partial z}$, parameterized as

$$\gamma(t) = (x_0 + a\cos t,\ y_0 + a\sin t, z_0 + bt),\ \ (x_0, y_0, z_0) \in \mathbb{E}^3,\ \ a, b \in \mathbb{R}.$$

A more difficult situation occurs in the study of magnetic curves determined by the rotational Killing vector field $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. The complete classification of these magnetic curves was done in [15] (see also [16]) and it consists in: planar curves situated in a vertical strip, circular helices and a class of curves for which the explicit parametrizations were provided, involving elliptic integrals. For this reason, in the next subsection we consider the translational Killing vector field $\frac{\partial}{\partial z}$.

3.2. *Magnetic Jacobi Fields in* $\mathbb{E}^3$

Let us denote by W a magnetic Jacobi field along the normal magnetic curve γ corresponding to the Killing vector field V. Then W satisfies:

$$W''(s) - V(s) \times W'(s) - (\nabla_W V) \times \dot{\gamma}(s) = 0. \tag{5}$$

In the following we consider $V = q\frac{\partial}{\partial z}$, $q \in \mathbb{R} \setminus \{0\}$. Notice that $\nabla_X V = 0$, $\forall X \in \mathfrak{X}(\mathbb{E}^3)$. Recall that the corresponding magnetic curve γ is a *helix*.

Thus, the magnetic Jacobi Equation (5) becomes:

$$W''(s) - q\frac{\partial}{\partial z} \times W'(s) = 0, \tag{6}$$

and its solutions are given in the following theorem.

Theorem 1. *Let $\gamma(s)$ be a normal magnetic curve corresponding to the Killing vector field $q\frac{\partial}{\partial z}$ in $\mathbb{E}^3$. Then, the magnetic Jacobi fields along γ are given by one of the following cases:*

(i) $W(s) = W_0 + as\frac{\partial}{\partial z}$,

(ii) $W(s) = W_0 + \frac{\sin qs}{q} v_0 - \cos qs\ \phi v_0 + as\frac{\partial}{\partial z}$,

where W_0 is a constant vector in $\mathbb{R}^3$, v_0 is a constant vector orthogonal to $\frac{\partial}{\partial z}$, and $a \in \mathbb{R}$.

Proof. In order to solve (6), we decompose $W'(s) = f(s)\frac{\partial}{\partial z} + H(s)$, such that $H(s) \perp \frac{\partial}{\partial z}$. The Equation (6) becomes:

$$f'(s)\frac{\partial}{\partial z} + H'(s) - q\frac{\partial}{\partial z} \times H(s) = 0. \tag{7}$$

Since $H'(s) \perp \frac{\partial}{\partial z}$ and $\frac{\partial}{\partial z} \times H(s) \perp \frac{\partial}{\partial z}$, one gets $f'(s) = 0$, i.e., f is a constant function, let us denote it by $f(s) = a \in \mathbb{R}$. Now, Equation (7) becomes:

$$H'(s) = q\frac{\partial}{\partial z} \times H(s), \tag{8}$$

which has two solutions: either $H(s) = 0$, or $H(s) = \cos qs\ v_0 + \sin qs\ \frac{1}{q}\phi v_0$, where v_0 is a constant vector orthogonal to $\frac{\partial}{\partial z}$. First, when $H(s) = 0$, one gets $W'(s) = a\frac{\partial}{\partial z}$

and hence solution (i) from the theorem is found. Second, if $H(s) \neq 0$, then $W'(s) = \cos qs\, v_0 + \sin qs \frac{1}{q}\phi v_0 + as \frac{\partial}{\partial z}$, which furnishes case (ii) from the theorem, concluding the proof. $\square$

Remark 1. *In fact, the constants $a \in \mathbb{R}$ and $W_0 \in \mathbb{R}^3$ are obtained from the initial conditions. As examples, we give the following Table 1.*

Table 1. Examples of magnetic Jacobi fields in $\mathbb{E}^3$.

Initial Conditions	**Magnetic Jacobi Field $W(s)$**
$W(0) = (0,0,1)$, $W'(0) = (0,0,0)$	$(0,0,1)$
$W(0) = (0,0,\lambda)$, $W'(0) = (0,0,1)$	$(0,0,s+\lambda)$, $\lambda \in \mathbb{R}$
$W(0) = (0,0,0)$, $W'(0) = (\cos\psi, \sin\psi, 0)$, $\psi \in \mathbb{R}$	$\frac{\sin qs}{q}(\cos\psi, \sin\psi, 0) + \frac{1-\cos qs}{q}(-\sin\psi, \cos\psi, 0)$.

Remark 2. *In Theorem 1, the expression of the magnetic trajectory γ is not, seemingly, explicitly involved. However, the conservation law holds true, it can be proved that the function $\langle W'(s), \dot{\gamma}\rangle$ is constant.*

On the contrary, if the Killing vector field changes to $V = q\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$, then the magnetic Jacobi equation becomes

$$W''(s) - V(s) \times W'(s) - q\left(\frac{\partial}{\partial z} \times W(s)\right) \times \dot{\gamma}(s) = 0.$$

Notice that, in the equation above, the expression of γ appears explicitly. To solve this ODE in $\mathbb{R}^3$ is a real challenge since the expression of γ is quite complicated. See [15]. Nevertheless, if W is a magnetic Jacobi field with respect to the new background, one can prove that the function $\langle W'(s), \dot{\gamma}\rangle$ is still constant. As we mentioned before, the conservation law holds true for magnetic Jacobi fields along trajectories of uniform magnetic fields on Riemannian manifolds. See e.g., [1].

