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Sobolev Regularity of Multilinear Fractional Maximal Operators on Infinite Connected Graphs

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Abstract: Let G be an infinite connected graph. We introduce two kinds of multilinear fractional maximal operators on G. By assuming that the graph G satisfies certain geometric conditions, we establish the bounds for the above operators on the endpoint Sobolev spaces and Hajłasz–Sobolev spaces on G.

Keywords: infinite connected graph; multilinear fractional maximal operator; endpoint Sobolev regularity; Hajłasz–Sobolev space

MSC: Primary 42B25; Secondary 46E35

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1. Introduction

In a very recent article [1], Liu and Zhang introduced the Hajłasz–Sobolev spaces on an infinite connected graph G and established the boundedness for the Hardy–Littlewood maximal operators on G and its fractional variant on the above function spaces and the endpoint Sobolev spaces. The main purpose of this paper is extending the above results to the multilinear setting. More precisely, we introduce two kinds of multilinear fractional maximal operators on G and to establish the bounds for the above operators on the Hajłasz–Sobolev spaces and endpoint Sobolev spaces on G. Although our arguments are greatly motivated by [1], our methods and techniques are more delicate and direct than those in [1]. Particularly, some technique details need to be overcome.

We firstly recall some necessary backgrounds. The centered Hardy–Littlewood maximal operator M is often defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n),$$

where the ball $B(x,r) \subset \mathbb{R}^n$, x is the ball center and r is the ball radius. The uncentered maximal function $\widetilde{M}f$ can be defined similarly. A famous result of harmonic analysis is the Hardy–Littlewood–Wiener theorem, which states that M is of type (p, p) for $1 < p \le \infty$ and of weak type $(1, 1)$. An active topic of current research is the investigation of the regularity properties of maximal operators. About the regularity theory of maximal operators, L^p-bound is one of the basic questions often considered: for $1 < p \le \infty$, whether the following inequality holds

$$\|\nabla Mf\|_{L^p(\mathbb{R}^n)} \le C\|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in W^{1,p}(\mathbb{R}^n), \tag{1}$$

where $W^{1,p}(\mathbb{R}^n)$ is the Sobolev space defined by $W^{1,p}(\mathbb{R}^n) = \{g : g \in L^p(\mathbb{R}^n), \nabla g \in L^p(\mathbb{R}^n)\}$, where ∇g refers to the weak gradient. The first work was due to Kinnunen [2] in 1997 when he established the inequality (1) and showed that M is bounded on $W^{1,p}(\mathbb{R}^n)$

for all $1 < p \leq \infty$. It was noticed that the $W^{1,p}$-bound for the uncentered maximal operator $\widetilde{M}$ also holds by a simple modification of Kinnunen's arguments or ([3], Theorem 1). Since then, Kinnunen's results were extended to a local version in [4], to a fractional version in [5] and to a multisublinear version in [6,7]. Other interesting works related to the regularity of maximal operators in Sobolev spaces and other function spaces are [8,9].

Due to the lack of reflexivity of L^1, the $W^{1,1}$-regularity for M is certainly a more delicate issue. The endpoint regularity of maximal operator has been an active topic of current research. A crucial question related to this topic was posed by Hajłasz and Onninen in [3]:

Question 1. ([3]) Is the map $f \mapsto |\nabla Mf|$ bounded from $W^{1,1}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$?

In the references [10–14], Question 1 in dimension $n = 1$ has been completely solved, and in [15,16], partial progress has been made on this issue for the general dimension $n \geq 2$. In 2002, Tanaka [14] first observed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{M}f$ is weakly differentiable and

$$\|(\widetilde{M}f)'\|_{L^1(\mathbb{R})} \leq 2\|f'\|_{L^1(\mathbb{R})}. \tag{2}$$

Tanaka's result was later sharpened by Aldaz and Pérez Lázaro [10]. The above authors proved that if f is of bounded variation on $\mathbb{R}$, then $\widetilde{M}f$ is absolutely continuous and

$$\mathrm{Var}(\widetilde{M}f) \leq \mathrm{Var}(f), \tag{3}$$

where $\mathrm{Var}(f)$ denotes the total variation of f on $\mathbb{R}$. This yields

$$\|(\widetilde{M}f)'\|_{L^1(\mathbb{R})} \leq \|f'\|_{L^1(\mathbb{R})} \tag{4}$$

