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Sehgal–Guseman-Type Fixed Point Theorem in *b*-Rectangular Metric Spaces

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Abstract: In this paper, we prove a Sehgal–Guseman-type fixed point theorem in b-rectangular metric spaces which provides a complete solution to an open problem raised by Zoran D. Mitrović (A note on a Banach's fixed point theorem in *b*-rectangular metric space and *b*-metric space). The result presented in the paper generalizes and unifies some results in fixed point theory.

Keywords: fixed point; *b*-metric space; rectangular metric space; *b*-rectangular metric space

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1. Introduction and Preliminaries

Fixed point theory is one of the most important and useful tools in nonlinear functional analysis and applied mathematics. Since the publication of the Banach contraction principle, many scholars have generalized and extended it. One of the generalizations is given by Sehgal. In [1], Sehgal initiated the study of fixed points for mappings with contractive iterates at a point. The main result of [1] is the following theorem.

Theorem 1. *Let (X, d) be a metric space and let $T : X \to X$ be a continuous mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that for each x there exists a positive integer $l(x)$ such that, for each $y \in X$,*

$$d(T^{l(x)}x, T^{l(x)}y) \leq kd(x, y).$$

Then, T has a unique fixed point.

Later, Guseman [2], Matkowsk [3] and others [4] discussed it in depth.

On the other hand, many authors discussed the Banach contraction principle in different generalized metric spaces. For example, Branciari [5] introduced the concept of rectangular metric spaces and proved an analogue of the Banach contraction principle in the setting of such a space. In [6], Bakhtin introduced the concept of *b*-metric spaces and also proved an analogue of the Banach contraction principle in the setting of such a space. In 2015, George et al. [7] introduced the concept of a *b*-rectangular metric space as a generalization of both rectangular metric space and *b*-metric space. Additionally, an analogue of the Banach contraction principle and Kannan's fixed point theorem have been proven in such a space. In the end of [7], the authors raised several open questions, one of which is whether analogues of the Chatterjee contraction, Reich contraction, Ciric contraction and Hardy–Rogers contraction theorems can be proven in *b*-rectangular metric spaces. Some other fixed point theorems in *b*-rectangular metric spaces can be seen [8,9]. In 2018, Mitrović [10] relaxed the contraction coefficient in the Banach contraction principle from $k \in (0, \frac{1}{s})$ to $k \in (0, 1)$ in a *b*-rectangular metric space. Furthermore, in the end of [10], the author raised an open question, which was to prove or disprove the following (Sehgal–Guseman theorem) in a *b*-rectangular metric space:

Let (X, d) be a complete b-rectangular metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping satisfying the condition: for each x there exists a positive integer $l(x)$ such that

$$d(T^{l(x)}x, T^{l(x)}y) \leq kd(x, y),$$

for all $y \in X$, where $k \in (0, 1)$. Then, T has a unique fixed point.

In this paper, we prove the Sehgal–Guseman-type theorem in b-rectangular metric spaces, which answers an open question raised by Mitrović. The result presented in the paper generalizes and unifies some results in fixed point theory.

Let us recall some definitions that will be used in the paper.

Definition 1 ([6,11])**.** *Let X be a nonempty set, $s \geq 1$ be a given real number and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping, such that for all $x, y, z \in X$, the following conditions hold:*

(b1) $d(x, y) = 0$ *if and only if* $x = y$*;*
(b2) $d(x, y) = d(y, x)$*;*
(b3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ *(b-triangular inequality).*

Then the pair (X, d) is called a b-metric space (metric type space).

For all definitions of such notions as b-convergence, b-completeness, and b-Cauchy in the frame of b-metric spaces, see [6,11].

In the last twenty years, many authors have discussed the fixed point theory on b-metric spaces. For instance, in [12], the author gave a survey of the recent fixed point results on b-metric spaces.