4. Cosymplectic Manifolds

4.1. Fundamentals

On a smooth manifold M, let φ be a tensor field of type $(1,1)$ such that $\varphi^2 = -I + \eta \otimes \xi$, where I is the identity, ξ a vector field and η a 1-form on M satisfying the conditions:

$$\varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1.$$

As a consequence, the dimension of M is odd $(= 2n+1)$. Then, (M, φ, ξ, η) is an almost contact manifold. Let g be a Riemannian metric on M compatible with the almost contact structure defined above, that is

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

The manifold M is called now an almost contact metric manifold. Note that an almost contact metric structure on an orientable $(2n+1)-$dimensional manifold is a reduction of the structure group of M to $U(n) \times 1$.

The fundamental 2-form Φ on M is defined by:

$$\Phi(X,Y) = g(X, \varphi Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

The almost contact structure is called normal if $N_\varphi + 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhius tensor defined by:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \quad \forall X, Y \in \mathfrak{X}(M).$$

A normal almost contact structure with $d\eta = 0$ and $d\Phi = 0$ is said to be a cosymplectic structure. The notion of cosymplectic manifold was introduced, independently, by Blair in his Ph.D. thesis [7] and by Ogiue in [17] using the terminology "cocomplex". If the normality condition is missing, the structure is known as *almost cosymplectic*.

It is known from [8] that the cosymplectic structure is characterized among the almost contact metric structures, by the parallelism of φ, that is $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection on M. It follows that η and ξ are also parallel.

Denote by R the Riemannian curvature tensor defined by

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

by Ric the Ricci tensor defined by

$$Ric(Z,Y) = trace_g(X \mapsto R(X,Y)Z),$$

and by Q the corresponding Ricci operator, defined by

$$g(QX,Y) = Ric(X,Y).$$

If X is a unit vector at $p \in M$, which is orthogonal to $\xi(p)$, we say that X and φX span a $\varphi-$section. The sectional curvature $K(X)$ of the $\varphi-$section defined by X is called the $\varphi-$sectional curvature defined by X (at p). If $K(X)$ is independent of X and p, we say that M is of constant $\varphi-$sectional curvature. A (complete) cosymplectic space form is a cosymplectic manifold of constant $\varphi-$sectional curvature.

Let us give now the following example from [18].

Example 1 ([18])**.** *Let $\overline{M} = (\overline{M}, \bar{g}, J)$ be an almost Kähler manifold. Consider a Riemannian product $M = (\overline{M} \times \mathbb{R}, g)$ with $g = \bar{g} + dt^2$, where t is the global coordinate on $\mathbb{R}$. Then, we can endow M with an almost cosymplectic structure by setting*

$$\xi = \frac{d}{dt}, \quad \eta = dt, \quad \varphi = J \circ d\pi,$$

where $\pi : \overline{M} \times \mathbb{R} \to \overline{M}$ is the projection map and t is the standard coordinate function on $\mathbb{R}$. The almost cosymplectic manifold M is cosymplectic if and only if $\overline{M}$ is Kähler. In particular, when $\overline{M}$ is a complex space form (i.e. a Kähler manifold of constant holomorphic sectional curvature), M is a cosymplectic manifold of constant φ-holomorphic sectional curvature. The cosymplectic manifolds:

$$\mathbb{C}P^n(c) \times \mathbb{R}, \;\; \mathbb{E}^{2n+1} = \mathbb{C}^n \times \mathbb{R}, \;\; \mathbb{C}H^n(c) \times \mathbb{R}$$

are cosymplectic space forms. Here $\mathbb{C}P^n(c)$ denotes the complex projective $n-$space of constant holomorphic sectional curvature $c > 0$, $\mathbb{C}^n$ is the complex Euclidean $n-$space, and $\mathbb{C}H^n(c)$ is the complex hyperbolic $n-$space of constant holomorphic sectional curvature $c < 0$.

Let us point out that a cosymplectic manifold can be regarded locally as endowed with a naturally local product structure of a Kähler manifold and a one-dimensional manifold, but there are also compact cosymplectic manifolds which are not global products. See e.g., [19].

4.2. Curvature

It has been proved that in a cosymplectic space form the curvature tensor can be expressed as

$$\begin{aligned} R_{XY}Z = \; & \frac{c}{4}\Big(g(Y,Z)X - g(X,Z)Y\Big) \\ & + \frac{c}{4}\Big(g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z\Big) \\ & + \frac{c}{4}\Big(\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\Big)\xi - \frac{c}{4}\eta(Z)\Big(\eta(Y)X - \eta(X)Y\Big). \end{aligned} \tag{9}$$

Probably this formula was first obtained by Blair, but it was not published; see also [20]. Remark that the curvature (9) looks very similar to the curvature of a Sasakian space form, obtained by Ogiue [21]; in the later case the coefficients $\frac{c}{4}, \frac{c}{4}, \frac{c}{4}, \frac{c}{4}$ should be replaced by $\frac{c+3}{4}, \frac{c-1}{4}, \frac{c-1}{4}, \frac{c-1}{4}$, respectively. Beautiful and easier expressions are given by Inoguchi [18]. In general, even it is not explicitly mentioned, the dimension of M is assumed to be ≥ 5. Nevertheless, the formula (9) is valid for dimension ≥ 3 and we will emphasize this fact here.

The dimension 3. In the study of almost contact metric manifolds, the three-dimensional case is rather exceptional. Olszak [22] proved a series of special formulas for arbitrary almost contact metric 3-manifolds. Three of them, that are of our interest,respectively express the covariant derivative of φ and the exterior derivatives $d\eta$ and $d\Phi$ as follows

$$(\nabla_X\varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi,$$

$$d\eta = \eta \wedge \nabla_\xi\eta + \frac{1}{2}\mathrm{tr}(\varphi\nabla\xi)\Phi,\; d\Phi = (\mathrm{div}\,\xi)\eta \wedge \Phi.$$

A normal almost contact 3-manifold is called *cosymplectic* (or coKähler) if

$$\mathrm{trace}(\varphi\nabla\xi) = 0 \quad \text{and} \quad \mathrm{div}\,\xi = 0.$$

Further, the Ricci tensor of a cosymplectic 3-manifold is given by

$$Ric(X,Y) = \frac{r}{2}g(X,Y) - \frac{r}{2}\eta(X)\eta(Y), \tag{10}$$

where r denotes the scalar curvature, i.e., $r = \mathrm{trace}\, Ric = \mathrm{trace}\, Q$.