if $f \in W^{1,1}(\mathbb{R})$. Notice that the constant $C = 1$ in inequalities (3) and (4) is sharp. Inequality (2) was recently extended to a fractional setting in ([17], Theorem 1) and to a multisublinear fractional setting in ([18], Theorems 1.3–1.4). Very recently, Carneiro et al. [19] proved that the map $f \mapsto (\widetilde{M}f)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. In the centered setting, Kurka [12] showed that if f is of bounded variation on $\mathbb{R}$, then inequality (3) holds for M (with constant $C = 240,004$). It was also shown in [12] that if $f \in W^{1,1}(\mathbb{R})$, then Mf is weakly differentiable and (2) holds for M with $C = 240{,}004$. It is currently unknown whether inequality (4) holds for M and the map $f \mapsto (Mf)'$ is continuous from $W^{1,1}(\mathbb{R})$ to $L^1(\mathbb{R})$. Recently, Beltran and Madrid [15] extended Kurka's result to the fractional version. Other interesting works can be found in [11,13,20–27], among others.

Next, we introduce the basic knowledge of graphs and the regularity properties of maximal operators on the graph settings. We assume that $G = (V_G, E_G)$ is the undirected combinatorial graph, where V_G denotes the set of vertices and E_G denotes the set of edges. Two vertices u, $v \in V_G$ are said to be neighbors if they are connected by an edge $u \sim v \in E_G$. We define $N_G(u)$ as the the set of neighbors of $u \in V_G$. The graph $G = (V_G, E_G)$ is said to be finite (resp., infinite) if $|V_G| < +\infty$ (resp., $|V_G| = +\infty$). The graph $G = (V_G, E_G)$ is said to be connected if there exists a finite sequence of vertices $\{u_i\}_{i=0}^{k}$, $k \in \mathbb{N} \setminus \{0\}$, so that $u = u_0 \sim u_1 \sim \cdots \sim u_k = v$, for any distinct u, $v \in V_G$, where $\mathbb{N}$ is the set of $\{0, 1, \ldots\}$.

In this paper, we always suppose that $G = (V_G, E_G)$ is an infinite connected graph. We use d_G to represent the metric induced by the edges in E_G, that is, for the given $u, w \in V_G$, we define the distance $d_G(u, w)$ by the number of edges in a shortest path connecting u and w. $B_G(u, t)$ represents the ball whose center is u and whose radius is t, i.e.,

$$B_G(u,t) = \{w \in V_G : d_G(u,w) \leq t\}.$$

For instance,

$$B_G(u,t) = \begin{cases} \{u\}, & \text{if } 0 \leq t < 1; \\ \{u\} \cup N_G(u), & \text{if } 1 \leq t < 2. \end{cases}$$

Set $S_G(v,r) = \{u \in V_G : d_G(u,v) = r\}$, and the notation $|A|$ means the cardinality of $A \subset V_G$.

Then, let us introduce two types of multilinear fractional maximal operators on the infinite connected graphs $G = (V_G, E_G)$.

Definition 1. *Suppose that $\alpha \geq 0$, $\kappa \geq 1$, $m \geq 1$ and the vector-valued function $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j : V_G \to \mathbb{R}$, the multilinear fractional maximal operator associated with $\vec{f}$ on G is defined by*

$$\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(u) = \sup_{t>0} |B_G(u,t)|^{\alpha} \prod_{j=1}^{m} \frac{1}{|B_G(u,\kappa t)|} \sum_{v \in B_G(u,t)} |f_j(v)|.$$

Another version is given by

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(u) = \sup_{t>0} t^{\alpha} \prod_{j=1}^{m} \frac{1}{|B_G(u,\kappa t)|} \sum_{v \in B_G(u,t)} |f_j(v)|.$$

Obviously, $d_G(u,v)$ can only be natural numbers. Then, the above two types of operators can be defined as follows; just take $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})$ as an example:

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(u) = \sup_{t \in \mathbb{N}} t^{\alpha} \prod_{j=1}^{m} \frac{1}{|B_G(u,\kappa t)|} \sum_{v \in B_G(u,t)} |f_j(v)|.$$

If $\kappa = 1$, we denote $\mathfrak{M}^{\kappa}_{\alpha,G} = \mathfrak{M}_{\alpha,G}$ and $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G} = \widetilde{\mathfrak{M}}_{\alpha,G}$. If $m = 1$, we denote $\mathfrak{M}^{\kappa}_{\alpha,G} = M^{\kappa}_{\alpha,G}$ and $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G} = \widetilde{M}^{\kappa}_{\alpha,G}$. When $\kappa = 1$, we denote $M^{\kappa}_{\alpha,G} = M_{\alpha,G}$ and $\widetilde{M}^{\kappa}_{\alpha,G} = \widetilde{M}_{\alpha,G}$. These operators $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ were firstly introduced by Liu and Zhang [1].

