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Bounds for Two New Subclasses of Bi-Univalent Functions Associated with Legendre Polynomials

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Abstract: In this article, two new subclasses of the bi-univalent function class σ related with Legendre polynomials are presented. Additionally, the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions belonging to these new subclasses are estimated.

Keywords: Legendre polynomials; coefficient estimations; starlike and convex functions; bi-univalent functions; subordination

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1. Introduction

In 1782, Adrien-Marie Legendre discovered Legendre polynomials, which have numerous physical applications. The Legendre polynomials $P_n(x)$, sometimes called Legendre functions of the first kind, are the particular solutions to the Legendre differential equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0,\ n \in \mathbb{N}_0, |x| < 1.$$

Here and in the following, let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The Legendre polynomials are defined by Rodrigues' formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \quad (n \in \mathbb{N}_0),$$

for arbitrary real or complex values of the variable x. The Legendre polynomials $P_n(x)$ are generated by the following function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1}$$

where the particular branch of $\left(1-2xt+t^2\right)^{-\frac{1}{2}}$ is taken to be 1 as $t \to 0$. The first few Legendre polynomials are

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}\left(3x^2-1\right),\ P_3(x) = \frac{1}{2}\left(5x^3-3x\right).$$

A general case of the Legendre polynomials and their applications can be found in [1,2]. Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the following Taylor–Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n\ , \tag{2}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in U. An important member of the class $\mathcal{S}$ is the Koebe function

$$K(z) = z(1-z)^{-2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n ,$$

for every $z \in U$. This function maps U in a one-to-one manner onto the domain D that consists of the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -\frac{1}{4}$. The function

$$\phi(z) = \frac{1-z}{\sqrt{1-2z\cos\alpha + z^2}},$$

is in $\mathcal{P}$ for every real α (see [[3] Page 102]), where $\mathcal{P}$ is the Caratheodory class defined by

$$\mathcal{P} = \{p(z) : \Re(p(z)) > 0, z \in U\},$$

$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$. By using (1), it is easy to check that

$$\begin{aligned} \phi(z) &= 1 + \sum_{n=1}^{\infty} [P_n(\cos\alpha) - P_{n-1}(\cos\alpha)]z^n, \\ &= 1 + \sum_{n=1}^{\infty} B_n z^n, z \in U. \end{aligned} \tag{3}$$

If we consider

$$\begin{aligned} \frac{1}{(\phi(z))^2} &= \frac{1-2z\cos\alpha + z^2}{(1-z)^2}, \\ &= 1 + 2(1-\cos\alpha)\frac{z}{(1-z)^2}. \end{aligned}$$

From the geometric properties of the Koebe function, the function ϕ maps the unit disc onto the right plane $\Re(w) > 0$ minus the slit along the positive real axis from $\frac{1}{|\cos\frac{\alpha}{2}|}$ to ∞. $\phi(U)$ is univalent, symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$. It is well known, by using the Koebe one-quarter theorem [4], that every univalent function $f \in \mathcal{S}$ has an inverse function f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in U),$$

and

$$f(f^{-1}(w)) = w \quad (w \in U^* = \{w \in \mathbb{C} : |w| < \frac{1}{4}\}),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots .$$

The function $f \in \mathcal{S}$ is said to be a bi-univalent function if its inverse f^{-1} is also univalent in U. Let σ be the class of all bi-univalent functions in U. Lewin [5] is the first author who introduced the class of analytic bi-univalent functions and estimated the second coefficient $|a_2|$. Many authors created several subclasses of analytic bi-univalent functions and found the bounds for the first two coefficients $|a_2|$ and $|a_3|$, see for example [6–23]. Let Ω be the class of all analytic functions ω in U which satisfy these conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$. A function f is said to be subordinate to g, written as $f(z) \prec g(z)$ if there exists a Schwarz function $\omega \in \Omega$ such that $f(z) = g(\omega(z))$. Furthermore, if the function g is univalent in U, then f is subordinate to g is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1. *A function $f \in \sigma$ belongs to the class $L_\sigma(\lambda, \phi)$ with $0 \le \lambda \le 1$ if the following subordination conditions are satisfied*

$$\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\left(\frac{zf'(z)}{f(z)}\right) \prec \phi(z) \quad (z \in U),$$

and

$$\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\lambda)\left(\frac{wg'(w)}{g(w)}\right) \prec \phi(w) \quad (w \in U),$$

where $g(w) = f^{-1}(w)$.