Definition 2 ([5])**.** *Let X be a nonempty set, and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x and y:*

(r1) $d(x, y) = 0$ *if and only if* $x = y$*;*
(r2) $d(x, y) = d(y, x)$*;*
(r3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *(rectangular inequality).*

Then, (X, d) is called a rectangular metric space or generalized metric space.

For all definitions of notions in the frame of rectangular metric spaces, see [5].

Definition 3 ([7])**.** *Let X be a nonempty set, $s \geq 1$ be a given real number and let $d : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y \in X$ and distinct points $u, v \in X$, each distinct from x and y:*

(rb1) $d(x, y) = 0$ *if and only if* $x = y$*;*
(rb2) $d(x, y) = d(y, x)$*;*
(rb3) $d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)]$ *(b-rectangular inequality).*

Then, (X, d) is called a b-rectangular metric space or b-generalized metric space.

Several authors have defined different contractive mappings for b-rectangular metric spaces and discussed the fixed point theory on them (for details, see [9,13,14] and the references therein).

From the above definitions, we know that every metric space is a rectangular metric space and a b-metric space. Additionally, every rectangular metric space or every b-metric space is a b-rectangular metric space. However the converse is not necessarily true [15]. To illustrate this, we give the following example, which is a modification of the example of [15].

Example 1. *Let $A = \{0, 2\}, B = \{\frac{1}{n} : n \in N\}$ and $X = A \bigcup B$. Define $d : X \times X \longrightarrow [0, +\infty)$ as follows:*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B \\ y^2, & \text{if } x \in A, y \in B \\ x^2, & \text{if } x \in B, y \in A. \end{cases}$$

Then, (X, d) is a complete b-rectangular metric space with coefficient $s = 3$, but which is neither a b-metric space nor a rectangular metric space. Meanwhile, it is easy to see that [15]:

(i) The sequence $\{\frac{1}{n}\}_{n \in N}$converges to both 0 and 2, and it is not a Cauchy sequence;
(ii) There is no $r > 0$ such that $B_r(0) \bigcap B_r(2) = \emptyset$. Hence, the corresponding topology is not Hausdorff;
(iii) $B_{1/3}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}$, however, there does not exist $r > 0$ such that $B_r(0) \subseteq B_{1/3}(\frac{1}{3})$;
(iv) $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\lim_{n\to\infty} d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})$. Hence, d is not a continuous function.

2. Main Results

Theorem 2. *Let (X, d) be a b-rectangular metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that for each x there exists a positive integer $l(x)$ such that, for each $y \in X$,*

$$d(T^{l(x)}x, T^{l(x)}y) \leq kd(x, y). \tag{1}$$

Then, T has a unique fixed point.

Proof. We first prove the theorem in case $0 \leq k < \frac{1}{s}$.

Let x_0 be an arbitrary point in X. Consider a sequence $\{x_n\}$ by $x_1 = T^{l(x_0)}x_0, \cdots\cdots$, $x_{n+1} = T^{l(x_n)}x_n$.

Step 1: If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in N$, then x_{n_0} is a fixed point of T.

Since $x_{n_0} = T^{l(x_{n_0})}x_{n_0}$, x_{n_0} is a fixed point of $T^{l(x_{n_0})}$.

To prove that x_{n_0} is a fixed point of T, we firstly show that x_{n_0} is the unique fixed point of $T^{l(x_{n_0})}$. Indeed, if $T^{l(x_{n_0})}v = v$ for some $v \neq x_{n_0}$, then

$d(x_{n_0}, v) = d(T^{l(x_{n_0})}x_{n_0}, T^{l(x_{n_0})}v) \leq kd(x_{n_0}, v)$,

which is a contradiction, since $0 \leq k < 1$.

Then, $Tx_{n_0} = TT^{l(x_{n_0})}x_{n_0} = T^{l(x_{n_0})}Tx_{n_0}$; that is, Tx_{n_0} is also a fixed point of $T^{l(x_{n_0})}$.

