Article

Steffensen-Type Inequalities with Weighted Function via (γ, a) -Nabla-Conformable Integral on Time Scales

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Abstract: The primary goal of this our research is to prove several new ∇ -conformable dynamic Steffensen inequalities that were demonstrated in recent works. Our results generalize and extend existing results in the literature. Many special cases of the proposed results are obtained and analyzed such as new conformable fractional sum inequalities and new classical conformable fractional integral inequalities.

Keywords: Steffensen's inequality; dynamic inequality; ∇ -conformable integral; time scales

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1. Introduction

Stefan Hilger initiated the theory of time scales in his PhD thesis [1] in order to unify discrete and continuous analysis (see [2]). Since then, this theory has received a lot of attention. The basic notion is to establish a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary closed subset of the reals $\mathbb{R}$. Recently, many new dynamic inequalities on time scales have been introduced by different authors who have been inspired with the aid of a few applications (see [3–6]). Some authors have obtained different results regarding fractional calculus on time scales to provide associated dynamic inequalities (see [7–10]).

In [11], the authors proved the following useful theorem:

Theorem 1. Suppose a positive integrable function $q : [m, n] \rightarrow \mathbb{R}$, and let $r, g : [m, n] \rightarrow \mathbb{R}$ be an integrable functions such that $0 \leq g(t) \leq 1$ for all $t \in [m, n]$ and $k \in (m, n)$. Moreover, let λ_1 and λ_2 be the solutions of the equations

$$\int_m^{m+\lambda_1} q(t)dt = \int_m^k q(t)g(t)dt,$$

and

$$\int_{n-\lambda_2}^n q(t)dt = \int_k^n q(t)g(t)dt.$$

If $r(t)/q(t) - At$ is nonincreasing on $[m, k]$ and nondecreasing on $[k, n]$, with a constant A and

$$\int_m^n tq(t)g(t)dt = \int_m^{m+\lambda_1} tq(t)dt + \int_{n-\lambda_2}^n tq(t)dt,$$

then

$$\int_m^n r(t)g(t)dt \leq \int_m^{m+\lambda_1} r(t)dt + \int_{n-\lambda_2}^n r(t)dt.$$

Anderson [12] extended the Steffensen inequality to a general time scale by

$$\int_{n-\lambda}^n r(t) \nabla t \leq \int_m^n r(t) g(t) \nabla t \leq \int_m^{m+\lambda} r(t) \nabla t.$$

where $m, n \in \mathbb{T}_\kappa$ with $m < n$, $r, g : [m, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ are nabla integrable with one sign and nonincreasing function r and $0 \leq g(t) \leq 1$ on $[m, n]_{\mathbb{T}}$ and $\lambda = \int_m^n g(t) \nabla t$, $n - \lambda, m + \lambda \in \mathbb{T}$.

Fractional calculus, the theory of integrals and derivatives of noninteger order, has an important role in mathematical analysis and applications. This field of research, with a history dating back to Abel, Riemann and Liouville (see [13] for a historical summary), has indeed received extensive investigation, especially in recent years. The integral operator

$$I_{a+}^\alpha \eta(q) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - q)^{\alpha-1} \eta(q) dq,$$

is called the Riemann–Liouville fractional integral. The corresponding fractional derivative is obtained by the composition of the fractional integral by an integer-order derivative.

The definitions of fractional integrals and derivatives are not unique, and many definitions of fractional derivative operators have been presented and successfully applied to give a solution for the complex systems in engineering science (see [14–16]). Recently, the study of fractional dynamic equations has become very widespread around the world and is useful in applied and pure mathematics, engineering, physics, biology and economics. The integral in the Cauchy's integral formula is used with some changes. Thus, inequalities, in some situations, are not easy calculations to obtain. By using the definitions of Caputo and Riemann–Liouville fractional derivatives, some theorems are formulated such as the mean value and Rolle's theorem.

Recently, depending on the basic limit definition of the derivative only, Khalil et al. [17] proposed the conformable derivative $T_\alpha f(q)$ ($\alpha \in (0, 1]$) of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$

$$T_\alpha f(q) = \lim_{\epsilon \rightarrow 0} \frac{f(q + \epsilon q^{1-\alpha}) - f(q)}{\epsilon},$$

for all $t > 0$, $\alpha \in (0, 1]$; this definition found wide resonance in the scientific community interested in fractional calculus (see [18–20]). Iyiola and Nwaeze in [18] proposed an extended mean value theorem and Racetrack-type principle for a class of α -differentiable functions. Therefore, calculating the derivative by this definition is easy compared to the definitions that are based on integration. The researchers in [17] also suggested a definition for the α -conformable integral of a function η as follows:

$$\int_a^b \eta(q) d_\alpha q = \int_a^b \eta(q) q^{\alpha-1} dq.$$

After that, Abdeljawad [21] studied the extensive research of the newly introduced conformable calculus. In his work, he introduced a generalization of the conformable derivative $T_\alpha^a f(q)$ definition. For $t > a \in \mathbb{R}^+$ as $f : \mathbb{R}^+ \rightarrow \mathbb{R}$

$$T_\alpha^a f(q) = \lim_{\epsilon \rightarrow 0} \frac{f(q + \epsilon(q - a)^{1-\alpha}) - f(q)}{\epsilon}.$$

Benkhettou et al. [22] introduced a conformable calculus on an arbitrary time scale, which is a natural extension of the conformable calculus.

However, in the last few decades, many authors have pointed out that derivatives and integrals of non-integer order are very suitable for the description of properties of various real materials; e.g., polymers. Fractional derivatives provides an excellent instrument for the description of memory and hereditary properties of various materials and processes (see [23]). This is the main advantages of fractional derivatives in comparison with classical integer-order models.