When $\alpha = 0$, the operators $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ reduce to the usual Hardy–Littlewood maximal operator on G, which is denoted by M_G. This type of maximal operator has been studied by many authors (see [28–32]), and the auhtors obtained a lot of wonderful results. See the literature here; we will not describe them one by one.

In fact, one can find the root of $\mathfrak{M}^{\kappa}_{\alpha,G}$ in the discrete harmonic analysis. Let $m \geq 1$, $0 \leq \alpha < m$ and $\kappa = 1$. Assume that $G_1 = (V_{G_1}, E_{G_1})$, where $V_{G_1} = \mathbb{Z}$ and $E_{G_1} = \{j \sim j+1 : j \in \mathbb{Z}\}$. $\mathfrak{M}^{\kappa}_{\alpha,G_1}$ is actually the usual one-dimensional discrete centered multilinear fractional maximal operator $\mathfrak{M}_{\alpha}$, i.e.,

$$\mathfrak{M}_{\alpha}(\vec{f})(n) = \sup_{r \in \mathbb{N}} \frac{1}{(2r+1)^{m-\alpha}} \prod_{j=1}^{m} \sum_{k=-r}^{r} |f_j(n+k)|, \quad n \in \mathbb{Z}.$$

When $\alpha = 0$, the operator $\mathfrak{M}^{\kappa}_{\alpha,G_1}$ means the usual one-dimensional discrete centered multilinear maximal operator $\mathfrak{M}$, i.e.,

$$\mathfrak{M}(\vec{f})(n) = \sup_{r \in \mathbb{N}} \frac{1}{(2r+1)^{m}} \prod_{j=1}^{m} \sum_{k=-r}^{r} |f_j(n+k)|, \quad n \in \mathbb{Z}.$$

Many authors have investigated the regularity properties of $\mathfrak{M}$ and $\mathfrak{M}_{\alpha}$ (for more details, see [33,34]).

In order to generalize results on $\mathbb{R}^n$ and its discrete setting to the graph setting, Liu and Xue [35] introduced the first-order Sobolev spaces on graphs and studied the Sobolev regularity of the Hardy–Littlewood maximal operator on a finite connected graph. Let us recall some definitions.

Definition 2. *For $0 < p \leq \infty$ and $G = (V_G, E_G)$, the Lebesgue space $L^p(V_G)$ consists of the functions $f : V_G \to \mathbb{R}$ satisfying $\|f\|_{L^p(V_G)} = (\sum_{u \in V_G} |f(u)|^p)^{1/p} < \infty$ for all $0 < p < \infty$ and $\|f\|_{L^{\infty}(V_G)} = \sup_{u \in V_G} |f(u)|$.*

Definition 3. *Denote $W^{1,p}(V_G)$ the first-order Sobolev space on $G = (V_G, E_G)$, it can be defined as follows for $1 \le p \le \infty$:*

$$W^{1,p}(V_G) := \{f : V_G \to \mathbb{R};\ \|f\|_{W^{1,p}(V_G)} := \|f\|_{L^p(V_G)} + \||\nabla f|\|_{L^p(V_G)} < \infty\},$$

where $|\nabla f|(u) := \Big(\sum_{v \in N_G(u)} |f(v) - f(u)|^2\Big)^{1/2}$, for $u \in V_G$.

It is not difficult to get that

$$\|f\|_{L^p(V_G)} \le \|f\|_{1,p} \le (2|V_G| - 1)\|f\|_{L^p(V_G)}, \quad 1 \le p \le \infty, \tag{5}$$

if the graph G is a finite connected graph.

According to (5), one can note that the space $W^{1,p}(V_G)$ is acctually the Lebesgue space $L^p(V_G)$ with an equivalent norm. The relationship between $W^{1,p}(V_G)$ and $L^p(V_G)(L^p(V_G) \subset W^{1,p}(V_G))$ is obvious, if $G = (V_G, E_G)$ is an infinite connected graph. However, generally speaking, the inclusion relation $L^p(V_G) \subset W^{1,p}(V_G)$ is not valid. As a matter of fact, we can cite a counterexample to illustrate this fact. Set $V_G = \mathbb{N}$, $E_G = \{0 \sim i : i \in \mathbb{N} \setminus \{0\}\}$ and $f(k) = \chi_{\{0\}}(k)$, $k \in \mathbb{N}$. It is easy to know $\|f\|_{L^p(V_G)} = 1$ and $\||\nabla f|\|_{L^p(V_G)} = +\infty$ for all $1 \le p \le \infty$; then, one can have

$$L^p(V_G) = W^{1,p}(V_G), \quad 1 \le p \le \infty, \tag{6}$$

if

$$\Delta_G := \sup_{v \in V_G} |N_G(v)| < +\infty, (\mathcal{UBD})$$

where the condtion $(\mathcal{UBD})$ is called the uniformly bounded degree conditon (for the proof of (6), see [1]). Therefore, under the $(\mathcal{UBD})$ condition, the boundedness of maximal operators on $W^{1,p}(V_G)$ is equivalent to the property of maximal operators on $L^p(V_G)$.

