Article

Hybrid Carbon Nanotube Flow near the Stagnation Region over a Permeable Vertical Plate with Heat Generation/Absorption

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Abstract: This research explored the mixed convection flow past a vertical plate immersed in a hybrid carbon nanotube near the stagnation point. The hybrid carbon nanotube was synthesized by the mixture of two nanoparticles, namely multi-wall (MWCNT) and single-wall (SWCNT) carbon nanotubes immersed in water (base fluid). In addition, attractive aspects of suction/injection and heat generation/absorption effects were incorporated. Similarity variables were used to convert the partial differential equations describing the fluid into ordinary (similarity) differential equations before being solved numerically using Matlab software. The simultaneous impact of several parameters on velocity and temperature profiles, skin friction coefficient, and local Nusselt number were represented with graphs. Dual solutions were observed for some pertinent parameters, which led to stability analysis. This analysis interpreted that merely the first numerical solution is stable. In addition, hybrid nanoparticle, injection effect, and heat-generation parameters led to a decreased range of solutions, whilst the suction effect and heat-absorption parameters acted in the opposite manner. Besides, it is noted that the rate of heat transfer for hybrid carbon nanotube was higher when compared with carbon nanotube and ordinary fluid. Additionally, the heat absorption and buoyancy-assisting flow parameters magnified the heat transfer rate.

Keywords: hybrid carbon nanotube; heat generation/absorption; suction/injection; dual solution; stability analysis

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1. Introduction

Many researchers are fascinated by nanofluid (i.e., mixing nanoparticles in the base fluid) due to its rapid advancement in nanotechnology and modern sciences. The work of Choi and Eastman [1] showed that copper nanoparticle helps enhance heat-transfer efficiency. Owing to their captivating convective thermal properties, nanofluids have been identified as potential thermofluids for future applications [2,3]. Compared with ordinary nanofluids, hybrid nanofluids (i.e., the mixture of two diverse nanoparticles in a base fluid) provide highly efficient thermal energy transfer. In an experimental investigation by Suresh et al. [4], they found that a 2% volume fraction of hybrid nanocomposites ($\text{Al}_2\text{O}_3\text{--Cu}$) in water caused a 12.11% increase in thermal conductivity. Afterwards, Takabi and Salehi [5] and Devi and Devi [6] conducted a numerical investigation on hybrid nanofluid with suspended nanoparticles (alumina and copper) in water (base fluid), and they showed that when compared with ordinary nanofluid, the heat-transfer rate of hybrid nanofluid produces an outstanding result. The idea of using hybrid nanofluid then flourished, attracting several researchers to study the thermal behavior of some nanoparticles such as metallic [7,8], ferrite [9,10], and carbon nanotube [11,12] in fluid flow through various surfaces and effects. However, research related to hybrid nanofluids that use carbon

nanotubes (i.e., MWCNT and SWCNT) as nanoparticles is limited. Since Iijima's [13] discovery of carbon nanotubes, carbon-based nanotechnology has promptly developed as a platform technology for a wide range of applications, including biomedical [14,15], wastewater treatments [16,17], and electronics [18,19]. As a result, the primary goal of this research was to investigate the behavior of MWCNT and SWCNT in the base fluid water.

Many technical and industrial applications use mixed convection flow (i.e., the simultaneous presence of free and forced convection flow). Examples of such applications are heat exchangers, nuclear reactors, solar central receivers, and electronic devices [20]. Mixed convection is induced by the motion of solid material, such as the suction/injection effect (forced convection) and buoyancy force generated by the heated vertical wall (natural convection). Injection or suction of fluid through the bounding surfaces can drastically alter the flow field and, as a result, affect the rate of heat transfer. Work on suction/injection in a vertical plate includes the study of Ishak et al. [21]. Their results showed that two solutions exist for assisting and opposing flow: injection delays the separation of the boundary layer while suction fastens in. This research was then continued by Ishak et al. [22] with an additional effect of magnetohydrodynamic (MHD), Tamim et al. [23] for MHD nanofluid flow, and by Rostami et al. [24] in aqueous silica-alumina hybrid nanofluid. In another work, Ghalambaz et al. [25] performed the stability analysis on hybrid nanofluid ($\text{Al}_2\text{O}_3\text{-Cu/water}$) in a vertical permeable plate and showed that the first solution is always a stable solution. It should be pointed out that most of these researchers consider their investigation near the stagnation point (i.e., the fluid flow near a circular body's stagnation region, which occurs in both fixed and moving bodies in a fluid) owing to its practical applications and numerous applications in the industry [26].