Next, some basic definitions and concepts about the time scales calculus and the α -conformable calculus, which are used in the next section, can be found in [22,24–38]. We suppose the time scale $\mathbb{T}$ has the standard topology on the real numbers $\mathbb{R}$. We assume the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(q) := \inf\{s \in \mathbb{T} : s > q\}, \quad q \in \mathbb{T}, \quad (1)$$

and the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\rho(q) := \sup\{s \in \mathbb{T} : s < q\}, \quad q \in \mathbb{T}. \quad (2)$$

In [39], the authors studied a version of the nabla conformable fractional derivative on arbitrary time scales. Namely, for a function $\eta : \mathbb{T} \rightarrow \mathbb{R}$, the nabla conformable fractional derivative, $T_{\nabla, \alpha} \eta(q) \in \mathbb{R}$ of order $\alpha \in (0, 1]$ at $q \in \mathbb{T}_\kappa$ and $t > 0$ was defined as follows: given any $\epsilon > 0$, there is a δ -neighborhood $U_t \subset \mathbb{T}$ of q , $\delta > 0$, such that

$$|[\eta(\rho(q)) - \eta(s)]q^{1-\alpha} - T_{\nabla, \alpha}(\eta)(q)[\rho(q) - s]| \leq \epsilon |\rho(q) - s|$$

for all $s \in U_t$. The nabla conformable fractional integral is defined by

$$\int \eta(q) \nabla_\alpha q = \int \eta(q) q^{\alpha-1} \nabla t.$$

Rahmat et al. [40] presented a new type of conformable nabla derivative and integral which involve the time scale power function $\widehat{G}_n(q, s)$ for $s, q \in \mathbb{T}$.

Definition 1 (γ -nabla integral from a). Assume that $0 < \gamma \leq 1$, $a, t_1, t_2 \in \mathbb{T}$, $a \leq t_1 \leq t_2$ and $f \in C_{ld}(\mathbb{T})$; then, the function f is called (γ, a) -nabla integrable on $[t_1, t_2]$. if:

$$\begin{aligned} \nabla_a^{-\gamma} f(q) &= \int_{t_1}^{t_2} f(\Pi) \nabla_a^\gamma \Pi \\ &= \int_{t_1}^{t_2} f(\Pi) \widehat{G}_{\gamma-1}(\sigma^{\gamma-1}(\Pi), a) \nabla \Pi, \end{aligned} \quad (3)$$

exists and is finite.

We need the relations between different types calculus on time scales $\mathbb{T}$ and continuous calculus, discrete calculus and quantum calculus as follows. Note that, for the case $\mathbb{T} = \mathbb{R}$, we get

$$\int_a^q f(\Pi) \nabla_a^\gamma \Pi = \int_a^q f(\Pi) (\Pi - a)^{\gamma-1} d\Pi. \quad (4)$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we get

$$\int_a^q f(\Pi) \nabla_a^\gamma \Pi = \sum_{\Pi \in (a, t]} h f(\Pi) (\rho^{\gamma-1}(\Pi) - a)_h^{(\gamma-1)}. \quad (5)$$

For $\mathbb{T} = q^{\mathbb{N}_0}$, we have

$$\int_a^q f(\Pi) \nabla_a^\gamma \Pi = \sum_{\Pi \in (a, t]} \Pi (1 - \tilde{q}) f(\Pi) (\rho^{\gamma-1}(\Pi) - a)_{\tilde{q}}^{(\gamma-1)}. \quad (6)$$

Theorem 2. Let $\gamma \in (0, 1]$. If $a, t_1, t_2, t_3 \in \mathbb{T}$, $a \leq t_1 \leq t_2 \leq t_3$, $\alpha \in \mathbb{R}$ and $f, g \in C_{ld}(\mathbb{T})$, then

- (i) $\int_{t_1}^{t_2} [f(q) + g(q)] \nabla_a^\gamma q = \int_{t_1}^{t_2} f(q) \nabla_a^\gamma t + \int_{t_1}^{t_2} g(q) \nabla_a^\gamma t.$
- (ii) $\int_{t_1}^{t_2} \alpha f(q) \nabla_a^\gamma q = \alpha \int_{t_1}^{t_2} f(q) \nabla_a^\gamma t.$
- (iii) $\int_{t_1}^{t_2} f(q) \nabla_a^\gamma q = - \int_{t_2}^{t_1} f(q) \nabla_a^\gamma t.$

$$\begin{aligned}
 (iv) \quad & \int_{t_1}^{t_3} f(\varrho) \nabla_a^\gamma s = \int_{t_1}^{t_2} f(\varrho) \nabla_a^\gamma t + \int_{t_2}^{t_3} f(\varrho) \nabla_a^\gamma t. \\
 (v) \quad & \int_{t_1}^{t_1} f(\varrho) \nabla_a^\gamma \varrho = 0.
 \end{aligned}$$

Recently, after the emergence of the conformable derivative, many authors have used this new definition to introduce several conformable inequalities and have investigated different inequalities, such as the inequality of Hardy [41,42], the inequality of Hermite–Hadamard [43–45], the inequality of Opial [46,47] and the inequality of Steffensen [48]. For instance, Anderson [49] proved the alpha conformable version of the inequality of Steffensen as follows:

Theorem 3 ([49]). Let $\alpha \in (0, 1]$ be $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 \leq r_2$. Suppose $\Pi : [r_1, r_2] \rightarrow [0, \infty)$ and $\Gamma : [r_1, r_2] \rightarrow [0, 1]$ be α -fractional integrable functions on $[r_1, r_2]$ with Π is decreasing; thus, we have

$$\int_{r_2-\aleph}^{r_2} \Pi(\iota) d_{\alpha} \iota \leq \int_{r_1}^{r_2} \Pi(\iota) \Gamma(\iota) d_{\alpha} \iota \leq \int_{r_1}^{r_1+\aleph} \Pi(\iota) d_{\alpha} \iota,$$

where $\aleph = \frac{\alpha(r_2-r_1)}{r_2^\alpha-r_1^\alpha} \int_{r_1}^{r_2} \Gamma(\iota) d_{\alpha} \iota \in [0, r_2 - r_1]$.