Recently, one of the authors and Xue [35] showed

$$\||\nabla M_G f|\|_{L^p(V_G)} \le C_{p,n} \||\nabla f|\|_{L^p(V_G)}, \quad 1 \le p \le \infty,$$

when $G = (V_G, E_G)$ is a finite connected graph with n vertices. When $G = (V_G, E_G)$ is an infinite connected graph, in [1], the authors studied the endpoint Sobolev regularity of the fractional maximal operator on G. More precisely, if G satisfies certain geometric conditions, they showed that

$$\max\{\||\nabla M_{\alpha,G} f|\|_{L^1(V_G)}, \||\nabla \widetilde{M}_{\alpha,G} f|\|_{L^1(V_G)}\} \le C\|f\|_{L^1(V_G)}.$$

The motivation of this paper is to develop the above results for the multilinear setting. More precisely, we shall prove that

$$\max\{\||\nabla \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)}, \||\nabla \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)}\} \le C \prod_{j=1}^{m} \|f_j\|_{L^1(V_G)},$$

provided that G satisfies certain geometric conditions. These results and their proofs can be found in Section 3. In Section 2, we give the proof of L^p boundedness of the multilinear maximal operator on graphs and its fractional variants on graphs. These together with (6) will lead to the bounds for the above operators on the Sobolev spaces. In Sections 3 and 4, for the multilinear maximal operator and its fractional variants on graphs, we establish their boundedness on endpoint Sobolev spaces and on the Hajłasz–Sobolev spaces, respectively.

In this article, we often use the following notation

$$f_B = \frac{1}{|B|} \sum_{v \in B} f(v)$$

for any arbitrary function $f : V_G \to \mathbb{R}$ and any subset B of V_G. Throughout this article, letters C or $C_{\alpha,\beta,\cdots}$ will denote positive constants that may change from one instance to another and depend on parameters α, β, $\cdots$ involved.

2. Boundedness on Lebesgue Spaces

Firstly, in this section, we want to study the bounds of the multilinear fractional maximal operators on Lebesgue spaces. We begin with some geometric conditions on graphs.

Definition 4. *Let $G = (V_G, E_G)$.*

(i) *G is said to be doubling condition if*

$$\mathcal{D}(G) := \sup\left\{\frac{|B_G(x,2t)|}{|B_G(x,t)|} : \ x \in V_G, \ t \in \mathbb{N}\right\} < \infty.(\mathcal{D})$$

(ii) *G is said to satisfy the lower bound condition if there is a constant $Q \geq 1$, such that*

$$\mathcal{B}_{1,Q} := \inf_{x\in V_G,\, t\in\mathbb{N}\setminus\{0\}} \frac{|B_G(x,t)|}{t^Q} > 0.(\mathcal{LB}-\mathcal{Q})$$

(iii) *G is said to satisfy the upper bound condition if there is a constant $Q \geq 1$, such that*

$$\mathcal{B}_{2,Q} := \sup_{x\in V_G,\, t\in\mathbb{N}\setminus\{0\}} \frac{|B_G(x,t)|}{t^Q} < \infty.(\mathcal{UB}-\mathcal{Q})$$

(iv) *Set $0 < \delta \leq 1$. G is said to satisfy the δ-annular decay property if*

$$\mathcal{B}_{3,\delta} := \sup_{\substack{x\in V_G,\\ s,t\in\mathbb{N}\setminus\{0\}, s<t}} \frac{|B_G(x,t)| - |B_G(x,t-s)|}{|B_G(x,t)|}\left(\frac{t}{s}\right)^{\delta} < \infty.(\mathcal{ADP}-\circ)$$

(v) *G is said to satisfy the upper bounded sphere condition if there is a constant $\xi > 0$, such that*

$$\mathcal{B}_{4,\xi} := \sup_{x\in V_G,\, t\in\mathbb{N}\setminus\{0\}} \frac{|S_G(x,t)|}{t^{\xi}} < \infty.(\mathcal{UBS}-\sim)$$

It was pointed out in [1] that the following facts are valid.

Remark 1.