By the uniqueness of fixed point of $T^{l(x_{n_0})}$, we have $Tx_{n_0} = x_{n_0}$, which shows that x_{n_0} is a fixed point of T.

In what follows, we can suppose that $x_n \neq x_{n+1}$ for all $n \in N$.

Step 2: We show that $x_n \neq x_m$ for $n \neq m$.

Without loss of generality, suppose $m > n$.

If $x_n = x_m$ for $n \neq m$, then

$$\begin{aligned} d(x_m, x_{m+1}) &= d(T^{l(x_{m-1})}x_{m-1}, T^{l(x_m)}T^{l(x_{m-1})}x_{m-1}) \\ &= d(T^{l(x_{m-1})}x_{m-1}, T^{l(x_{m-1})}T^{l(x_m)}x_{m-1}) \\ &\leq kd(x_{m-1}, T^{l(x_m)}x_{m-1}) \\ &= kd(T^{l(x_{m-2})}x_{m-2}, T^{l(x_m)}T^{l(x_{m-2})}x_{m-2}) \\ &= kd(T^{l(x_{m-2})}x_{m-2}, T^{l(x_{m-2})}T^{l(x_m)}x_{m-2}) \\ &\leq k^2d(x_{m-2}, T^{l(x_m)}x_{m-2}) \\ &\leq \cdots\cdots \\ &\leq k^{m-n}d(x_n, T^{l(x_m)}x_n) \\ &= k^{m-n}d(x_m, T^{l(x_m)}x_m) \\ &= k^{m-n}d(x_m, x_{m+1}), \end{aligned}$$

which is a contradiction since $0 \leq k < 1$.

Step 3: For $x \in X$, $r(x) = \sup_n d(T^n x, x)$ is finite.

Let $x \in X$ and let

$$z(x) = \max\{d(T^k x, x) : k = 1, 2, \cdots, l(x), l(x)+1, \cdots, 2l(x)\}.$$

If n is a positive integer, then there exists an integer $t \geq 0$ such that $tl(x) < n \leq (t+1)l(x)$. We can assume that $T^n x, T^{l(x)}x, T^{2l(x)}x, x$ are different from each other. Otherwise, the conclusion is obvious.

$$\begin{array}{rcl}
d(T^n x, x) & \leq & s[d(T^n x, T^{l(x)}x) + d(T^{l(x)}x, T^{2l(x)}x) + d(T^{2l(x)}x, x)] \\
& \leq & s[kd(T^{n-l(x)}x, x) + kd(x, T^{l(x)}x) + d(T^{2l(x)}x, x)] \\
& \leq & skd(T^{n-l(x)}x, x) + skz(x) + sz(x) \\
& \leq & s^2k[d(T^{n-l(x)}x, T^{l(x)}x) + d(T^{l(x)}x, T^{2l(x)}x) + d(T^{2l(x)}x, x)] + skz(x) + sz(x) \\
& \leq & s^2k^2 d(T^{n-2l(x)}x, x) + s^2k^2 d(x, T^{l(x)}x) + s^2kd(T^{2l(x)}x, x) + skz(x) + sz(x) \\
& \leq & s^2k^2 d(T^{n-2l(x)}x, x) + s^2k^2 z(x) + s^2kz(x) + skz(x) + sz(x) \\
& \leq & \cdots\cdots \\
& \leq & s^t k^t d(T^{n-tl(x)}x, x) + (sz(x) + s^2kz(x) + s^3k^2z(x) + \cdots) \\
& + & (skz(x) + s^2k^2z(x) + s^3k^3z(x) + \cdots) \\
& \leq & s^t k^t z(x) + \frac{sz(x)}{1-sk} + \frac{skz(x)}{1-sk} \\
& \leq & z(x) + \frac{sz(x)}{1-sk} + \frac{skz(x)}{1-sk}
\end{array}$$

Hence, $r(x) = \sup_n d(T^n x, x)$ is finite.