In 2017, Sarikaya et al. [48] gave a generalization for 3 as follows:

Theorem 4. Suppose $\alpha \in (0, 1]$ and $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 \leq r_2$. Suppose $\Pi, \Gamma, \Sigma : [r_1, r_2] \rightarrow [0, \infty)$ are α -fractional integrable functions on $[r_1, r_2]$ with Π is decreasing and $0 \leq \Gamma \leq \Sigma$, we have

$$\int_{r_2-\aleph}^{r_2} \Sigma(\iota) \Pi(\iota) d_{\alpha} \iota \leq \int_{r_1}^{r_2} \Pi(\iota) \Gamma(\iota) d_{\alpha} \iota \leq \int_{r_1}^{r_1+\aleph} \Sigma(\iota) \Pi(\iota) d_{\alpha} \iota,$$

where $\aleph = \frac{(r_2-r_1)}{\int_{r_1}^{r_2} \Sigma(\iota) d_{\alpha} \iota} \int_{r_1}^{r_2} \Gamma(\iota) d_{\alpha} \iota \in [0, r_2 - r_1]$.

Next, some important theorems are stated which are helpful for the evidence presented in the next section.

Theorem 5. Suppose q is a positive integrable function, $q(t) \geq g(t) \geq 0$ for all $t \in [m, n]_{\mathbb{T}}$ and λ is the solution of the equation $\int_m^n g(t) \nabla_a^\gamma t = \int_m^{m+\lambda} q(t) \nabla_a^\gamma t$, then

$$\int_m^n r(t) g(t) \nabla_a^\gamma t \leq \int_m^{m+\lambda} r(t) q(t) \nabla_a^\gamma t. \quad (7)$$

Furthermore, given $0 \leq g(t) \leq q(t)$ and $\int_{n-\lambda}^n q(t) \nabla_a^\gamma t = \int_m^n g(t) \nabla_a^\gamma t$, we have

$$\int_{n-\lambda}^n r(t) q(t) \nabla_a^\gamma t \leq \int_m^n r(t) g(t) \nabla_a^\gamma t. \quad (8)$$

If r/q is nondecreasing, then the reverse inequalities in (7) and (8) hold.

Theorem 6. Suppose ψ is an integrable function on $[m, n]_{\mathbb{T}}$, $0 \leq \psi(t) \leq g(t) \leq q(t) - \psi(t)$ for all $t \in [m, n]_{\mathbb{T}}$ with $\int_m^{m+\lambda} q(t) \nabla_a^\gamma t = \int_m^n g(t) \nabla_a^\gamma t = \int_{n-\lambda}^n q(t) \nabla_a^\gamma t$ and r, g and q are ∇_a^γ -integrable functions, $0 \leq g(t) \leq q(t)$, we have

$$\begin{aligned}
 & \int_{n-\lambda}^n r(t) q(t) \nabla_a^\gamma t + \int_m^n \left| (r(t) - r(n - \lambda)) \psi(t) \right| \nabla_a^\gamma t \\
 & \leq \int_m^n r(t) g(t) \nabla_a^\gamma t \\
 & \leq \int_m^{m+\lambda} r(t) q(t) \nabla_a^\gamma t - \int_m^n \left| (r(t) - r(m + \lambda)) \psi(t) \right| \nabla_a^\gamma t,
 \end{aligned} \quad (9)$$

and

$$\begin{aligned}
 \int_{n-\lambda}^n r(t)q(t)\nabla_a^\gamma t &\leq \int_{n-\lambda}^n [r(t)q(t) - (r(t) - r(n-\lambda))] [q(t) - g(t)] \nabla_a^\gamma t \\
 &\leq \int_m^n r(t)g(t)\nabla_a^\gamma t \\
 &\leq \int_m^{m+\lambda} [r(t)q(t) - (r(t) - r(m+\lambda))] [q(t) - g(t)] \nabla_a^\gamma t \\
 &\leq \int_m^{m+\lambda} r(t)q(t)\nabla_a^\gamma t.
 \end{aligned} \tag{10}$$

Proof. The techniques used to prove Theorems 5 and 6 are similar to those of the techniques of the diamond- α version in [50] and are omitted. $\square$

In this paper, we formulate several speculations of fundamental dynamic inequalities of Steffensen's type on time scales and set up numerous new sharpened types of ∇ -conformable dynamic Steffensen's inequality. As unique instances of our inequalities, we recover the crucial inequalities given in those papers. New discrete Steffensen's inequalities can be established by our results via ∇ -conformable integrals on time scales.

2. Main Results

Give us a chance to start by presenting a class of functions that extends the convex function class.

Definition 2. Let $\phi, q : [m, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ be positive functions, $r : [m, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a function and $k \in (m, n)$. We say that f/h belongs to the class $\mathfrak{N}_1^k[m, n]$ (respectively, $\mathfrak{N}_2^k[m, n]$) with a constant A so that the function $r(t)/q(t) - A\phi(t)$ is nonincreasing (respectively, nondecreasing) on $[m, k]_{\mathbb{T}}$ and nondecreasing (respectively, nonincreasing) on $[k, n]_{\mathbb{T}}$.

3. ∇ -Conformable Dynamic Steffensen's Inequality

Next, some important lemmas are stated which are helpful for the evidence presented by our results.

Lemma 1. Suppose the integrable function $q > 0$ on $[m, n]_{\mathbb{T}}$ and r, g are integrable functions on $[m, n]_{\mathbb{T}}$ where r/q is nonincreasing and $0 \leq g(t) \leq 1$ for all $t \in [m, n]_{\mathbb{T}}$. Then,

$$(a_1) \quad \int_m^n r(t)g(t)\nabla_a^\gamma t \leq \int_m^{m+\lambda} r(t)\nabla_a^\gamma t, \tag{11}$$

such that λ is obtained from the equation

$$\int_m^n q(t)g(t)\nabla_a^\gamma t = \int_m^{m+\lambda} q(t)\nabla_a^\gamma t.$$

$$(a_2) \quad \int_{n-\lambda}^n r(t)\nabla_a^\gamma t \leq \int_m^n r(t)g(t)\nabla_a^\gamma t, \tag{12}$$

such that

$$\int_{n-\lambda}^n q(t)\nabla_a^\gamma t = \int_m^n q(t)g(t)\nabla_a^\gamma t.$$

The reverse inequalities in (11) and (12) are satisfied if r/q is nondecreasing.