It is known that on a Riemannian 3-manifold (M,g), the curvature tensor is described by the Ricci tensor field Ric and the corresponding Ricci operator Q as follows

$$\begin{aligned} R_{XY}Z = \; & Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ & - \frac{r}{2}\Big(g(Y,Z)X - g(X,Z)Y\Big). \end{aligned} \tag{11}$$

Combining the two formulas (10) and (11), one obtains the following expression for the curvature tensor of a cosymplectic 3-manifold

$$\begin{aligned} R_{XY}Z = \frac{r}{2}\Big(g(Y,Z)X - g(X,Z)Y\Big) \\ + \frac{r}{2}\eta(Z)\Big(\eta(X)Y - \eta(Y)X\Big) + \frac{r}{2}\Big(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\Big)\xi. \end{aligned} \tag{12}$$

If the sectional curvature is a constant c, then we have the expression for the curvature tensor of a cosymplectic 3-dimensional space form

$$R_{XY}Z = c\Big(g(Y,Z)X - g(X,Z)Y\Big) + c\eta(Z)\Big(\eta(X)Y - \eta(Y)X\Big) + c\Big(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\Big)\xi. \tag{13}$$

It is straightforward to show that the formula (13) leads to the same result as (9), despite the fact that the two expressions look very different. To prove this, we need a φ-basis $\{\xi, e, \varphi e\}$, where $e \in \ker \eta$ is unitary. No matter which formula we use, one gets

$$\begin{array}{lllllllll} R_{e\,\varphi e}e & = & -c\varphi e & R_{e\,\xi}e & = & 0 & R_{\varphi e\,\xi}e & = & 0 \\ R_{e\,\varphi e}\varphi e & = & ce & R_{e\,\xi}\varphi e & = & 0 & R_{\varphi e\,\xi}\varphi e & = & 0 \\ R_{e\,\varphi e}\xi & = & 0 & R_{e\,\xi}\xi & = & 0 & R_{\varphi e\,\xi}\xi & = & 0. \end{array}$$

As R is $C^\infty(M)$ trilinear, the two expressions must coincide.

4.3. First Results

Let $(M, \varphi, \xi, \eta, g)$ be a cosymplectic manifold. We denote by ∇ the Levi-Civita connection associated to the metric g. We consider $\gamma : I \to M$ a normal contact magnetic curve parametrized by its arc length s satisfying the following differential equation

$$\frac{D}{ds}\dot{\gamma} = \phi\dot{\gamma},$$

where ϕ represents the Lorentz force. We take

$$\phi = q\varphi, \tag{14}$$

where $q \in \mathbb{R}$ is the *strength* (the *charge*) of the magnetic field. Moreover, the *magnetic field* is defined as $F = -q\Phi$, where

$$\Phi(X, Y) = g(X, \varphi Y). \tag{15}$$

The following property of the magnetic curves in cosymplectic manifolds holds according to [11].

Remark 3. *Every contact magnetic curve in a cosymplectic manifold is slant, that is, the angle θ between $\dot{\gamma}(s)$ and $\xi(\gamma(s))$ is constant (i.e., independent of s).*

In Proposition 1 of [11] the authors provide the list of all normal magnetic curves corresponding to the contact magnetic field F, as follows:

(a) geodesics, obtained as integral curves of ξ;
(b) Legendre $\varphi-$circles of curvature $\kappa_1 = |q|$;
(c) $\varphi-$helices of order 3, with curvatures $\kappa_1 = |q|\sin\theta$, $\kappa_2 = |q\cos\theta|$ and such that $\mathrm{sgn}(\tau_{01}) = -\mathrm{sgn}(q)$, where $\theta \neq \frac{\pi}{2}$ is the constant contact angle of γ.

In the case when the ambient is cosymplectic, the (1,1)- type tensor field is parallel, namely $\nabla\varphi = 0$, equivalently, the Lorentz force is parallel, and therefore the magnetic field is uniform. As we mentioned in Section 2, in this case we retrieve the equation of a magnetic Jacobi field given by (2).

Replacing (14) in (2), the *magnetic Jacobi equation* writes as:

$$\mathcal{J}_F(W) := \frac{D^2}{ds^2}W + R(W, \dot{\gamma})\dot{\gamma} - q\varphi\frac{D}{ds}W = 0. \tag{16}$$

Let us point out the following known fact.

Proposition 1. *On an arbitrary Riemannian manifold (M, g) endowed with a magnetic field, the (unit) speed vector $\dot{\gamma}$ is always a magnetic Jacobi field along the magnetic curve γ.*

The next result may be interpreted as a conservation law.

Proposition 2. *Let W be a magnetic Jacobi field along the contact magnetic curve γ in a cosymplectic manifold of arbitrary dimension. Then, the function $g(\frac{D}{ds}W(s), \dot{\gamma}(s))$ is constant.*

Proof. We have

$$\begin{aligned} \tfrac{d}{dt}g(\tfrac{D}{ds}W(s),\dot{\gamma}(s)) = &\ g(\tfrac{D^2}{ds^2}W(s),\dot{\gamma}(s)) + g(\tfrac{D}{ds}W(s),\tfrac{D}{ds}\dot{\gamma}(s)) \\ = &\ g(q\varphi\tfrac{D}{ds}W(s) + R(\dot{\gamma}(s),W(s))\dot{\gamma}(s),\dot{\gamma}(s)) + g(\tfrac{D}{ds}W(s),q\varphi\dot{\gamma}(s)) \\ = &\ 0. \end{aligned}$$

Hence the function $g(\frac{D}{ds}W(s), \dot{\gamma}(s))$ is constant. □

If the constant obtained in Proposition 2 is zero, the magnetic Jacobi field along γ is called *normal*. See e.g., [1]. Furthermore, we can state the following result.