Step 4: $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

$$\begin{array}{rcl}
d(x_n, x_{n+1}) & = & d(T^{l(x_{n-1})}x_{n-1}, T^{l(x_n)}T^{l(x_{n-1})}x_{n-1}) \\
& = & d(T^{l(x_{n-1})}x_{n-1}, T^{l(x_{n-1})}T^{l(x_n)}x_{n-1}) \\
& \leq & kd(x_{n-1}, T^{l(x_n)}x_{n-1}) \\
& \leq & \cdots\cdots \\
& \leq & k^n d(x_0, T^{l(x_n)}x_0) \\
& \leq & k^n r(x_0)
\end{array}$$

Then $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Step 5: $\{x_n\}$ is a Cauchy sequence in X.

For the sequence $\{x_n\}$, we consider $d(x_n, x_{n+p})$ in two cases. For the sake of convenience, we denote $r(x_0)$ by r_0.

If p is odd, say $2m+1$, then by step 2 and (rb3)

$$\begin{array}{rcl}
d(x_n, x_{n+2m+1}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \\
& \leq & sk^n r_0 + sk^{n+1}r_0 + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})] \\
& \leq & \cdots\cdots \\
& \leq & sk^n r_0 + sk^{n+1}r_0 + s^2k^{n+2}r_0 + s^2k^{n+3}r_0 + s^3k^{n+4}r_0 + s^3k^{n+5}r_0 + \cdots + s^m k^{n+2m} r_0 \\
& \leq & sk^n[1 + sk^2 + s^2k^4 + \cdots]r_0 + sk^{n+1}[1 + sk^2 + s^2k^4 + \cdots]r_0 \\
& \leq & \frac{1+k}{1-sk^2} sk^n r_0
\end{array}$$

If p is even, say $2m$, then by step 2 and (rb3)

$$\begin{array}{rcl}
d(x_n, x_{n+2m}) & \leq & s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})] \\
& \leq & sk^n r_0 + sk^{n+1}r_0 + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})] \\
& \leq & \cdots\cdots \\
& \leq & sk^n r_0 + sk^{n+1}r_0 + s^2k^{n+2}r_0 + s^2k^{n+3}r_0 + s^3k^{n+4}r_0 + s^3k^{n+5}r_0 \\
& + & \cdots + s^{m-1}k^{n+2m-4}r_0 + s^{m-1}k^{n+2m-3}r_0 + +s^{m-1}d(x_{n+2m-2}, x_{n+2m}) \\
& \leq & sk^n r_0 + sk^{n+1}r_0 + s^2k^{n+2}r_0 + s^2k^{n+3}r_0 + s^3k^{n+4}r_0 + s^3k^{n+5}r_0 \\
& + & \cdots + s^{m-1}k^{n+2m-4}r_0 + s^{m-1}k^{n+2m-3}r_0 + s^{m-1}k^{n+2m-2}d(x_0, T^{n+2m-1}T^{n+2m-2}x_0) \\
& \leq & sk^n[1 + sk^2 + s^2k^4 + \cdots]r_0 + sk^{n+1}[1 + sk^2 + s^2k^4 + \cdots]r_0 + s^{m-1}k^{n+2m-2}r_0 \\
& \leq & \frac{1+k}{1-sk^2} sk^n r_0 + s^{m-1}k^{n+2m-2}r_0 \\
& \leq & \frac{1+k}{1-sk^2} sk^n r_0 + (sk)^{2m}k^{n-2}r_0 \\
& \leq & \frac{1+k}{1-sk^2} sk^n r_0 + k^{n-2}r_0.
\end{array}$$

Then, it follows from the above argument that

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = 0 \text{ for all } p > 0.$$