Proof. Substitute $g(t) \mapsto q(t)g(t)$ and $r(t) \mapsto r(t)/q(t)$ in (7), (8) to get (a_1) and (a_2) simultaneously. $\square$

Lemma 2. Assume that r, q, g are defined as in Lemma 1 and ψ is an integrable function on $[m, n]_{\mathbb{T}}$ with $1 - \psi \geq g(t) \geq \psi(t) \leq 0 \forall t \in [m, n]_{\mathbb{T}}$. Then,

$$\begin{aligned} & \int_{n-\lambda}^n r(t) \nabla_a^\gamma t + \int_m^n \left| \left(\frac{r(t)}{q(t)} - \frac{r(n-\lambda)}{q(n-\lambda)} \right) q(t) \psi(t) \right| \nabla_a^\gamma t \\ & \leq \int_m^n r(t) g(t) \nabla_a^\gamma t \\ & \leq \int_m^{m+\lambda} r(t) \nabla_a^\gamma t - \int_m^n \left| \left(\frac{r(t)}{q(t)} - \frac{r(m+\lambda)}{q(m+\lambda)} \right) q(t) \psi(t) \right| \nabla_a^\gamma t, \end{aligned}$$

where λ is obtained from the equation

$$\int_m^{m+\lambda} h(t) \nabla_a^\gamma t = \int_m^n q(t) g(t) \nabla_a^\gamma t = \int_{n-\lambda}^n q(t) \nabla_a^\gamma t.$$

Proof. Put $g(t) \mapsto q(t)g(t)$, $r(t) \mapsto r(t)/h(t)$ and $\psi(t) \mapsto q(t)\psi(t)$ in (9). $\square$

Lemma 3. Furthermore, r/q is nonincreasing and $1 \geq g(t) \geq 0$ for all $t \in [m, n]_{\mathbb{T}}$. Then,

$$\begin{aligned} \int_{n-\lambda}^n r(t) \nabla_a^\gamma t & \leq \int_{n-\lambda}^n \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(n-\lambda)}{q(n-\lambda)} \right] q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ & \leq \int_m^n r(t) g(t) \nabla_a^\gamma t \\ & \leq \int_m^{m+\lambda} \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(m+\lambda)}{q(m+\lambda)} \right] q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ & \leq \int_m^{m+\lambda} r(t) \nabla_a^\gamma t, \end{aligned}$$

where λ is given by

$$\int_m^{m+\lambda} q(t) \nabla_a^\gamma t = \int_m^n g(t) \nabla_a^\gamma t = \int_{n-\lambda}^n q(t) \nabla_a^\gamma t.$$

Proof. Put $g(t) \mapsto q(t)g(t)$ and $r(t) \mapsto r(t)/q(t)$ in (10). $\square$

Theorem 7. Suppose that q, r, g are mentioned as in Lemma 3 such that $k \in (m, n)$ and λ_1 and λ_2 are the solutions of the equations

$$\begin{aligned} (a_3) \quad & \int_m^{m+\lambda_1} q(t) \nabla_a^\gamma t = \int_m^k q(t) g(t) \nabla_a^\gamma t, \\ & \int_{n-\lambda_2}^n q(t) \nabla_a^\gamma t = \int_k^n q(t) g(t) \nabla_a^\gamma t. \end{aligned}$$

If $r/q \in \mathfrak{N}_1^k[m, n]$ and

$$\int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t = \int_m^{m+\lambda_1} \phi(t) q(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n \phi(t) q(t) \nabla_a^\gamma t, \quad (13)$$

then

$$\int_m^n r(t) g(t) \nabla_a^\gamma t \leq \int_m^{m+\lambda_1} r(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n r(t) \nabla_a^\gamma t. \quad (14)$$

The inequality in (14) is reversed in the case of $r/q \in \mathfrak{N}_2^k[m, n]$ and (13).

$$(a_4) \quad \int_{k-\lambda_1}^k q(t) \nabla_a^\gamma t = \int_m^k q(t) g(t) \nabla_a^\gamma t,$$

$$\int_k^{k+\lambda_2} q(t) \nabla_a^\gamma t = \int_k^n q(t) g(t) \nabla_a^\gamma t.$$

If $r/q \in \aleph_1^k[m, n]$ and

$$\int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t = \int_{k-\lambda_1}^{k+\lambda_2} \phi(t) q(t) \nabla_a^\gamma t, \quad (15)$$

then

$$\int_m^n r(t) g(t) \nabla_a^\gamma t \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t) \nabla_a^\gamma t. \quad (16)$$

If $r/q \in \aleph_2^k[m, n]$ and (15) holds, the inequality in (16) is reversed in the case of

(a₅) If λ_1, λ_2 are the same as in (a₃) and $r/q \in \aleph_1^k[m, n]$ so that

$$\begin{aligned} & \int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t \\ &= \int_m^{m+\lambda_1} \left(\phi(t) q(t) - [\phi(t) - m - \lambda_1] q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ &+ \int_{n-\lambda_2}^n \left(\phi(t) q(t) - [\phi(t) - n + \lambda_2] q(t) [1 - g(t)] \right) \nabla_a^\gamma t, \end{aligned} \quad (17)$$

then

$$\begin{aligned} & \int_m^n r(t) g(t) \nabla_a^\gamma t \\ &\leq \int_m^{m+\lambda_1} \left(r(t) - \left| \frac{r(t)}{q(t)} - \frac{r(m+\lambda_1)}{q(m+\lambda_1)} \right| q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ &+ \int_{n-\lambda_2}^n \left(r(t) - \left| \frac{r(t)}{q(t)} - \frac{r(n-\lambda_2)}{q(n-\lambda_2)} \right| q(t) [1 - g(t)] \right) \nabla_a^\gamma t. \end{aligned} \quad (18)$$

If $r/q \in \aleph_2^k[m, n]$ and (17) holds, the inequality in (18) is reversed.