In this research, the use of heat generation (source) and absorption (sink) may have a crucial impact on the heat transfer characteristics and optimize the fluid flow. These effects are essential in cooling problems associated with nuclear reactions, combustion processes, and neurobiology (in the study of the human brain function) [27]. Some important investigations concerning the heat source or sink impact in boundary layer flow were made by Bhattacharyya [28], Ahmed et al. [29], Makinde et al. [30], and Agrawal et al. [31]. A detailed study covering the heat generation/absorption influence on MHD radiative porous vertical moving plate was conducted by Anwar et al. [32]. In their investigation, they discovered that increasing values of heat sink parameter reduces the rate of heat transfer. In very recent work, Reddy et al. [33] considered the influence of heat source/sink on MHD flow when it is stretched exponentially through a porous medium and demonstrated that the local Nusselt number rises as the value of heat source increases due to the increased amount of heat energy stored.

In several boundary layer flow problems, the existence of two or more solutions is another interesting flow phenomenon. The researchers have started performing and breaking down the results in the last two or three decades, which have non-unique solutions. The occurrence of non-unique solution is due to several physical parameters' impacts, such as mixed convection parameter, suction parameter, shrinking parameter, and moving (opposing flow to the free stream) parameter. Usually, the first solution is represented by the solid line or upper branch solution, while the second solution is represented by the dash line or lower branch solution.

According to the work cited earlier, the topic of hybrid carbon nanotube (using SWCNT and MWCNT as nanoparticles) in permeable vertical plates has not been thoroughly investigated up until now. Therefore, this paper extends the research done by Ghalambaz et al. [25] by introducing an additional parameter heat generation or absorption parameter in a hybrid carbon nanotube. With the aid of similarity transformations, we will use `bvp4c` in Matlab to solve the ordinary differential equations derived from the governing boundary layer equations. Surprisingly, the current research revealed the occurrence of non-unique solutions, and thus the stability analysis initiated by Merkin [34] was carried out. The stability analysis of non-unique solutions has attracted much attention from researchers because of the different behavior of the solutions in temperature and velocity

profiles. This kind of analysis is essential to avoid any misleading interpretation of flow. Hence, many researchers (see Refs. [35–38]) have mathematically performed the stability analysis of solutions in their recent work. Furthermore, this theoretical study would benefit engineers who are conducting experiments on hybrid carbon nanotubes, and the findings are expected to lower the cost of future experiments.

2. Mathematical Analysis

2.1. Flow Formulation

Let us investigate the steady hybrid carbon nanotube flow near the stagnation region towards a permeable vertical plate in the existence of heat generation/absorption. In the present research, a hybrid carbon nanotube (SWCNT–MWCNT/water) was formed by dispersing 1% of multi-wall carbon nanotube (MWCNT) nanoparticles into the base fluid (water). Then, the single-wall carbon nanotube (SWCNT) is scattered in MWCNT/water nanofluid to produce the targeted hybrid carbon nanotube. In addition, the inviscid flow velocity is $U_e(x) = ax$, where $a(>0)$ is the stagnation point's velocity, while the sheet's temperature is $T_w(x) = T_\infty + bx$, given that T_∞ is the ambient fluid at a constant temperature and b is a constant. Both heating ($T_w(x) > T_\infty$) and cooling ($T_w(x) < T_\infty$) of the sheet are considered, which correspond to assisting and opposing flows, respectively. The assisting flow ($b > 0$) situation takes place if the lower half of the plate surface is cooled and the upper half of the plate surface is heated. Nevertheless, opposing flow ($b < 0$) takes place in the opposite manner. The buoyancy force is caused by temperature variance in the fluid; meanwhile, a heat generation or absorption is situated within the flow to allow for heat source or sink effects. To model the present problem, a Cartesian coordinate system (x, y) is chosen in such a way that the plate is parallel along the x -axis, and therefore, the y -axis is perpendicular to it, the flow being at $y \geq 0$, as illustrated in Figure 1.