Thus, $\{x_n\}$ is a Cauchy sequence in X.
Since X is complete, there exists a point $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Step 6: u is a fixed point of T.
By (2.1), $d(T^{l(u)}u, T^{l(u)}x_n) \le d(u, x_n)$, then $\lim_{n\to\infty} d(T^{l(u)}u, T^{l(u)}x_n) = 0$.
On the other hand,

$$\begin{aligned} d(T^{l(u)}x_n, x_n) &= d(T^{l(u)}T^{l(x_{n-1})}x_{n-1}, T^{l(x_{n-1})}x_{n-1}) \\ &= d(T^{l(x_{n-1})}x_{n-1}, T^{l(x_{n-1})}T^{l(u)}x_{n-1}) \\ &\le kd(x_{n-1}, T^{l(u)}x_{n-1}) \\ &\le \cdots\cdots \\ &\le k^n d(x_0, T^{l(u)}x_0). \end{aligned}$$

That is, $\lim_{n\to\infty} d(T^{l(u)}x_n, x_n) = 0$.
By (rb3), $d(T^{l(u)}u, x_{n+1}) \le s(d(T^{l(u)}u, T^{l(u)}x_n) + d(T^{l(u)}x_n, x_n) + d(x_n, x_{n+1}))$.
Then, $\lim_{n\to\infty} d(T^{l(u)}u, x_{n+1}) = 0$.
Therefore, by (rb3) we have
$d(u, T^{l(u)}u) \le s(d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, T^{l(u)}u))$.
Passing the limit to the inequality, we have
$d(u, T^{l(u)}u) = 0$.
This means that $T^{l(u)}u = u$; that is, u is a fixed point of $T^{l(u)}$.

Now, $d(u, Tu) = d(T^{l(u)}u, TT^{l(u)}u) = d(T^{l(u)}u, T^{l(u)}Tu) \le kd(u, Tu)$, then $d(u, Tu) = 0$, that is u is a fixed point of T.

Step 7: u is the unique fixed point of T.

To prove that u is the unique fixed point of T, we show that u is the unique fixed point of $T^{l(u)}$ firstly. Indeed, if $T^{l(u)}v = v$ for some $v \ne u$, then

$d(u, v) = d(T^{l(u)}u, T^{l(u)}v) \le kd(u, v)$, which is a contradiction since $0 \le k < 1$.

If w is another fixed point of T, then $w = Tw = T^2w = \cdots\cdots = T^{l(u)}w$, that is, w is a fixed point of $T^{l(u)}$ too. By the uniqueness of fixed point of $T^{l(u)}$, we have $u = w$.

Next, we prove that the theorem is still valid in case $\frac{1}{s} \le k < 1$.

Since $\frac{1}{s} \le k < 1$, there exists $n_0 \in N$ such that $k^{n_o} \in (0, \frac{1}{s})$.

For each $x \in X$, there exists a positive integer $l(x)$ such that, for each $y \in X$,

$$d(T^{l(x)}x, T^{l(x)}y) \le kd(x, y).$$

Write $x[1] = T^{l(x)}x, x[2] = T^{l(x[1])}x[1] = T^{l(x[1])+l(x)}x, \ldots,$
$x[n_0] = T^{l(x[n_0-1])}x[n_0 - 1] = T^{l(x[n_0-1])+l(x[n_0-2])+\cdots+l(x)}x.$
Denote $m(x) = l(x[n_0 - 1]) + l(x[n_0 - 2]) + \cdots + l(x)$, then

$$d(T^{m(x)}x, T^{m(x)}y) \le k^{n_0} d(x, y).$$

Since $k^{n_o} \in (0, \frac{1}{s})$, from the above proof, we know that T has a unique fixed point.
□

Remark 1. *From the proof of Theorem 2, we can see that we do not require T to be continuous. Using our proof method, we can know that the continuity condition of Theorem 1 can be removed, as shown in [2].*

To illustrate the validity of our theorem, we give the following example, which is just a modification of Example 1.