(a₆) If λ_1, λ_2 be defined as in (a₄) and $r/q \in \aleph_1^k[m, n]$ and

$$\begin{aligned} & \int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t \\ &= \int_{k-\lambda_1}^k \left(\phi(t) q(t) - [\phi(t) - k + \lambda_1] q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ &= \int_m^{m+\lambda_1} \left(\phi(t) q(t) - [\phi(t) - k - \lambda_2] q(t) [1 - g(t)] \right) \nabla_a^\gamma t, \end{aligned} \quad (19)$$

then

$$\begin{aligned} & \int_m^n r(t) g(t) \nabla_a^\gamma t \\ &\geq \int_{k-\lambda_1}^k \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(k-\lambda_1)}{q(k-\lambda_1)} \right] q(t) [1 - g(t)] \right) \nabla_a^\gamma t \\ &+ \int_k^{k+\lambda_2} \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(k+\lambda_2)}{q(k+\lambda_2)} \right] q(t) [1 - g(t)] \right) \nabla_a^\gamma t. \end{aligned} \quad (20)$$

If $r/q \in \aleph_2^k[m, n]$ and (19) holds, the inequality in (20) is reversed.

Proof. (a_3) Consider $r/q \in \mathbb{N}_1^k[m, n]$ and $R_1(x) = r(x) - A\phi(x)q(x)$, where A is the constant from Definition 2. In view of the nonincreasing nature of $R_1/q : [m, k]_{\mathbb{T}} \rightarrow \mathbb{R}$, from Lemma 1(a_1), we deduce

$$\begin{aligned} 0 &\leq \int_m^{m+\lambda_1} R_1(t) \nabla_a^\gamma t - \int_m^k R_1(t) g(t) \nabla_a^\gamma t \\ &= \int_m^{m+\lambda_1} r(t) \nabla_a^\gamma t - \int_m^k r(t) g(t) \nabla_a^\gamma t \\ &\quad - A \left(\int_m^{m+\lambda_1} \phi(t) q(t) \nabla_a^\gamma t - \int_m^k \phi(t) q(t) g(t) \nabla_a^\gamma t \right). \end{aligned} \quad (21)$$

As $R_1/q : [k, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ is nondecreasing, from Lemma 1 (a_2), we get

$$\begin{aligned} 0 &\geq \int_k^n R_1(t) g(t) \nabla_a^\gamma t - \int_{n-\lambda_2}^n R_1(t) \nabla_a^\gamma t \\ &= \int_k^n r(t) g(t) \nabla_a^\gamma t - \int_{n-\lambda_2}^n r(t) \nabla_a^\gamma t \\ &\quad - A \left(\int_k^n \phi(t) q(t) g(t) \nabla_a^\gamma t - \int_{n-\lambda_2}^n \phi(t) q(t) \nabla_a^\gamma t \right). \end{aligned} \quad (22)$$

(21) and (22) imply that

$$\begin{aligned} &\int_m^{m+\lambda_1} r(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n r(t) \nabla_a^\gamma t - \int_m^n r(t) g(t) \nabla_a^\gamma t \\ &\geq A \left(\int_m^{m+\lambda_1} \phi(t) q(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n \phi(t) q(t) \nabla_a^\gamma t - \int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t \right) \end{aligned}$$

Hence, if (13) is satisfied, then (14) holds. A similar procedure will be followed for $r/q \in \mathbb{N}_2^k[m, n]$.

(a_4) Let $r/q \in \mathbb{N}_1^k[m, n]$, also $R_1(x) = r(x) - A\phi(x)q(x)$, where A is the constant from Definition 2. $R_1/q : [m, k]_{\mathbb{T}} \rightarrow \mathbb{R}$ is nonincreasing, so from Lemma 1(a_1) we obtain

$$\begin{aligned} 0 &\leq \int_m^k r(t) g(t) \nabla_a^\gamma t - \int_{k-\lambda_1}^k r(t) \nabla_a^\gamma t \\ &\quad - A \left(\int_m^k \phi(t) h(t) g(t) \nabla_a^\gamma t - \int_{c-\lambda_1}^k \phi(t) q(t) \nabla_a^\gamma t \right). \end{aligned} \quad (23)$$

$R_1/q : [k, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ is nondecreasing, so from Lemma 1(a_1) we obtain

$$\begin{aligned} 0 &\geq \int_k^{k+\lambda_2} r(t) \nabla_a^\gamma t - \int_k^n r(t) g(t) \nabla_a^\gamma t \\ &\quad - A \left(\int_k^{k+\lambda_2} \phi(t) q(t) \nabla_a^\gamma t - \int_k^n \phi(t) q(t) g(t) \nabla_a^\gamma t \right). \end{aligned} \quad (24)$$

Hence, from (23) and (24), we have

$$\begin{aligned} &\int_m^n r(t) g(t) \nabla_a^\gamma t - \int_{k-\lambda_1}^{k+\lambda_2} r(t) \nabla_a^\gamma t \\ &\geq A \left(\int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t - \int_{k-\lambda_1}^{k+\lambda_2} \phi(t) q(t) \nabla_a^\gamma t \right) \end{aligned}$$

Therefore, if $\int_m^n \phi(t) q(t) g(t) \nabla_a^\gamma t = \int_{k-\lambda_1}^{k+\lambda_2} \phi(t) q(t) \nabla_a^\gamma t$ is satisfied, then (18) holds. Follow the same steps for $r/q \in \mathbb{N}_2^k[m, n]$.