Proposition 3. *Let W_1 and W_2 be two magnetic Jacobi fields along the contact magnetic curve γ in a cosymplectic manifold of arbitrary dimension. Then, the function*

$$g\left(\frac{D}{ds}W_1(s), W_2(s)\right) - g\left(\frac{D}{ds}W_2(s), W_1(s)\right) + qg(W_1(s), \varphi W_2(s))$$

is constant.

Proof. Similar computations should be done as in the proof of Proposition 2. □

As a matter of fact, the conclusion of Proposition 2 follows from Proposition 3 considering Proposition 1.

Proposition 4. *The characteristic vector field ξ of a cosymplectic manifold M^3 is a magnetic Jacobi field along any normal contact magnetic curve.*

Proof. In order to prove the proposition, we should check if ξ verifies (2). Replacing W by ξ, the magnetic Jacobi Equation (2) becomes:

$$\mathcal{J}_F(\xi) = \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}\xi + R(\xi,\dot{\gamma})\dot{\gamma} - q\varphi\nabla_{\dot{\gamma}}\xi = 0. \quad (17)$$

Since, on one side $\nabla_{\dot{\gamma}}\xi = 0$, and on the other side, by straightforward computations, one can show that $R(\xi,\dot{\gamma})\dot{\gamma} = 0$, the magnetic Jacobi equation is satisfied by ξ, concluding the proof. □

Let us turn our attention now on the cosymplectic manifolds of dimension 3.

Proposition 5. *Let γ be a contact magnetic curve on the cosymplectic three dimensional manifold M^3, such that γ is not an integral curve of ξ. Then, $\varphi\dot{\gamma}$ is a magnetic Jacobi field along γ if and only if M^3 is a cosymplectic space form $M^3(c)$ with $c = 0$.*

Proof. We replace W by $\varphi\dot{\gamma}$ and we compute the operator $\mathcal{J}_F$ for $\varphi\dot{\gamma}$:

$$\mathcal{J}_F(\varphi\dot{\gamma}) = \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}(\varphi\dot{\gamma}) + R(\varphi\dot{\gamma},\dot{\gamma})\dot{\gamma} - q\varphi\nabla_{\dot{\gamma}}(\varphi\dot{\gamma}). \quad (18)$$

Using the cosymplectic structure and the Lorentz equation, by straightforward computations one can compute

$$\nabla_{\dot\gamma}(\varphi\dot\gamma) = \varphi\nabla_{\dot\gamma}\dot\gamma = q(-\dot\gamma + \cos\theta\xi),$$

and subsequently

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}(\varphi\dot\gamma) = q(-\nabla_{\dot\gamma}\dot\gamma + \cos\theta\nabla_{\dot\gamma}\xi) = -q^2\varphi\dot\gamma.$$

Here θ is the contact angle, that is $\cos\theta = \eta(\dot\gamma)$. Computing also the curvature, we obtain

$$R(\varphi\dot\gamma, \dot\gamma)\dot\gamma = \frac{r}{2}\sin^2\theta\varphi\dot\gamma.$$

Hence, the magnetic Jacobi operator is given by:

$$\mathcal{J}_F(\varphi\dot\gamma) = \frac{r}{2}\sin^2\theta\varphi\dot\gamma. \tag{19}$$

Notice that $\varphi\dot\gamma$ is an eigenvector of the operator $\mathcal{J}_F$ with the corresponding eigenvalue $\frac{r}{2}\sin^2\theta$. Thus, $\varphi\dot\gamma$ is a magnetic Jacobi field along γ if and only if $\mathcal{J}_F(\varphi\dot\gamma) = 0$. Equivalently, one of the next cases takes place:

- either $\dot\gamma \| \xi$ (hence $\dot\gamma = \pm\xi$ since $\dot\gamma$ and ξ are unitary), equivalently to $\theta = 0$ or π;
- or, $\dot\gamma$ and ξ are linearly independent (equivalently $\sin\theta \neq 0$), case when $r = 0$.

The scalar curvature of M^3 is constant, and so, the φ-sectional curvature of M^3 is a constant c, with $c = \frac{r}{2} = 0$.

This ends the proof. □

5. Magnetic Jacobi Fields in Three-Dimensional Cosymplectic Space Forms

We first investigate magnetic Jacobi fields along the integral curves of ξ (first item in Proposition 1 of [11]). Since magnetic Jacobi fields are natural generalization of Jacobi fields (along geodesics), we would like to emphasize a link between Jacobi fields along the Reeb vector field and Ricci solitons on K-contact manifolds. See e.g., Theorem 3.1 of [23].