Example 2. *Let* $A = \{0, 2\}, B = \{\frac{1}{n} : n \in N\}, C = \{n : n \geq 3 \text{ and } n \in N\}$ *and* $X = A \bigcup B$. *Define* $d : X \times X \longrightarrow [0, +\infty)$ *as follows:*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B \\ y^2, & \text{if } x \in A, y \in B \\ x^2, & \text{if } x \in B, y \in A \\ \frac{1}{2}, & \text{if } x = 0, y \in C \text{ or } y = 0, x \in C \\ |x-y|, & \text{otherwise.} \end{cases}$$

Then (X, d) is a complete b-rectangular metric space with coefficient $s = 3$.
Let

$$f(x) = \begin{cases} 0, & \text{if } x \in A \cup B \text{ or } x = 3 \\ x - 1, & \text{if } x \in C - \{3\}. \end{cases}$$

If $x \in A \cup B$, put $l(x) = 1$; if $x = n \in C$, put $l(x) = n$. Then, for each x, there exists a positive integer $l(x)$ such that, for each $y \in X$,

$$d(T^{l(x)}x, T^{l(x)}y) \leq \frac{1}{2} d(x,y).$$

Therefore, all the hypotheses of Theorem 2 are satisfied; thus, T has a fixed point. In this example $x = 0$ is the unique fixed point.

Note that since every rectangular metric space or every b-metric space is a b-rectangular metric space, then we have the following results.

Corollary 1. *Let* (X, d) *be a rectangular metric space. Let* $T : X \to X$ *be a mapping which satisfies the condition that there exists a real number* $k, 0 \leq k < 1$ *such that for each* x *there exists a positive integer* $l(x)$ *such that, for each* $y \in X$,

$$d(T^{l(x)}x, T^{l(x)}y) \leq kd(x,y).$$

Then, T *has a unique fixed point.*

Corollary 2. *Let* (X, d) *be a b-metric space with coefficient* $s \geq 1$, *and let* $T : X \to X$ *be a mapping which satisfies the condition that there exists a real number* $k, 0 \leq k < 1$ *such that for each* x *there exists a positive integer* $l(x)$ *such that, for each* $y \in X$,

$$d(T^{l(x)}x, T^{l(x)}y) \leq kd(x,y).$$

Then, T *has a unique fixed point.*

If, for each $x \in X$, there exists a fixed positive integer l such that $l(x) = l$, then we have the following results.

Corollary 3. *Let* (X, d) *be a rectangular b-metric space with coefficient* $s \geq 1$, *and let* $T : X \to X$ *be a mapping which satisfies the condition that there exists a real number* $k, 0 \leq k < 1$ *and a fixed positive integer* l *such that, for each* $x, y \in X$,

$$d(T^l x, T^l y) \leq kd(x,y).$$

Then, T *has a unique fixed point.*

Corollary 4. *Let (X, d) be a rectangular metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ and a fixed positive integer l such that, for each $x, y \in X$,*

$$d(T^l x, T^l y) \leq kd(x, y).$$

Then, T has a unique fixed point.

Corollary 5. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ and a fixed positive integer l such that, for each $x, y \in X$,*

$$d(T^l x, T^l y) \leq kd(x, y).$$

Then, T has a unique fixed point.

Corollary 6. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that, for each $x, y \in X$,*

$$d(T^l x, T^l y) \leq kd(x, y).$$

Then, T has a unique fixed point.

Specifically, if $l = 1$, then we have

Corollary 7. *Let (X, d) be a rectangular b-metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq kd(x, y).$$

Then, T has a unique fixed point.

Corollary 8. *Let (X, d) be a rectangular metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq kd(x, y).$$

Then, T has a unique fixed point.

Corollary 9. *Let (X, d) be a b-metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq kd(x, y).$$

Then, T has a unique fixed point.

Corollary 10. *Let (X, d) be a metric space and let $T : X \to X$ be a mapping which satisfies the condition that there exists a real number $k, 0 \leq k < 1$ such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq kd(x, y).$$

Then, T has a unique fixed point.

3. Conclusions

A Sehgal–Guseman-type fixed point theorem in b-rectangular metric spaces was proven, which answered an open question raised by Mitrović. The result presented in the present paper generalized and unified some results in fixed point theory.

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