For $R_1/q : [m, k]_{\mathbb{T}} \rightarrow \mathbb{R}$ nonincreasing and for $R_1/q : [k, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ nondecreasing, we apply Lemma 3 and the same argument of Theorem 7(a_3) and Theorem 7(a_4) in the proof of (a_5) and (a_6) respectively. $\square$

Corollary 1. The inequalities (14), (16), (18) and (20) of Theorem 7 by putting $\mathbb{T} = \mathbb{R}$ take the following form of the delta version:

$$(i) \quad \int_m^n f(t)g(t)(t-a)^{\gamma-1}dt \leq \int_m^{m+\lambda_1} r(t)(t-a)^{\gamma-1}dt + \int_{n-\lambda_2}^n r(t)(t-a)^{\gamma-1}dt. \quad (25)$$

$$(ii) \quad \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t)(t-a)^{\gamma-1}dt. \quad (26)$$

$$(iii) \quad \begin{aligned} & \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \\ & \leq \int_m^{m+\lambda_1} \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(m+\lambda_1)}{q(m+\lambda_1)} \right] q(t)[1-g(t)] \right) (t-a)^{\gamma-1}dt \\ & \quad + \int_{n-\lambda_2}^n \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(n-\lambda_2)}{q(n-\lambda_2)} \right] q(t)[1-g(t)] \right) (t-a)^{\gamma-1}dt. \end{aligned} \quad (27)$$

$$(iv) \quad \begin{aligned} & \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \\ & \geq \int_{k-\lambda_1}^k \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(k-\lambda_1)}{q(k-\lambda_1)} \right] q(t)[1-g(t)] \right) (t-a)^{\gamma-1}dt \\ & \quad + \int_k^{k+\lambda_2} \left(r(t) - \left[\frac{r(t)}{q(t)} - \frac{r(k+\lambda_2)}{q(k+\lambda_2)} \right] q(t)[1-g(t)] \right) (t-a)^{\gamma-1}dt. \end{aligned} \quad (28)$$

Corollary 2. The inequalities can be recollected ([11], Theorems 8, 10, 21 and 22), $\alpha = 1$ and $\phi(t) = t$ in Corollary 1 [(i), (ii), (iii), (iv)] simultaneously.

Corollary 3. Putting $\mathbb{T} = \mathbb{Z}$ in Corollary 1, the inequalities (25)–(28) are equivalent to

$$(i) \quad \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \leq \sum_{t=m}^{m+\lambda_1-1} r(t+1) + \sum_{t=n-\lambda_2}^{n-1} r(t+1)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)}.$$

$$(ii) \quad \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \geq \sum_{t=k-\lambda_1}^{k+\lambda_2-1} r(t+1)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)}.$$

$$(iii) \quad \begin{aligned} & \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\ & \leq \sum_{t=m}^{m+\lambda_1-1} \left(r(t+1) - \left[\frac{r(t+1)}{q(t)} - \frac{r(m+\lambda_1+1)}{q(m+\lambda_1+1)} \right] q(t)[1-g(t)] \right) (\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\ & \quad + \sum_{t=n-\lambda_2}^{n-1} \left(r(t+1) - \left[\frac{r(t+1)}{q(t)} - \frac{r(n-\lambda_2+1)}{q(n-\lambda_2+1)} \right] q(t)[1-g(t)] \right) (\rho^{\gamma-1}(t)-a)^{(\gamma-1)}. \end{aligned}$$

$$(iv) \quad \begin{aligned} & \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\ & \geq \sum_{t=k-\lambda_1}^{k-1} \left(r(t+1) - \left[\frac{r(t+1)}{q(t)} - \frac{r(k-\lambda_1+1)}{q(k-\lambda_1+1)} \right] q(t)[1-g(t)] \right) (\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\ & \quad + \sum_{t=k}^{k+\lambda_2-1} \left(r(t+1) - \left[\frac{r(t+1)}{q(t)} - \frac{r(k+\lambda_2+1)}{q(k+\lambda_2+1)} \right] q(t)[1-g(t)] \right) (\rho^{\gamma-1}(t)-a)^{(\gamma-1)}. \end{aligned}$$

4. Further ∇ -Conformable Dynamic Steffensen's Inequality

In this section, we introduce further results of the ∇ -conformable dynamic Steffensen inequality.

Theorem 8. Let k, r, q, g be satisfied as in Lemma 3 such that $0 \leq g(t) \leq q(t)$ for all $t \in [m, n]_{\mathbb{T}}$ and λ_1, λ_2 be defined as

$$(a_7) \quad \int_m^{m+\lambda_1} q(t) \nabla_a^\gamma t = \int_m^k g(t) \nabla_a^\gamma t,$$

$$\int_{n-\lambda_2}^n q(t) \nabla_a^\gamma t = \int_k^n g(t) \nabla_a^\gamma t.$$

If $r/q \in \mathbb{N}_1^k[m, n]$ and

$$\int_m^n \phi(t) g(t) \nabla_a^\gamma t = \int_m^{m+\lambda_1} \phi(t) q(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n \phi(t) q(t) \nabla_a^\gamma t, \quad (29)$$

then

$$\int_m^n r(t) g(t) \nabla_a^\gamma t \leq \int_m^{m+\lambda_1} r(t) q(t) \nabla_a^\gamma t + \int_{n-\lambda_2}^n r(t) q(t) \nabla_a^\gamma t. \quad (30)$$

$$(a_8) \quad \int_{k-\lambda_1}^k q(t) \nabla_a^\gamma t = \int_m^k g(t) \nabla_a^\gamma t,$$

$$\int_k^{k+\lambda_2} q(t) \nabla_a^\gamma t = \int_k^n g(t) \nabla_a^\gamma t.$$

If $r/q \in \mathbb{N}_1^k[m, n]$ and

$$\int_m^n \phi(t) g(t) \nabla_a^\gamma t = \int_{k-\lambda_1}^{k+\lambda_2} \phi(t) q(t) \nabla_a^\gamma t, \quad (31)$$

then

$$\int_m^n r(t) g(t) \nabla_a^\gamma t \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t) q(t) \nabla_a^\gamma t. \quad (32)$$

If $r/q \in \mathbb{N}_2^k[m, n]$ and (29), (31) hold, the inequalities in (30) and (32) are reversed.

Proof. The proof of (a₇) and (a₈) is followed by a similar application of Theorem 7 [(a₃), (a₄)] and by taking $g \mapsto g/h$ and $f \mapsto fh$. $\square$

Corollary 4. Taking $\mathbb{T} = \mathbb{R}$ in Theorem 8 [(a₇), (a₈)], we obtain the delta version of inequalities (30) and (32) as follows:

$$(i) \quad \int_m^n r(t) g(t) (t-a)^{\gamma-1} dt \leq \int_m^{m+\lambda_1} r(t) q(t) (t-a)^{\gamma-1} dt + \int_{n-\lambda_2}^n r(t) q(t) (t-a)^{\gamma-1} dt. \quad (33)$$

$$(ii) \quad \int_m^n r(t) g(t) (t-a)^{\gamma-1} dt \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t) q(t) (t-a)^{\gamma-1} dt. \quad (34)$$

Corollary 5. If $\alpha = 1$ and $\phi(t) = t$ in Corollary 4 [(i), (ii)] then we can reclaim ([11], Theorems 16 and 17).