1 The characteristic vector field ξ as magnetic field: $\dot\gamma(s) = \xi(s)$.

Theorem 2. *Let γ be an integral curve of ξ in a cosymplectic manifold $(M^3, \varphi, \xi, \eta, g)$ endowed with $F = -q\Phi$, the magnetic field of strength q defined by the fundamental 2-form Φ. The magnetic Jacobi field along γ is given:*

$$\begin{aligned} \text{either by} \quad & W(s) = W_0(s) + (f_0 + f_1 s)\xi(\gamma(s)) + \sin qs v_0(s) - \cos qs\varphi v_0(s), \\ \text{or by} \quad & W(s) = W_0(s) + (f_0 + f_1 s)\xi(\gamma(s)), \end{aligned} \tag{20}$$

where $v_0(s)$ is a vector field parallel along $\gamma(s)$ lying in the contact distribution $\ker\eta$ *and W_0 is a linear combination, with constant coefficients, of $v_0(s)$ and $\varphi v_0(s)$, and $f_0, f_1 \in \mathbb{R}$.*

Proof. Since $R(\cdot,\cdot)\xi = 0$, the magnetic Jacobi Equation (16) can be written as

$$\nabla_\xi\nabla_\xi W - q\varphi\nabla_\xi W = 0. \tag{21}$$

In the sequel we adopt the same strategy as in [9] (or as in [24]). We can take an orthonormal basis parallel along γ of the following form: $\{\xi(\gamma(s)), E(s), \varphi E(s)\}$, where $E(s) \in \ker\eta(\gamma(s))$. We decompose

$$W(s) = f(s)\xi(\gamma(s)) + a(s)E(s) + b(s)\varphi E(s), \tag{22}$$

where $f,\ a,\ b$ are smooth functions on I. As ξ, E, φE are parallel,

$$\nabla_\xi W(s) = f'(s)\xi + a'(s)E(s) + b'(s)\varphi E(s), \tag{23}$$
$$\nabla_\xi \nabla_\xi W(s) = f''(s)\xi + a''(s)E(s) + b''(s)\varphi E(s), \tag{24}$$
$$\varphi \nabla_\xi W(s) = a'(s)\varphi E(s) - b'(s)E(s). \tag{25}$$

Thus, equation (21) becomes:

$$f''(s)\xi + a''(s)E(s) + b''(s)\varphi E(s) - q(a'(s)\varphi E(s) - b'(s)E(s)) = 0$$

and the following system of ODEs is obtained

$$\begin{cases} f''(s) = 0, \\ a''(s) + qb'(s) = 0, \\ b''(s) - qa'(s) = 0, \end{cases} \tag{26}$$

having the solutions

$$\begin{aligned} &f(s) = f_0 + f_1 s, \\ &a(s) = a_0 + \frac{1}{q}(c_1 \sin qs - c_2 \cos qs), \\ &b(s) = b_0 - \frac{1}{q}(c_1 \cos qs + c_2 \sin qs), \quad f_0,\ f_1,\ a_0,\ b_0,\ c_1,\ c_2 \in \mathbb{R}. \end{aligned} \tag{27}$$

Thus, $W(s) = W_0(s) + (f_0 + f_1 s)\xi + \frac{\sin qs}{q}(c_1 E(s) - c_2 \varphi E(s)) - \frac{\cos qs}{q}(c_1 \varphi E(s) + c_2 E(s))$. Let $c_1^2 + c_2^2 \neq 0$. Denoting $v_0(s) = \frac{1}{q}\big(c_1 E(s) - c_2 \varphi E(s)\big) \in \ker \eta$ a vector field parallel along $\gamma(s)$, and subsequently $\varphi v_0(s) = \frac{1}{q}\big(c_1 \varphi E(s) + c_2 E(s)\big)$, we get the first expression in (20) for the magnetic Jacobi field W. The vector field $W_0(s) = a_0 E(s) + b_0 \varphi E(s)$ (along $\gamma(s)$) is parallel and it can be expressed as a linear combination of $v_0(s)$ and $\varphi v_0(s)$. If the constants c_1 and c_2 both vanish, then $a(s) = a_0$, $b(s) = b_0$ and W is given by the second expression in (20), $W(s) = W_0(s) + (f_0 + f_1 s)\xi$, where $W_0(s)$ is a linear combination of $v_0(s)$ and $\varphi v_0(s)$ with $v_0(s) = E(s)$. This ends the proof. □

2 The case when $\dot\gamma(s) \nparallel \xi$; in particular, γ can be a Legendre curve, $\dot\gamma \perp \xi$.

Theorem 3. *Let γ be a normal contact magnetic curve in the cosymplectic space form $(M^3(c), \varphi, \xi, \eta, g)$ endowed with the magnetic field $F = -q\Phi$. Then, the magnetic Jacobi field W along γ derived from the uniform magnetic field F is given by:*

$$W(s) = A(s)\dot\gamma(s) + B(s)\varphi\dot\gamma(s) + C(s)\xi(s),$$

where $A, B, C \in C^\infty(I)$ have the following expressions:

$$q^2 + c\sin^2\theta = 0 : \begin{cases} A(s) = -q^2 c_0 \frac{s^3}{6} + c_1 q \frac{s^2}{2} + (c_0 + c_2 q)s + c_3, \\ B(s) = -qc_0 \frac{s^2}{2} + c_1 s + c_2, \\ C(s) = q^2 \cos\theta c_0 \frac{s^3}{3} - q\cos\theta c_1 \frac{s^2}{2} + (c_4 - c_2 q\cos\theta)s + c_5. \end{cases} \tag{28}$$

$$q^2 + c\sin^2\theta := k^2 > 0 : \begin{cases} A(s) = \dfrac{qc_1}{k}\sin ks - \dfrac{qc_2}{k}\cos ks + \dfrac{c_0 c\sin^2\theta}{k^2}s + c_3, \\ B(s) = c_1\cos ks + c_2\sin ks - \dfrac{qc_0}{k^2}, \\ C(s) = -\dfrac{q\cos\theta}{k}(c_1\sin ks - c_2\cos ks) + c_4 s + c_5. \end{cases} \tag{29}$$

$$q^2 + c\sin^2\theta := -k^2 < 0 : \begin{cases} A(s) = \dfrac{qc_1}{k}\sinh ks + \dfrac{qc_2}{k}\cosh ks - \dfrac{c_0 c\sin^2\theta}{k^2}s + c_3, \\ B(s) = c_1\cosh ks + c_2\sinh ks + \dfrac{qc_0}{k^2}, \\ C(s) = -\dfrac{q\cos\theta}{k}(c_1\sinh ks + c_2\cosh ks) + c_4 s + c_5. \end{cases} \tag{30}$$

In all the three cases, $c_i \in \mathbb{R}$, $i = \overline{0,5}$, and k is a real positive constant. Moreover, $q \in \mathbb{R}$ denotes the strength of the magnetic field, Φ is the fundamental 2-from, θ is the contact angle and c denotes the sectional curvature of M.