Figure 1. Physical model.

The modelled equations for the steady flow of incompressible water-based hybrid carbon nanotube can be expressed as (see Refs. [6,25]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_e \frac{\partial U_e}{\partial x} + \frac{\mu_{hcnt}}{\rho_{hcnt}} \frac{\partial^2 u}{\partial y^2} + \beta_{hcnt} g(T - T_\infty), \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{hcnt}}{(\rho C_p)_{hcnt}} \frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{(\rho C_p)_{hcnt}} (T - T_\infty), \quad (3)$$

associated with its conditions:

$$\begin{aligned} u &= 0, \quad v = v_w, \quad T = T_w(x) \text{ at } y = 0, \\ u &\rightarrow U_e(x), \quad T \rightarrow T_\infty \text{ as } y \rightarrow \infty \end{aligned} \quad (4)$$

where the velocity components in x - and y -direction are given by u and v , respectively, T is the temperature of the fluid, Q_0 is the uniform volumetric heat source/sink, g is the acceleration due to gravity, and v_w is the mass-transfer velocity.

Further, μ refers to the dynamic viscosity, β is the coefficient of thermal expansion, ρ is the density, ρC_p and k denote the heat capacity and thermal conductivity, respectively, where the subscript 'hcnt' refers to the hybrid carbon nanotube. The following thermo-physical characteristics for hybrid carbon nanotube are given in Table 1 (see Takabi and Salehi [5]).

Table 1. Thermo-physical characteristic of hybrid nanofluid.

Properties	Hybrid Nanofluid
Dynamic viscosity	$\mu_{hcnt} = \frac{\mu_f}{(1-\varphi)^{2.5}}$
Heat capacity	$(\rho C_p)_{hcnt} = (1-\varphi)(\rho C_p)_f + \varphi(\rho C_p)_{hp}$ where $(\rho C_p)_{hp} = \frac{\varphi_1(\rho C_p)_1 + \varphi_2(\rho C_p)_2}{\varphi}$
Thermal expansion	$\beta_{hcnt} = (1-\varphi)\beta_f + \varphi\beta_{hp}$ where $\beta_{hp} = \frac{\varphi_1\beta_1 + \varphi_2\beta_2}{\varphi}$
Density	$\rho_{hcnt} = (1-\varphi)\rho_f + \varphi\rho_{hp}$ where $\rho_{hp} = \frac{\varphi_1\rho_1 + \varphi_2\rho_2}{\varphi}$
Thermal conductivity	$\frac{k_{hcnt}}{k_f} = \frac{1-\varphi+2\varphi\frac{k_{hp}}{k_f}\ln\left(\frac{k_{hp}+k_f}{2k_f}\right)}{1-\varphi+2\varphi\frac{k_f}{k_{hp}-k_f}\ln\left(\frac{k_{hp}+k_f}{2k_f}\right)}$ where $k_{hp} = \frac{\varphi_1k_1 + \varphi_2k_2}{\varphi}$

Here, φ is the summation of the volume concentration of two dissimilar kind of nanoparticles, i.e., $\varphi = \varphi_1 + \varphi_2$ where the subscripts '1' and '2' stand for the first and second nanoparticles, respectively. In addition, the subscript 'f' and 'hp' denote the base fluid and hybrid particle. It should be emphasized that the first nanoparticle is denoted by MWCNT while the second nanoparticle refers to SWCNT. Here, the volume fraction of SWCNT nanoparticle fluctuates from 0% to 2%. The physical characteristics of the nanosized particles and the base fluid are shown in Table 2 (see Aman et al. [39]).