Definition 2. *A function $f \in \sigma$ belongs to the class $L_\sigma(\gamma, \rho, \phi)$ with $0 \le \gamma, \rho \le 1$ if the following subordination conditions are satisfied*

$$(1-\gamma+2\rho)\frac{f(z)}{z}+(\gamma-2\rho)f'(z)+\rho z f''(z) \prec \phi(z) \quad (z \in U),$$

and

$$(1-\gamma+2\rho)\frac{g(w)}{w}+(\gamma-2\rho)g'(w)+\rho w g''(w) \prec \phi(w) \quad (w \in U),$$

where $g(w) = f^{-1}(w)$.

Remark 1. *In Definition 1, if $\lambda = 1$ and $\alpha = \pi$, then the subclass in [15] will be obtained. If $\lambda = 0$ and $\alpha = \pi$, then the subclass in [24] will be obtained. In addition, putting $\alpha = \pi$, this yields to the subclass in [25].*

Remark 2. *In Definition 2, taking $\rho = 0$ and $\alpha = \pi$, the subclass in [26] will be obtained. In addition, putting $\gamma = 1, \rho = 0$ and $\alpha = \pi$, this yields to the subclass in [27].*

In this paper, the estimates for initial coefficients of functions in the two classes $L_\sigma(\lambda, \phi)$ and $L_\sigma(\gamma, \rho, \phi)$ are found.

2. The Estimate of the Coefficients for the Classes $L_\sigma(\lambda, \phi)$ and $L_\sigma(\gamma, \rho, \phi)$

Lemma 1 ([4]). *Let $\omega \in \Omega$ with $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$ $(z \in U)$. Then*

$$|\omega_1| \le 1, \quad |\omega_n| \le 1-|\omega_1|^2 \quad (n \in \mathbb{N}\backslash\{1\}).$$

Theorem 1. *Let the function $f \in L_\sigma(\lambda, \phi)$. Then*

$$|a_2| \le \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{(1+\lambda)[3\lambda(\cos\alpha+1)+5+\cos\alpha]}},$$

and

$$|a_3| \le \begin{cases} \frac{1-\cos\alpha}{2(1+2\lambda)} & if\ \cos\alpha \ge 1-\frac{(1+\lambda)^2}{2(1+2\lambda)} \\ \frac{(1-\cos\alpha)[2(1+2\lambda)(1-\cos\alpha)-(1+\lambda)^2]}{(1+2\lambda)(1+\lambda)[3\lambda(\cos\alpha+1)+5+\cos\alpha]}+\frac{1-\cos\alpha}{2(1+2\lambda)} & if\ \cos\alpha < 1-\frac{(1+\lambda)^2}{2(1+2\lambda)} \end{cases}$$

Proof. Since $f \in L_\sigma(\lambda, \phi)$, from Definition 1, we have

$$\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\lambda)\left(\frac{zf'(z)}{f(z)}\right) = \phi(u(z)) \quad (z \in U), \tag{4}$$

and

$$\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)+(1-\lambda)\left(\frac{wg'(w)}{g(w)}\right)=\phi(v(w)) \quad (w\in U), \tag{5}$$

for some $0\leq\lambda\leq 1$, where $g(w)=f^{-1}(w)$ and $u,v\in\Omega$ such that

$$u(z)=\sum_{n=1}^{\infty}b_nz^n,$$

and

$$v(w)=\sum_{n=1}^{\infty}c_nw^n.$$

Then

$$\phi(u(z))=1+B_1b_1z+\left(B_1b_2+B_2b_1^2\right)z^2+\left(B_1b_3+2b_1b_2B_2+B_3b_1^3\right)z^3+..., \tag{6}$$

and

$$\phi(v(w))=1+B_1c_1w+\left(B_1c_2+B_2c_1^2\right)w^2+\left(B_1c_3+2c_1c_2B_2+B_3c_1^3\right)w^3+..., \tag{7}$$

where

$$B_1=\cos\alpha-1,\ B_2=\frac{1}{2}(\cos\alpha-1)(1+3\cos\alpha) \text{ and } B_3=\frac{1}{2}\left(5\cos^3\alpha-3\cos^2\alpha-3\cos\alpha+1\right). \tag{8}$$