Proof. In the sequel, we will develop two approaches.

A We will proceed as in case 1, that is we use an orthonormal basis $\{\xi, E, \varphi E\}$ parallel along γ. Notice that ξ is parallel along γ. This can be obtained if we consider $E(0) \perp \xi(\gamma(0))$ and after that take $E(s)$ parallel along γ.

Computing $\dfrac{d}{ds}g(E(s), \xi(\gamma(s))) = g(\nabla_{\dot\gamma}E, \xi) + g(E, \nabla_{\dot\gamma}\xi) = 0$, it follows that $g(E(s), \xi(\gamma(s)))$ is constant, thus $E(s) \in \ker\eta(\gamma(s))$. Next, $\varphi E(s)$ is parallel along γ and orthogonal to ξ. If $E(0)$ is unitary, then $E(s)$ is also unitary. Indeed, computing $\dfrac{d}{ds}g(E(s), E(s)) = 2g(\nabla_{\dot\gamma}E, E) = 0$, we conclude with $g(E(s), E(s))$ is constant.

Let us decompose

$$W(s) = f(s)\xi(\gamma(s)) + a(s)E(s) + b(s)\varphi E(s). \tag{31}$$

Recall the magnetic Jacobi equation from (16)

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}W + R(W, \dot\gamma)\dot\gamma - q\varphi\nabla_{\dot\gamma}W = 0, \tag{32}$$

in order to replace the expression of W given by (31). One can compute each term in (32) as follows:

$$\begin{aligned} \nabla_{\dot\gamma}W &= f'(s)\xi + a'(s)E(s) + b'(s)\varphi E(s), \\ \nabla_{\dot\gamma}\nabla_{\dot\gamma}W &= f''(s)\xi + a''(s)E(s) + b''(s)\varphi E(s), \\ \varphi\nabla_{\dot\gamma}W &= a'(s)\varphi E(s) - b'(s)E(s), \end{aligned}$$

and the curvature

$$R(W, \dot\gamma)\dot\gamma = \frac{r}{2}\sin^2\theta W + \frac{r}{2}(-g(W, \dot\gamma) + \eta(W)\cos\theta)\dot\gamma + \frac{r}{2}(\cos\theta g(W, \dot\gamma) - \eta(W))\xi,$$

where r denotes the scalar curvature and θ is the constant angle. Next, we decompose $\dot\gamma$ in the basis $\{\xi, E, \varphi E\}$ as:

$$\dot\gamma(s) = \cos\theta\xi(s) + \sin\theta(\cos(qs + \alpha_0)E(s) + \sin(qs + \alpha_0)\varphi E(s)). \tag{33}$$

This fact is a direct consequence of the Lorentz equation for the normal magnetic curve γ, that is, it follows by straightforward computations from the equation $\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma$. In the expression above $\alpha_0 \in \mathbb{R}$.

Solving the magnetic Jacobi Equation (32) becomes now equivalent to solving the next ODEs system:

$$\begin{cases} f''(s) = 0 \\ a''(s) + \frac{r}{2}\sin^2\theta\Big(a(s)\sin^2(qs+\alpha_0) - b(s)\sin(qs+\alpha_0)\cos(qs+\alpha_0)\Big) + qb'(s) = 0 \\ b''(s) + \frac{r}{2}\sin^2\theta\Big(b(s)\cos^2(qs+\alpha_0) - a(s)\sin(qs+\alpha_0)\cos(qs+\alpha_0)\Big) - qa'(s) = 0. \end{cases} \tag{34}$$

Immediately, the solution of the first equation in (34) is

$$f(s) = f_0 + f_1 s, \quad f_0, f_1 \in \mathbb{R}. \tag{35}$$

Next, we should find the expressions of $a(s)$ and $b(s)$. Adding the second equation multiplied by $\cos(qs+\alpha_0)$ and the third equation multiplied by $\sin(qs+\alpha_0)$ one gets $\frac{d}{ds}\Big(a'(s)\cos(qs+\alpha_0) + b'(s)\sin(qs+\alpha_0)\Big) = 0$. So, there exists $\lambda \in \mathbb{R}$ such that $a'(s)\cos(qs+\alpha_0) + b'(s)\sin(qs+\alpha_0) = \lambda$. Thus, we have

$$\begin{cases} a'(s) = \lambda\cos(qs+\alpha_0) + \rho(s)\sin(qs+\alpha_0) \\ b'(s) = \lambda\sin(qs+\alpha_0) - \rho(s)\cos(qs+\alpha_0), \end{cases} \tag{36}$$

for a certain $\rho \in C^\infty(I)$. Now, returning to (34), one gets

$$\rho'(s) + \frac{r}{2}\sin^2\theta\Big(a(s)\sin(qs+\alpha_0) - b(s)\cos(qs+\alpha_0)\Big) = 0. \tag{37}$$

At this point it is time to ask $r = \mathit{constant}$, i.e., M is a cosymplectic space form with $c = \frac{r}{2}$. Taking successive derivatives in (37), after some straightforward computations, we get

$$\rho'''(s) + (q^2 + c\sin^2\theta)\rho'(s) + qc\lambda\sin^2\theta = 0. \tag{38}$$

Denote $\mu := q^2 + c\sin^2\theta$. The solutions of (38) depend on the sign of μ.