Corollary 6. Putting $\mathbb{T} = \mathbb{Z}$, the inequalities (33) and (34) become

$$(i) \quad \sum_{t=m}^{n-1} r(t+1) g(t) (\rho^{\gamma-1}(t) - a)^{(\gamma-1)} \leq \sum_{t=m}^{m+\lambda_1-1} r(t+1) h(t) + \sum_{t=n-\lambda_2}^{n-1} r(t+1) h(t) (\rho^{\gamma-1}(t) - a)^{(\gamma-1)}.$$

$$(ii) \quad \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)} \geq \sum_{t=k-\lambda_1}^{k+\lambda_2-1} r(t+1)q(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)}.$$

Theorem 9. Assume that r, q, g, k are fulfilled as in Lemma 3. Moreover, $w : [m, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a ∇_a^γ -integrable function, $0 \leq g(t) \leq w(t) \forall t \in [m, n]_{\mathbb{T}}$ and λ_1, λ_2, k be given as

$$(a_9) \quad \int_m^{m+\lambda_1} w(t)q(t)\nabla_a^\gamma t = \int_m^k q(t)g(t)\nabla_a^\gamma t, \\ \int_{n-\lambda_2}^n w(t)q(t)\nabla_a^\gamma t = \int_k^n q(t)g(t)\nabla_a^\gamma t.$$

If $r/q \in \mathfrak{N}_1^k[m, n]$ and

$$\int_m^n \phi(t)q(t)g(t)\nabla_a^\gamma t = \int_m^{m+\lambda_1} \phi(t)w(t)q(t)\nabla_a^\gamma t + \int_{n-\lambda_2}^n \phi(t)w(t)q(t)\nabla_a^\gamma t, \quad (35)$$

then

$$\int_m^n r(t)g(t)\nabla_a^\gamma t \leq \int_m^{m+\lambda_1} r(t)w(t)\nabla_a^\gamma t + \int_{n-\lambda_2}^n r(t)w(t)\nabla_a^\gamma t. \quad (36)$$

$$(a_{10}) \quad \int_{k-\lambda_1}^k w(t)q(t)\nabla_a^\gamma t = \int_m^k q(t)g(t)\nabla_a^\gamma t, \\ \int_k^{k+\lambda_2} w(t)q(t)\nabla_a^\gamma t = \int_k^n q(t)g(t)\nabla_a^\gamma t.$$

If $r/q \in \mathfrak{N}_1^k[m, n]$ and

$$\int_m^n \phi(t)q(t)g(t)\nabla_a^\gamma t = \int_{k-\lambda_1}^{k+\lambda_2} \phi(t)w(t)q(t)\nabla_a^\gamma t, \quad (37)$$

$$\int_m^n r(t)g(t)\nabla_a^\gamma t \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t)w(t)\nabla_a^\gamma t. \quad (38)$$

The inequalities in (36) and (38) are reversible if $r/q \in \mathfrak{N}_2^k[a, b]$ and (35) and (37) are satisfied.

Proof. Replace $q \mapsto wq$, $g \mapsto g/w$ and $r \mapsto rw$ in Theorem 7 [(a₃), (a₄)]. $\square$

Corollary 7. For the delta kind of inequalities (36) and (38) by setting $\mathbb{T} = \mathbb{R}$ in Theorem 9 [(a₉), (a₁₀)], we have

$$(i) \quad \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \leq \int_m^{m+\lambda_1} r(t)w(t)(t-a)^{\gamma-1}dt + \int_{n-\lambda_2}^n r(t)w(t)(t-a)^{\gamma-1}dt. \quad (39)$$

$$(ii) \quad \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t)w(t)(t-a)^{\gamma-1}dt. \quad (40)$$

Corollary 8. Putting $\alpha = 1$ and $\phi(t) = t$ in Corollary 7 [(i), (ii)], we recapture ([11], Theorems 18 and 19) respectively.

Corollary 9. If $\mathbb{T} = \mathbb{Z}$, the inequalities (39) and (40) are converted into

$$(i) \quad \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)} \leq \sum_{t=m}^{m+\lambda_1-1} r(t+1)w(t) + \sum_{t=n-\lambda_2}^{n-1} r(t+1)w(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)}.$$

$$(ii) \quad \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)} \geq \sum_{t=k-\lambda_1}^{k+\lambda_2-1} r(t+1)w(t)(\rho^{\gamma-1}(t) - a)^{(\gamma-1)}.$$

Theorem 10. Let r, q, g be given as in Lemma 3, λ_1, λ_2, k be defined as in Theorem 7 $[(a_3), (a_4)]$ and $\psi : [m, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ be a ∇_a^γ -integrable function such that $0 \leq \psi(t) \leq g(t) \leq 1 - \psi(t)$ for all $t \in [m, n]_{\mathbb{T}}$.

(a₁₁) If $r/q \in \mathfrak{N}_1^k[m, n]$ and

$$\begin{aligned} & \int_m^n \phi(t)q(t)g(t)\nabla_a^\gamma t \\ &= \int_m^{m+\lambda_1} \phi(t)q(t)\nabla_a^\gamma t - \int_m^k |\phi(t) - m - \lambda_1|q(t)\psi(t)\nabla_a^\gamma t + \int_{n-\lambda_2}^n \phi(t)q(t)\nabla_a^\gamma t \\ & \quad + \int_k^n |\phi(t) - n + \lambda_2|q(t)\psi(t)\nabla_a^\gamma t, \end{aligned} \quad (41)$$

then

$$\begin{aligned} & \int_m^n r(t)g(t)\nabla_a^\gamma t \\ & \leq \int_m^{m+\lambda_1} r(t)\nabla_a^\gamma t - \int_m^k \left| \frac{r(t)}{q(t)} - \frac{r(m+\lambda_1)}{q(m+\lambda_1)} \right| q(t)\psi(t)\nabla_a^\gamma t + \int_{n-\lambda_2}^n r(t)\nabla_a^\gamma t \\ & \quad + \int_k^n \left| \frac{r(t)}{q(t)} - \frac{r(n-\lambda_2)}{q(n-\lambda_2)} \right| q(t)\psi(t)\nabla_a^\gamma t. \end{aligned} \quad (42)$$