Then, Equations (4) and (5) become

$$\begin{aligned}&\lambda\left[1+2a_2z+\left(6a_3-4a_2^2\right)z^2+...\right]+(1-\lambda)\left[1+a_2z+\left(2a_3-a_2^2\right)z^2+...\right]\\&=1+B_1b_1z+\left(B_1b_2+B_2b_1^2\right)z^2+\left(B_1b_3+2b_1b_2B_2+B_3b_1^3\right)z^3+...,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\lambda\left[1-2a_2w+\left(8a_2^2-6a_3\right)w^2+...\right]+(1-\lambda)\left[1-a_2w+\left(3a_2^2-2a_3\right)w^2+...\right]\\&=1+B_1c_1w+\left(B_1c_2+B_2c_1^2\right)w^2+\left(B_1c_3+2c_1c_2B_2+B_3c_1^3\right)w^3+....\end{aligned} \tag{10}$$

Now, equating the corresponding coefficients in (9) and (10), we get

$$(1+\lambda)a_2=B_1b_1, \tag{11}$$

$$2(1+2\lambda)a_3-(1+3\lambda)a_2^2=B_1b_2+B_2b_1^2, \tag{12}$$

$$-(1+\lambda)a_2=B_1c_1, \tag{13}$$

$$(3+5\lambda)a_2^2-2(1+2\lambda)a_3=B_1c_2+B_2c_1^2. \tag{14}$$

(11) and (13) yield

$$b_1=-c_1, \tag{15}$$

and

$$b_1^2+c_1^2=\frac{2(1+\lambda)^2}{B_1^2}a_2^2. \tag{16}$$

From (12), (14) and (16), we have

$$a_2^2=\frac{B_1^3}{2(1+\lambda)\left[B_1^2-(1+\lambda)B_2\right]}(b_2+c_2).$$

By using Lemma 1, (11) and (15), we obtain

$$\begin{aligned}|a_2|^2 &\leq \frac{|B_1|^3\left(1-|b_1|^2\right)}{(1+\lambda)\left|B_1^2-(1+\lambda)B_2\right|},\\ &= \frac{|B_1|^3}{(1+\lambda)\left[\left|B_1^2-(1+\lambda)B_2\right|+(1+\lambda)|B_1|\right]}.\end{aligned}$$

Therefore,

$$|a_2| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{(1+\lambda)\left[\left|B_1^2-(1+\lambda)B_2\right|+(1+\lambda)|B_1|\right]}}. \tag{17}$$

By noting that,

$$\begin{aligned}B_1^2-(1+\lambda)B_2 &= \frac{1}{2}(1-\cos\alpha)\{(1+3\lambda)\cos\alpha+3+\lambda\},\\ &\geq (1-\cos\alpha)(1-\lambda)\geq 0,\end{aligned}$$

now substituting the values of B_1 and B_2 from (8) in (17), we obtain

$$|a_2| \leq \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{(1+\lambda)[3\lambda(\cos\alpha+1)+5+\cos\alpha]}},$$

which is the required estimation for $|a_2|$.

Next, in order to estimate $|a_3|$, subtracting (14) from (12), we obtain

$$a_3 = a_2^2 + \frac{B_1}{4(1+2\lambda)}(b_2-c_2).$$

By using Lemma 1 and (11),we find

$$|a_3| \leq \left[1-\frac{(1+\lambda)^2}{2(1+2\lambda)|B_1|}\right]|a_2|^2 + \frac{|B_1|}{2(1+2\lambda)}.$$