Let us consider the easiest case when $\mu = 0$, i.e., $q^2 = -c\sin^2\theta$, meaning that $c < 0$. The Equation (38) becomes $\rho'''(s) = -qc\lambda\sin^2\theta = \lambda q^3$. If $\lambda \neq 0$, then $\rho(s)$ is a third order polynomial in s, be it $\rho(s) = \frac{\lambda q^3}{6}s^3 + c_2 s^2 + c_1 s + c_0$. The functions a and b are given by

$$\begin{aligned} a(s) &= T(s)\cos(qs+\alpha_0) + U(s)\sin(qs+\alpha_0), \\ b(s) &= T(s)\sin(qs+\alpha_0) - U(s)\cos(qs+\alpha_0), \end{aligned} \tag{39}$$

where $T(s) = -\frac{q^2\lambda}{6}s^3 - \frac{c_2}{q}s^2 + \left(\lambda - \frac{c_1}{q}\right)s + \frac{2c_2 - c_0 q^2}{q^3}$ is a third order polynomial and $U(s) = \frac{q\lambda}{2}s^2 + \frac{2c_2}{q^2}s + \frac{c_1}{q^2}$ is a second order polynomial, with $c_2, c_1, c_0 \in \mathbb{R}$.

As one can notice, the solution are not so easy to be formulated explicitly, and in the case $\mu \neq 0$ things are getting even more tricky. For this reason, we attack the system of ODEs (34) in a different approach, using the basis $\{\dot{\gamma}, \varphi\dot{\gamma}, \xi\}$.

(B) We will proceed in the following using the basis $\{\dot{\gamma}, \varphi\dot{\gamma}, \xi\}$ (see also [9]). Let us decompose

$$W(s) = A(s)\dot{\gamma}(s) + B(s)\varphi\dot{\gamma}(s) + C(s)\xi(s), \quad A, B, C \in C^\infty(I). \tag{40}$$

As in the previous case, we can compute

$$\begin{aligned}
\nabla_{\dot\gamma} W &= (A'(s) - qB(s))\dot\gamma(s) + (B'(s) + qA(s))\varphi\dot\gamma + (C'(s) + q\cos\theta B(s))\xi, \\
\nabla_{\dot\gamma}\nabla_{\dot\gamma} W &= (A''(s) - 2qB'(s) - q^2A(s))\dot\gamma + (B''(s) + 2qA'(s) - q^2B(s))\varphi\dot\gamma \\
&\quad +(C''(s) + 2q\cos\theta B'(s) + q^2\cos\theta A(s))\xi, \\
q\varphi\nabla_{\dot\gamma} W &= -q(B'(s) + qA(s))\dot\gamma + q(A'(s) - qB(s))\varphi\dot\gamma + q(B'(s) + qA(s))\cos\theta\xi, \\
R(W,\dot\gamma)\dot\gamma &= B(s)c\sin^2\theta\varphi\dot\gamma.
\end{aligned}$$

Now, the magnetic Jacobi Equation (32) becomes equivalent to the following system of ODEs

$$\begin{cases} A''(s) - qB'(s) = 0, \\ B''(s) + qA'(s) + cB(s)\sin^2\theta = 0, \\ C''(s) + q\cos\theta B'(s) = 0. \end{cases} \tag{41}$$

The first equation yields $A'(s) = qB(s) + c_0$, for a certain $c_0 \in \mathbb{R}$. Replacing into the second equation of (41) we get

$$B''(s) + \mu B(s) + qc_0 = 0, \tag{42}$$

where we denoted $\mu := q^2 + c\sin^2\theta$ as previously.

We need to distinguish some cases:

Case I. $\mu = 0$: The solution of (42) is given by a second order polynomial as in formula (28). Next, the other two solutions of (41) are also polynomials, of degree three, as given in (28).

Case II. $\mu > 0$: Let us denote $\mu = k^2$ ($k > 0$). The solutions of (41) are given by (29).

Case III. $\mu < 0$: Let us denote $\mu = -k^2$ ($k > 0$). The solutions of (41) are given by (30), concluding the proof. □

6. Jacobi Magnetic Fields on $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

The motivation of this section has roots in the classification theorem given by Perrone in Theorem 4.1 of [25]. *Let $(M, \varphi, \xi, \eta, g)$ be a simply connected homogeneous almost cosymplectic 3-manifold. Then, either M is a Lie group G equipped with a left invariant almost cosymplectic structure, or a Riemannian product of type $N \times \mathbb{R}$, where N is a simply connected Kähler surface of constant curvature.*

Let $N = N^2(\bar c)$ be one of the two two-dimensional manifolds $\mathbb{S}^2$ or $\mathbb{H}^2$ where $\bar c = \pm 1$ is the (constant) sectional curvature. Let (M^3, g) be the Riemannian product manifold $M^3 = N \times \mathbb{R}$, with the metric $g = \bar g + dt^2$, where $\bar g$ is the metric on N and t is the global coordinate on $\mathbb{R}$. Since N has a natural Kähler structure, one can naturally define a cosymplectic structure (φ, ξ, η, g) on M as follows

$$\varphi X = JX, \quad \varphi\frac{\partial}{\partial t} = 0, \quad \xi = \frac{\partial}{\partial t}, \quad \eta = dt,$$

where J is the complex structure on N and X is tangent to N. See for more details [26].

Let $\gamma : I \to M$, $\gamma(s) = (\bar\gamma(s), t(s))$ be a curve on the product manifold $M = N \times \mathbb{R}$ parametrized by the arc length, i.e., $\bar g(\bar\gamma'(s), \bar\gamma'(s)) + t'(s)^2 = 1$. We exclude the case when γ is the characteristic flow, i.e., when $\dot\gamma(s) = \xi(\gamma(s))$. This ensures that $\bar\gamma$ does not reduce to a point (on N). Suppose that γ is a (normal) contact magnetic curve on the cosymplectic manifold M, that is γ satisfies the Lorentz equation $\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma$, where ∇ is the Levi-Civita connection of g and q is the strength. We can write this equation as

$$\begin{cases} \overline{\nabla}_{\bar\gamma'}\bar\gamma' = qJ\bar\gamma', \\ t''(s) = 0, \end{cases} \tag{43}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\bar g$. This shows that $\bar\gamma$ is a (Kähler) magnetic curve on N, not necessary normal.