Table 2. Properties of base fluid and carbon nanotube.

	Physical Properties			
	$k (\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1})$	$C_p (\text{J} \cdot \text{kg}^{-1} \cdot \text{K}^{-1})$	$\rho (\text{kg} \cdot \text{m}^{-3})$	$\beta (\times 10^{-5} \text{K}^{-1})$
Base fluid:				
water	0.613	4179	997.1	21
Carbon nanotube:				
SWCNT	6600	425	2600	27
MWCNT	3000	796	1600	44

Further, we define the following transformation:

$$\psi = \sqrt{av_f x} f(\eta), \quad \eta = \sqrt{\frac{a}{v_f}} y, \quad \theta(\eta) = \frac{T - T_\infty}{T_w(x) - T_\infty} \quad (5)$$

where v_f , f , θ , and η refer to the kinematic viscosity, dimensionless velocity, dimensionless temperature, and similarity variable, respectively. ψ is the stream function, which helps to

transform velocity components in the form of dimensionless parameter and defined in the usual way as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. So that:

$$u = a x f_\eta, \quad v = -\sqrt{av_f} f \quad (6)$$

On using Equations (5) and (6), the continuity equation, i.e., Equation (1), is identically satisfied. Now, Equations (2) and (3) are reduced to:

$$\frac{\mu_{hcnt}/\mu_f}{\rho_{hcnt}/\rho_f} f_{\eta\eta\eta} - f_\eta^2 + f f_{\eta\eta} + \frac{\beta_{hcnt}}{\beta_f} \lambda \theta + 1 = 0 \quad (7)$$

$$\frac{1}{(\rho C_p)_{hcnt}/(\rho C_p)_f} \left(\frac{k_{hcnt}/k_f}{Pr} \theta_{\eta\eta} + Q\theta \right) + f\theta_\eta - f_\eta\theta = 0 \quad (8)$$

Conditions (4) give:

$$\begin{aligned} f(0) &= s, \quad f_\eta(0) = 0, \quad \theta(0) = 1 \\ \theta(\eta) &\rightarrow 0, \quad f_\eta(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (9)$$

From the above equations, $Pr (= \mu_f(C_p)_f/k_f)$ is the Prandtl number, $\lambda (= Gr_x/Re_x^2)$ denotes the mixed convection parameter with $\lambda > 0$, and $\lambda < 0$ represents the assisting and opposing buoyancy flow where $Gr_x (= g\beta_f(T_w(x) - T_\infty)x^3/\nu_f^2)$ and $Re_x (= U_e(x)x/\nu_f)$ are the Grashoff number and Reynold number, respectively, $Q (= Q_0/(a(\rho C_p)_f))$ is the heat sink ($Q < 0$) or source ($Q > 0$) parameter and $s (= -v_w/\sqrt{av_f})$ is the injection ($s < 0$) or suction ($s > 0$) parameter.

2.2. Physical Quantities

The skin friction and Nusselt number are defined as:

$$C_f = \frac{\mu_{hcnt} \left(\frac{\partial u}{\partial y} \right)_{y=0}}{\rho_f U_e^2}, \quad Nu_x = -\frac{x k_{hcnt} \left(\frac{\partial T}{\partial y} \right)_{y=0}}{k_f (T_w - T_\infty)} \quad (10)$$

By the aid of similarity transformation of Equation (5) and using Equations (6) and (10), one obtains:

$$C_f Re_x^{1/2} = \frac{\mu_{hcnt}}{\mu_f} f''(0), \quad Nu_x Re_x^{-1/2} = -\frac{k_{hcnt}}{k_f} \theta'(0) \quad (11)$$

where prime is the differentiation with respect to η .