Case 1. If $1-\frac{(1+\lambda)^2}{2(1+2\lambda)|B_1|}\leq 0$, then

$$|a_3| \leq \frac{|B_1|}{2(1+2\lambda)}.$$

Case 2. If $1-\frac{(1+\lambda)^2}{2(1+2\lambda)|B_1|}> 0$, then

$$|a_3| \leq \left[1-\frac{(1+\lambda)^2}{2(1+2\lambda)|B_1|}\right]|a_2|^2 + \frac{|B_1|}{2(1+2\lambda)}.$$

Therefore,

$$|a_3| \leq \begin{cases} \frac{1-\cos\alpha}{2(1+2\lambda)} & if\ \cos\alpha \geq 1-\frac{(1+\lambda)^2}{2(1+2\lambda)} \\ \frac{(1-\cos\alpha)[2(1+2\lambda)(1-\cos\alpha)-(1+\lambda)^2]}{(1+2\lambda)(1+\lambda)[3\lambda(\cos\alpha+1)+5+\cos\alpha]} + \frac{1-\cos\alpha}{2(1+2\lambda)} & if\ \cos\alpha < 1-\frac{(1+\lambda)^2}{2(1+2\lambda)} \end{cases}$$

which completes the proof. $\square$

Theorem 2. *Let the function $f \in L_\sigma(\gamma,\rho,\phi)$. Then*

$$|a_2| \leq \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{2(1+2\gamma+2\rho)(1-\cos\alpha)+3(1+\gamma)^2(1+\cos\alpha)}},$$

and

$$|a_3| \leq \begin{cases} \frac{1-\cos\alpha}{(1+2\gamma+2\rho)} & if\ \cos\alpha \geq 1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)} \\ \frac{2[(1+2\gamma+2\rho)(1-\cos\alpha)-(1+\gamma)^2](1-\cos\alpha)}{(1+2\gamma+2\rho)[2(1+2\gamma+2\rho)(1-\cos\alpha)+3(1+\gamma)^2(1+\cos\alpha)]} + \frac{1-\cos\alpha}{1+2\gamma+2\rho} & if\ \cos\alpha < 1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)} \end{cases}$$

Proof. Since $f \in L_\sigma(\gamma,\rho,\phi)$, from Definition 2, we have

$$(1-\gamma+2\rho)\frac{f(z)}{z} + (\gamma-2\rho)f'(z) + \rho z f''(z) = \phi(u(z)) \quad (z \in U), \tag{18}$$

and

$$(1-\gamma+2\rho)\frac{g(w)}{w} + (\gamma-2\rho)g'(w) + \rho w g''(w) = \phi(v(w)) \quad (w \in U), \tag{19}$$

for $0 \leq \gamma,\rho \leq 1$, where $g(w) = f^{-1}(w)$ and $u,v \in \Omega$ are defined as in Theorem 1. Then, rewriting (18) and (19) as

$$\begin{aligned} &(1-\gamma+2\rho)\left[1+a_2z+a_3z^2+...\right] + (\gamma-2\rho)\left[1+2a_2z+3a_3z^2+...\right] \\ &+\rho z[2a_2+6a_3z+...] = 1+B_1b_1z+\left(B_1b_2+B_2b_1^2\right)z^2 \\ &+\left(B_1b_3+2b_1b_2B_2+B_3b_1^3\right)z^3+..., \end{aligned} \tag{20}$$

and

$$\begin{aligned} &(1-\gamma+2\rho)\left[1-2a_2w+\left(2a_2^2-a_3\right)w^2+...\right] + (\gamma-2\rho)\left[1-2a_2w+\left(6a_2^2-3a_3\right)w^2+...\right] \\ &+\rho w\left[-2a_2+\left(12a_2^2-6a_3\right)w+...\right] = 1+B_1c_1w+\left(B_1c_2+B_2c_1^2\right)w^2 \\ &+\left(B_1c_3+2c_1c_2B_2+B_3c_1^3\right)w^3+..., \end{aligned} \tag{21}$$

where B_1, B_2 and B_3 are defined as in (8). Now, equating the coefficients in (20) and (21) yields