(a₁₂) If $r/q \in \mathfrak{N}_1^k[m, n]$ and

$$\begin{aligned} & \int_m^n \phi(t)q(t)g(t)\nabla_a^\gamma t \\ &= \int_{k-\lambda_1}^k \phi(t)q(t)\nabla_a^\gamma t - \int_m^k |\phi(t) - k + \lambda_1|q(t)\psi(t)\nabla_a^\gamma t \\ & \quad + \int_k^n |\phi(t) - k - \lambda_1|q(t)\psi(t)\nabla_a^\gamma t, \end{aligned} \quad (43)$$

then

$$\begin{aligned} & \int_m^n r(t)g(t)\nabla_a^\gamma t \\ & \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t)\nabla_a^\gamma t + \int_m^k \left| \frac{r(t)}{q(t)} - \frac{r(k-\lambda_1)}{q(k-\lambda_1)} \right| q(t)\psi(t)\nabla_a^\gamma t \\ & \quad - \int_k^n \left| \frac{r(t)}{q(t)} - \frac{r(k+\lambda_2)}{q(k+\lambda_2)} \right| q(t)\psi(t)\nabla_a^\gamma t. \end{aligned} \quad (44)$$

If $r/q \in \mathfrak{N}_2^k[m, n]$ and (41) and (43) hold, the inequalities in (42) and (44) are switched.

Proof. The proof is the same as that of Theorem 7 $[(a_3), (a_4)]$ with Lemma 3, $R_1/q : [m, k]_{\mathbb{T}} \rightarrow \mathbb{R}$ nonincreasing and $R_1/q : [k, n]_{\mathbb{T}} \rightarrow \mathbb{R}$ nondecreasing. $\square$

Corollary 10. Setting $\mathbb{T} = \mathbb{R}$ in Theorem 10 $[(a_{11}), (a_{12})]$, we obtain the delta version of inequality (42) and (44) as follows:

$$\begin{aligned} (i) \quad & \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \\ & \leq \int_m^{m+\lambda_1} r(t)(t-a)^{\gamma-1}dt - \int_m^k \left| \frac{r(t)}{q(t)} - \frac{r(m+\lambda_1)}{q(m+\lambda_1)} \right| q(t)\psi(t)(t-a)^{\gamma-1}dt + \sum_{t=n-\lambda_2}^{t=n-1} r(t)(t-a)^{\gamma-1}dt \\ & \quad + \int_k^n \left| \frac{r(t)}{q(t)} - \frac{r(n-\lambda_2)}{q(n-\lambda_2)} \right| q(t)\psi(t)(t-a)^{\gamma-1}dt. \end{aligned} \quad (45)$$

$$\begin{aligned}
(ii) \quad & \int_m^n r(t)g(t)(t-a)^{\gamma-1}dt \\
& \geq \int_{k-\lambda_1}^{k+\lambda_2} r(t)(t-a)^{\gamma-1}dt + \int_m^k \left| \frac{r(t)}{q(t)} - \frac{r(k-\lambda_1)}{q(k-\lambda_1)} \right| q(t)\psi(t)(t-a)^{\gamma-1}dt \\
& \quad - \int_k^n \left| \frac{r(t)}{q(t)} - \frac{r(k+\lambda_2)}{q(k+\lambda_2)} \right| q(t)\psi(t)(t-a)^{\gamma-1}dt.
\end{aligned} \tag{46}$$

Corollary 11. If $\alpha = 1$, $\phi(t) = t$ in (45) and (46), we recapture ([11], Theorems 23 and 24) respectively.

Corollary 12. The inequalities (45) and (46), with $\mathbb{T} = \mathbb{Z}$ become

$$\begin{aligned}
(i) \quad & \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\
& \leq \sum_{t=m}^{m+\lambda_1-1} r(t+1)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} - \sum_{t=m}^{k-1} \left| \frac{r(t+1)}{q(t)} - \frac{r(m+\lambda_1+1)}{q(m+\lambda_1)} \right| q(t)\psi(t)\nabla t \\
& \quad + \int_{n-\lambda_2}^n r(t+1)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} + \sum_{t=k}^{n-1} \left| \frac{r(t+1)}{q(t)} - \frac{r(n-\lambda_2+1)}{q(n-\lambda_2)} \right| q(t)\psi(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)}. \\
(ii) \quad & \sum_{t=m}^{n-1} r(t+1)g(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\
& \geq \sum_{t=k-\lambda_1}^{k+\lambda_2-1} r(t+1)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} + \sum_{t=m}^{k-1} \left| \frac{r(t+1)}{q(t)} - \frac{r(k-\lambda_1+1)}{q(k-\lambda_1)} \right| q(t)\psi(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)} \\
& \quad - \sum_{t=k}^{n-1} \left| \frac{r(t+1)}{q(t)} - \frac{r(k+\lambda_2+1)}{q(k+\lambda_2)} \right| h(t)\psi(t)(\rho^{\gamma-1}(t)-a)^{(\gamma-1)}.
\end{aligned}$$

5. Conclusions and Discussion

In this paper, we proved some new nabla Steffensen dynamic inequalities by using the conformable fractional nabla conformable-integral on time scales. In addition, as special cases, we obtained the discrete version and integral version inequalities. Our results give a new generalization of the conformable nabla derivative and integral which involve the time scale power function $\widehat{G}_n(t, s)$ for $s, t \in \mathbb{T}$. The time scale power function takes the form $(t-a)^\eta$ for $\mathbb{T} = \mathbb{R}$, which reduces to the definition of the conformable fractional derivative defined by Khalil. Furthermore, in the future, we intend to extend these results to quantum calculus and give new results regarding Steffensen-type inequalities.

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