2.3. Stability Analysis

The existence of more than one solution led us to dictate which of these solutions is the stable solution. To perform this analysis, the unsteady governing equations were considered as:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial U_e}{\partial t} + U_e \frac{\partial U_e}{\partial x} + \frac{\mu_{hcnt}}{\rho_{hcnt}} \frac{\partial^2 u}{\partial y^2} + \beta_{hcnt} g(T - T_\infty), \quad (12)$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{k_{hcnt}}{(\rho C_p)_{hcnt}} \frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{(\rho C_p)_{hcnt}} (T - T_\infty), \quad (13)$$

which correspond to its boundary condition:

$$\begin{aligned} t < 0 : v = u = 0, T = T_\infty \text{ for any } x, y \\ t \geq 0 : v = v_w, u = 0, T = T_w(x) \text{ at } y = 0, \\ u \rightarrow U_e(x), T \rightarrow T_\infty \text{ as } y \rightarrow \infty \end{aligned} \quad (14)$$

where t refers to time. The following new transformation in the presence of time variable τ for the above time dependent problem are introduced:

$$\psi = \sqrt{av_f} x f(\eta, \tau), \eta = \sqrt{\frac{a}{v_f}} y, \theta(\eta, \tau) = \frac{T - T_\infty}{T_w(x) - T_\infty}, \tau = at \quad (15)$$

Implementing the new transformation given in Equation (15) into Equations (12)–(14), we obtain:

$$\frac{\mu_{hcnt}/\mu_f}{\rho_{hcnt}/\rho_f} f_{\eta\eta\eta} + f f_{\eta\eta} - (f_\eta)^2 + 1 + \frac{\beta_{hcnt}}{\beta_f} \lambda \theta - f_\tau \eta = 0 \quad (16)$$

$$\frac{1}{(\rho C_p)_{hcnt}/(\rho C_p)_f} \left(\frac{k_{hcnt}/k_f}{Pr} \theta_{\eta\eta} + Q \theta \right) + f \theta_\eta - f_\eta \theta - \theta_\tau = 0 \quad (17)$$

together with conditions:

$$\begin{aligned} f(0, \tau) = s, \theta(0, \tau) = 1, f_\eta(0, \tau) = 0 \\ \theta(\eta, \tau) \rightarrow 0, f_\eta(\eta, \tau) \rightarrow 1 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (18)$$

Next, the following equations are used to detect the stability of the flow [40]:

$$\theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma\tau} H(\eta, \tau), f(\eta, \tau) = f_0(\eta) + e^{-\gamma\tau} F(\eta, \tau) \quad (19)$$

where $H(\eta, \tau)$ and $F(\eta, \tau)$ are small relative disturbances to $\theta_0(\eta)$ and $f_0(\eta)$, respectively, and γ is the unknown eigenvalue parameter. Using Equation (19) into (16), (18) and letting $\tau \rightarrow 0$, in which $H(\eta) = H_0(\eta)$ and $F(\eta) = F_0(\eta)$, we have the following linearized equation:

$$\frac{\mu_{hcnt}/\mu_f}{\rho_{hcnt}/\rho_f} F_0''' + f_0 F_0'' + F_0 f_0'' - 2f_0' F_0' + \frac{\beta_{hcnt}}{\beta_f} \lambda H_0 + \gamma F_0' = 0 \quad (20)$$

$$\frac{1}{(\rho C_p)_{hcnt}/(\rho C_p)_f} \left(\frac{k_{hcnt}/k_f}{Pr} H_0'' + Q H_0 \right) + f_0 H_0' - f_0' H_0 + \gamma H_0 + F_0 \theta_0' - F_0' \theta_0 = 0 \quad (21)$$

and conditions (18) become:

$$\begin{aligned} F_0'(0) = 0, F_0(0) = 0, H_0(0) = 0, \\ F_0'(\eta) \rightarrow 0, H_0(\eta) \rightarrow 0 \text{ as } \eta \rightarrow \infty \end{aligned} \quad (22)$$

Following the work of Harris et al. [41], one of the boundary conditions need to be relaxed. Hence, we replaced condition $F_0'(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$ with the new condition $F_0''(0) = 1$.