$$(1+\gamma)a_2 = B_1b_1, \tag{22}$$

$$(1+2\gamma+2\rho)a_3 = B_1b_2+B_2b_1^2, \tag{23}$$

$$-(1+\gamma)a_2 = B_1c_1, \tag{24}$$

$$(1+2\gamma+2\rho)\left(2a_2^2-a_3\right) = B_1c_2+B_2c_1^2. \tag{25}$$

From (22) and (24), it is easy to see that

$$b_1 = -c_1, \tag{26}$$

and

$$b_1^2+c_1^2 = \frac{2(1+\gamma)^2}{B_1^2}a_2^2. \tag{27}$$

From (23), (25) and (27), we have

$$a_2^2 = \frac{B_1^3}{2\left[(1+2\gamma+2\rho)B_1^2-(1+\gamma)^2B_2\right]}(b_2+c_2).$$

By using Lemma 1, (22) and (26), we obtain

$$|a_2|^2 \leq \frac{|B_1|^3\left(1-|b_1|^2\right)}{\left|(1+2\gamma+2\rho)B_1^2-(1+\gamma)^2B_2\right|},$$
$$= \frac{|B_1|^3}{\left|(1+2\gamma+2\rho)B_1^2-(1+\gamma)^2B_2\right|+(1+\gamma)^2|B_1|}.$$

Therefore,

$$|a_2| \leq \frac{|B_1|\sqrt{|B_1|}}{\sqrt{\left|(1+2\gamma+2\rho)B_1^2-(1+\gamma)^2B_2\right|+(1+\gamma)^2|B_1|}}. \tag{28}$$

By noting that,

$$\begin{aligned}&(1+2\gamma+2\rho)B_1^2-(1+\gamma)^2B_2\\ &=\frac{1}{2}(1-\cos\alpha)\{2(1+2\gamma+2\rho)(1-\cos\alpha)+(1+\gamma)^2(1+3\cos\alpha)\},\\ &=\frac{1}{2}(1-\cos\alpha)\{2(1+\gamma)^2(1+\cos\alpha)+[1+\gamma(2-\gamma)+4\rho](1-\cos\alpha)\}>0,\end{aligned}$$

now substituting the values of B_1 and B_2 from (8) in (28), we obtain

$$|a_2| \leq \frac{\sqrt{2}(1-\cos\alpha)}{\sqrt{2(1+2\gamma+2\rho)(1-\cos\alpha)+3(1+\gamma)^2(1+\cos\alpha)}},$$

which is the desired estimation for $|a_2|$.

Next, in order to estimate $|a_3|$, subtracting (25) from (23) ,we obtain

$$a_3 = a_2^2 + \frac{B_1}{2(1+2\gamma+2\rho)}(b_2-c_2).$$

By using Lemma 1 and (22), we find

$$|a_3| \leq \left[1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)|B_1|}\right]|a_2|^2+\frac{|B_1|}{(1+2\gamma+2\rho)}.$$

Case 1. If $1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)|B_1|} \leq 0$, then

$$|a_3| \leq \frac{|B_1|}{(1+2\gamma+2\rho)}.$$

Case 2. If $1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)|B_1|} > 0$, then

$$|a_3| \leq \left[1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)|B_1|}\right]|a_2|^2+\frac{|B_1|}{(1+2\gamma+2\rho)}.$$

Therefore,

$$|a_3| \leq \begin{cases} \frac{1-\cos\alpha}{(1+2\gamma+2\rho)} & if\ \cos\alpha \geq 1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)} \\ \frac{2[(1+2\gamma+2\rho)(1-\cos\alpha)-(1+\gamma)^2](1-\cos\alpha)}{(1+2\gamma+2\rho)[2(1+2\gamma+2\rho)(1-\cos\alpha)+3(1+\gamma)^2(1+\cos\alpha)]}+\frac{1-\cos\alpha}{1+2\gamma+2\rho} & if\ \cos\alpha < 1-\frac{(1+\gamma)^2}{(1+2\gamma+2\rho)} \end{cases}$$

which completes the proof. □

3. Conclusions

In this paper, we have used the Legendre polynomials to define and study two new subclasses of the bi-univalent function class σ. Moreover, we have provided the estimations for the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions belonging to these new subclasses. Some special cases have been discussed as applications of our main results.

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