Due to the homogeneity of M we can consider $t(0) = 0$. Hence $t(s) = s\cos\theta$, where θ is the constant angle between $\dot\gamma(s)$ and $\xi(\gamma(s))$, called the contact angle of γ. See e.g., [11].

A vector field $W(s)$ along γ may be expressed as $W(s) = (\overline{W}(s), a(s))$, where $\overline{W}$ is a vector field along $\bar\gamma$ on N. We prove the following result.

Theorem 4. *Let $W(s) = (\overline{W}(s), a(s))$ be a magnetic Jacobi field along the normal contact magnetic curve $\gamma(s) = (\bar\gamma(s), t(s))$ in the product manifold $M^3 = N \times \mathbb{R}$, where N denotes the 2-sphere $\mathbb{S}^2$ or the hyperbolic plane $\mathbb{H}^2$. Then $\overline{W}$ is a magnetic Jacobi field along $\bar\gamma$ on N and a is an affine function. The converse also holds.*

Proof. We need to analyze all terms in the magnetic Jacobi equation

$$\frac{D^2}{ds^2}W(s) - R(\dot\gamma(s), W(s))\dot\gamma(s) - q\varphi\frac{D}{ds}W(s) = 0.$$

We have

$$\frac{D^2}{ds^2}W(s) = \Big(\frac{D^2}{ds^2}\overline{W}(s), a''(s)\Big) \quad \text{and} \quad \varphi\Big(\frac{D}{ds}W(s)\Big) = \Big(J\frac{D}{ds}\overline{W}(s), 0\Big).$$

Since M is a product manifold, the curvature tensors R and $\overline{R}$ are respectively related by the following relations

$$R(X,Y)Z = \overline{R}(X,Y)Z,\ R(X,Y)\frac{\partial}{\partial t} = 0,\ R\left(X, \frac{\partial}{\partial t}\right)Z = 0,\ R\left(X, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial t} = 0,$$

where X, Y, Z are tangent to N. It follows that the magnetic Jacobi equation is equivalent to the following system

$$\begin{cases} \dfrac{D^2}{ds^2}\overline{W}(s) - \overline{R}(\bar\gamma'(s), \overline{W}(s))\bar\gamma'(s) - qJ\dfrac{D}{ds}\overline{W}(s) = 0, \\ a''(s) = 0. \end{cases} \tag{44}$$

The first equation means that $\overline{W}$ is a magnetic Jacobi field on N along $\bar\gamma$, while the second equation says that a is an affine function. □

In the following we will sketch how $\overline{W}$ can be obtained. We mention here two (from a series of some) papers of Adachi [1,27].

If $\bar\gamma = 0$ the curve γ is an integral curve of ξ and hence the contact angle is 0 or π. In such a case

$$\frac{D}{ds}W(s) = (0, a'(s)),$$

where $W(s) = (\overline{W}(s), a(s))$ as before. This means that one has no condition for $\overline{W}(s)$.

Suppose that $\bar\gamma(s) \neq 0$. Then we can decompose $\overline{W}(s)$ in the basis $\{\bar\gamma'(s), J\bar\gamma'(s)\}$

$$\overline{W}(s) = \alpha(s)\bar\gamma'(s) + \beta(s)J\bar\gamma'(s),$$

where α and β are smooth functions of s. We can compute successively

$$\frac{D}{ds}\overline{W}(s) = \big(\alpha'(s) - q\beta(s)\big)\bar{\gamma}'(s) + \big(\beta'(s) + q\alpha(s)\big)J\bar{\gamma}'(s),$$
$$\frac{D^2}{ds^2}\overline{W}(s) = \big(\alpha''(s) - 2q\beta'(s) - q^2\alpha(s)\big)\bar{\gamma}'(s) + \big(\beta''(s) + 2q\alpha'(s) - q^2\beta(s)\big)J\bar{\gamma}'(s),$$
$$\overline{R}(\bar{\gamma}(s), \overline{W}(s))\bar{\gamma}'(s) = -\eta\bar{c}\sin^2\theta\, J\bar{\gamma}'(s).$$

The first equation in (44) becomes

$$\big(\alpha''(s) - q\beta'(s)\big)\bar{\gamma}'(s) + \big(\beta''(s) + q\alpha'(s) + \beta\bar{c}\sin^2\theta\big)J\bar{\gamma}'(s) = 0,$$

which implies

$$\begin{aligned} \alpha''(s) &= q\beta'(s) \\ \beta''(s) + q\alpha'(s) + \beta\bar{c}\sin^2\theta &= 0. \end{aligned} \tag{45}$$

Remark that the first equation in (45) is a consequence of the fact that $\bar{g}(\frac{D}{ds}\overline{W}(s), \bar{\gamma}'(s))$ is a constant. A first integration in this equation leads to

$$\alpha'(s) = q\beta(s) + \lambda_0, \tag{46}$$

with $\lambda_0 \in \mathbb{R}$. Replacing in the second equation of (45) we obtain the following second order differential equation in β

$$\beta''(s) + (q^2 + \bar{c}\sin^2\theta)\beta(s) + q\lambda_0 = 0. \tag{47}$$

The Equation (47) can be solved taking into consideration the sign of $q^2 + \bar{c}\sin^2\theta$. See e.g., Equation (42). Then, with β obtained from (47) we easily obtain the function α from (46).

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