3. Analysis of Results

The ordinary differential Equations (7)–(9) were solved through numerical technique bvp4c in Matlab. Equations (7) and (8) reduce to those obtained by Ghalambaz et al. [25] in the omission of heat generation or absorption effect. The current results obtained are validated with Ramachandran et al. [42] and Ghalambaz et al. [25] in the absence of hybrid nanoparticles ($\varphi = 0$), suction/injection ($s = 0$), and heat source/sink ($Q = 0$) as given in Tables 3 and 4. The current findings are confirmed to be in good agreement with the results of Ramachandran et al. [42] and Ghalambaz et al. [25]. As a result, the proposed mathematical model can be used confidently to scrutinize the fluid flow and heat transfer

of the current problem. Throughout this study, nanoparticle volume fraction (φ_1, φ_2) are taken over the range of $0 \leq \varphi_1, \varphi_2 \leq 0.02$, heat source/sink parameter (Q) over the range of $-0.1 \leq Q \leq 0.1$, the values of mixed convection (λ) are taken over the range of $\lambda_c \leq \lambda \leq 4$ where λ_c is the bifurcation point (critical value) and the Prandtl number Pr is fixed to 6.2 (water), except for comparisons to the previous study. Furthermore, the value of s is set to 0.2 for suction, -0.2 for injection, and 0 for the impermeable case.

Table 3. Values of $f''(0)$ when $\varphi_1 = \varphi_2 = 0, s = 0, Q = 0$.

Pr	Assisting Flow ($\lambda = 1$)			Opposing Flow ($\lambda = -1$)		
	Ref. [42]	Ref. [25]	Present Result	Ref. [42]	Ref. [25]	Present Result
0.7	1.7063	1.7063	1.706323 [1.238720]	0.6917	0.6917	0.691661 [−0.28505]
7.0	1.5179	1.5179	1.517913 [0.582401]	0.9235	0.9235	0.923481 [−0.375337]
20	1.4485	1.4485	1.448483	1.0031	1.0031	1.003108

"[]" Second solution.

Table 4. Comparisons of $-\theta'(0)$ when $\varphi_1 = \varphi_2 = 0, s = 0, Q = 0$.

Pr	Assisting Flow ($\lambda = 1$)			Opposing Flow ($\lambda = -1$)		
	Ref. [42]	Ref. [25]	Present Result	Ref. [42]	Ref. [25]	Present Result
0.7	0.7641	0.7641	0.764063 [1.022633]	0.6332	0.6332	0.633247 [−0.222165]
7.0	−1.7224	−1.7224	1.722382 [2.219194]	1.5460	1.5460	1.546032 [−1.285559]
20	2.4576	2.4576	2.457590	2.2683	2.2683	2.268272

"[]" Second solution.

This section discusses the various outcomes of the current study, which are depicted graphically in Figures 2–14. The behavior of the SWCNT nanoparticle φ_2 for $s = 0.2$ (suction) and $s = -0.2$ (injection) on the velocity $f'(\eta)$ and temperature $\theta(\eta)$ fields are displayed in Figures 2 and 3. The conditions at infinity (9) are clearly verified by these figures. This demonstrates the correctness of our results for the boundary-value problem (7)–(9). Clearly, $f'(\eta)$ is declining against the increasing value of φ_2 while $\theta(\eta)$ is enhanced with it for first and second solutions. Here, the solid line represents the first solution, whereas the second solution is given by the dash line. Additionally, from this figure, it is noticed that the presence of suction effect (red line) results in the decreasing of momentum and thermal boundary layer thickness while the injection effect (blue line) does contrary.

Figure 2. Impact of SWCNT nanoparticle on $f'(\eta)$.

Figure 3. Influence of SWCNT nanoparticle on $\theta(\eta)$.

Figure 4. $C_f Re_x^{1/2}$ for selected value of the nanoparticle when $s = 0.2$.

Figure 5. $C_f Re_x^{1/2}$ for some value of the nanoparticle when $s = -0.2$.

Figure 6. $Nu_x Re_x^{-1/2}$ for different nanoparticles when $s = 0.2$.

Figure 7. $Nu_x Re_x^{-1/2}$ for selected nanoparticles when $s = -0.2$.

Figure 8. Influence of heat source/sink on $f'(\eta)$.

Figure 9. Impact of heat source/sink on $\theta(\eta)$.

Figure 10. $C_f Re_x^{1/2}$ for different heat generation/absorption when $s = 0.2$.

Figure 11. $C_f Re_x^{1/2}$ for different heat generation/absorption when $s = -0.2$.

Figure 12. $Nu_x Re_x^{-1/2}$ for selected value of heat generation/absorption when $s = 0.2$.

Figure 13. $Nu_x Re_x^{-1/2}$ for selected value of heat generation/absorption when $s = -0.2$.

Figure 14. Smallest eigenvalues γ_1 against λ .

Figures 4–7, in turn, illustrate the variation of $C_f Re_x^{1/2}$ (skin friction coefficient) and $Nu_x Re_x^{-1/2}$ (local Nusselt number) with λ (mixed convection parameter) in the existence of heat generation ($Q = 0.1$) for selected values of the nanoparticle volume fraction (ϕ_1 and ϕ_2) when $s = 0.2$ (Figures 4 and 6) and $s = -0.2$ (Figures 5 and 7). From these figures, two numerical solutions were observed for assisting flow ($0 < \lambda \leq 4$) and opposing flow ($\lambda_c < \lambda < 0$), where λ_c is the bifurcation point in which the second and first solutions intersect. In contrast, no solution was discovered when $\lambda < \lambda_c$, whereas one solution was obtained when $\lambda = \lambda_c$. It is spotted that the bifurcation points for SWCNT–MWCNT/water hybrid carbon nanotube ($\phi_1 = \phi_2 = 0.01$) are $\lambda_c = -4.55435$

(suction) and $\lambda_c = -2.04779$ (injection), while the bifurcation points for MWCNT/water nanofluid ($\varphi_1 = 0.01$, $\varphi_2 = 0$) are $\lambda_c = -4.87471$ (suction) and $\lambda_c = -2.05397$ (injection). Additionally, in the absence of nanoparticle volume fraction ($\varphi_1 = \varphi_2 = 0$), the bifurcation points λ_c are -5.25952 (suction) and -2.06981 (injection). It was thus confirmed that the hybrid nanoparticle influenced the acceleration of the boundary layer separation. Furthermore, the presence of suction effect in the flow could hinder the separation of the boundary layer, whereas injection effect facilitates it. Further, it was discovered that $Nu_x Re_x^{-1/2}$ and $C_f Re_x^{1/2}$ increases with an upsurge value of nanoparticle volume fraction except when λ approaches its bifurcation point for the first solution. This implies that hybrid carbon nanotube acts better in the cooling process than MWCNT/water nanofluid and ordinary fluid. Besides, the graphical behavior also shows that assisting flow ($\lambda > 0$) tends to offer higher values of $C_f Re_x^{1/2}$ and $Nu_x Re_x^{-1/2}$ compared to opposing flow ($\lambda < 0$). This is explained by the fact that in the assisting flow, buoyancy force behaves as a favorable pressure gradient, causing the flow to accelerate and, as a result, there is a faster heat transfer rate and larger skin friction coefficient than in the opposing case.

The $f'(\eta)$ (velocity) and $\theta(\eta)$ (temperature) fields with some values of $Q (= -0.1, 0, 0.1)$ for the suction and injection effects in SWCNT-MWCNT/water hybrid carbon nanotube have been displayed in Figures 8 and 9, respectively when $\lambda = -1$ (opposing flow). It is shown in the figure that for the first solution, the transition from $Q = -0.1$ (heat absorption) to $Q = 0.1$ (heat generation) enhanced the momentum and thermal boundary layer thickness, whilst the contrary manner is observed for the second solution. From the definition of Q , the rise of the internal heat of fluid will directly enhance the kinetic energy and thus fluid flow. In addition, it was discovered that the first solution and presence of suction possess thinner boundary layer thicknesses relative to the second solution and presence of injection.

Next, the behavior of Q (transition from heat absorption to heat generation) on $C_f Re_x^{1/2}$ (skin friction coefficient) and $Nu_x Re_x^{-1/2}$ (local Nusselt number) in the existence of suction (Figures 10 and 12) and injection (Figures 11 and 13) were analyzed. Once again, it is clearly seen that dual solution exists for opposing ($\lambda_c < \lambda < 0$) and assisting flow ($0 < \lambda \leq 4$). Surprisingly, the boundary layer separation is affected by the occurrence of heat generation or absorption. The bifurcation points are $\lambda_c = -5.41800$, -4.97950 and -4.55435 for $Q = -0.1, 0$ and 0.1 in the presence of suction, whilst in the presence of injection, the bifurcation points are $\lambda_c = -2.57420$, -2.30529 and -2.04779 . Hence, it is proved that heat absorption and the presence of suction in the flow could delay the boundary layer separation. Besides, upsurge values of Q causes the values of $C_f Re_x^{1/2}$ to enhance for $\lambda > 0$ but depreciate for $\lambda < 0$, whereas the values of $Nu_x Re_x^{-1/2}$ depreciate for all range of λ in the first solution. The heat generation process generates a layer of hot fluid near the surface, and the fluid's temperature eventually exceeds the surface temperature, causing a decline in the rate of heat transfer from the surface.

It should be pointed out that the two solutions are likely to appear for some governing parameters, as shown in Figures 2–13. As a result, it is worthwhile to comment on which of the two solutions is physically relevant. This, however, primarily depends on the stability of the solution. Thus, the linearized Equations (20)–(22) were numerically solved using the bvp4c solver. These linearized equations produce a set of infinite eigenvalues γ such that $\gamma_1 < \gamma_2 < \gamma_3 < \dots$ where γ_1 indicates the smallest eigenvalue. It should be noted that the positive value of γ_1 represents the initial decay of the disturbance, and the flow associated with such a solution is considered stable. Meanwhile, the negative value of γ_1 shows the growth of the smallest disturbance which accordingly cause the solutions to be unstable. Figure 14 depict the variation of smallest eigenvalues γ_1 with mixed convection parameter λ when $Q = 0.1$ and $s = 0.2$ in the hybrid carbon nanotube. It is apparent that the first solution was stable solution since the value of γ_1 is positive, whilst the negative value of γ_1 for the second solution which clearly proved that it is an unstable solution. In addition, as the value of λ approaches its bifurcation point λ_c , the shift from negative (unstable) to positive (stable) values of γ_1 is bound to occur at the turning point ($\lambda = \lambda_c$).

4. Concluding Remarks

The present research contributes to the analysis of SWCNT–MWCNT/water hybrid flow when the plate is vertical with suction/injection and heat generation/absorption effects. The ordinary differential equations were obtained by implementing the similarity transformation and being solved numerically. The conclusions of the current investigation are as follows:

- Two numerical solutions are discoverable for assisting buoyancy flow parameter ($0 < \lambda \leq 4$) and opposing buoyancy flow parameter ($\lambda_c < \lambda < 0$).
- The stability analysis demonstrates that the first solution is stable whilst the second solution is not.
- The domains of the similarity solutions decrease with the use of SWCNT–MWCNT/water hybrid nanofluid, injection effect, and heat generation. Therefore, it fastens the boundary layer separation. However, suction effect and heat absorption parameter delay the boundary layer separation.
- The energy transport rate and skin friction coefficient of SWCNT–MWCNT/water hybrid nanofluid is more advanced than MWCNT/water nanofluid and ordinary fluid.
- Heat absorption enhances the rate of heat transfer while heat generation decreases it.
- The magnitude of local Nusselt number and skin friction coefficient are higher for assisting buoyancy flow than the opposing buoyancy flow case.
- The heat-transfer rate is elevated in the presence of suction effect compared to the presence of injection effect.

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