(i) *If $\Delta_G \leq \mathcal{B}_{4,\xi}$, $(\mathcal{UBS}-\sim)$ can deduce $(\mathcal{UBD})$, but $(\mathcal{UBD})$ cannot deduce $(\mathcal{UBS}-\sim)$.*

(ii) *$(\mathcal{UBS}-\sim)$ may imply $(\mathcal{UB}-\mathcal{Q})$ with $Q = \xi + 1$ and $\mathcal{B}_{2,Q} \leq \frac{2^{\xi+1}}{\xi+1}\mathcal{B}_{4,\xi}$, as well as the condition $(\mathcal{UB}-\mathcal{Q})$ means $(\mathcal{UBS}-\sim)$ where $\xi \geq Q$ and $\mathcal{B}_{4,\xi} \leq \mathcal{B}_{2,Q}$.*

(iii) *Obviously, if $0 < \delta_1 \leq \delta_2 \leq 1$, $(\mathcal{ADP}-\circ_2)$ means $(\mathcal{ADP}-\circ_1)$.*

(iv) *There exsits some $\delta \in (0,1]$ satisfying $\mathcal{B}_{3,\delta} < 2^{\delta}$ so that $(\mathcal{ADP}-\circ)$ means $(\mathcal{D})$ with $\mathcal{D}(G) \leq \frac{2^{\delta}}{2^{\delta}-\mathcal{B}_{3,\delta}}$.*

When the graph $G = (V_G, E_G)$ satisfies $(\mathcal{D})$, it is easy to check that

$$\mathcal{D}(G)^{-[\log_2^{\kappa}]-1} M_{\alpha,G} f(v) \leq M_{\alpha,G}^{\kappa} f(v) \leq M_{\alpha,G} f(v), \quad \forall v \in V_G,$$

$$\mathcal{D}(G)^{-[\log_2^{\kappa}]-1} \widetilde{M}_{\alpha,G} f(v) \leq \widetilde{M}_{\alpha,G}^{\kappa} f(v) \leq \widetilde{M}_{\alpha,G} f(v), \quad \forall v \in V_G.$$

In [1], the authors established the boundedness of the fractional maximal operator $M_{\alpha,G}$ and $\widetilde{M}_{\alpha,G}$ on L^p. This together with the above estimates implies the following theorem.

Theorem 1. *Suppose that $\kappa \geq 1$, $1 < p < \infty$ and $G = (V_G, E_G)$ satisfies $(\mathcal{D})$.*

(i) *If $0 \le \alpha \le 1/p$ and $q = p/(1-p\alpha)$, then for $f \in L^p(V_G)$ we have*

$$\|M^{\kappa}_{\alpha,G} f\|_{L^q(V_G)} \le C\|f\|_{L^p(V_G)},$$

where C depends on $\alpha, p, \mathcal{D}(G)$.

(ii) *Assume that $G = (V_G, E_G)$ satisfies $(\mathcal{LB} - \mathcal{Q})$, $Q \ge 1$, $0 \le \alpha \le Q/p$ and $q = pQ/(Q - \alpha p)$, then for $f \in L^p(V_G)$ we have*

$$\|\widetilde{M}^{\kappa}_{\alpha,G} f\|_{L^q(V_G)} \le C\|f\|_{L^p(V_G)},$$

where C depends on $\alpha, p, Q, \mathcal{D}(G), \mathcal{B}_{1,Q}$.

Applying above theorem, we can obtain the following result immediately.

Theorem 2. *Assume that $\kappa \ge 1$, $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^{p_j}(V_G)$ for $1 < p_j < \infty$ and $G = (V_G, E_G)$ satisfies $(\mathcal{D})$.*

(i) *Suppose that $0 \le \alpha \le \sum_{i=1}^m 1/p_i$, $1/q = \sum_{i=1}^m 1/p_i - \alpha \le 1$, we have*

$$\|\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})\|_{L^q(V_G)} \le C_{\alpha,p_1,\ldots,p_m,\mathcal{D}(G)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(V_G)}.$$

(ii) *Let $Q \ge 1$ and G satisfy $(\mathcal{LB} - \mathcal{Q})$. If $0 \le \alpha \le \sum_{i=1}^m Q/p_i$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/Q \le 1$, then*

$$\|\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})\|_{L^q(V_G)} \le C_{\alpha,p_1,\ldots,p_m,Q,\mathcal{B}_{1,Q},\mathcal{D}(G)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(V_G)}.$$

Proof. Let $1/q = 1/q_1 + \cdots + 1/q_m$, where $1 < q_i < \infty$, $1/q_i = 1/p_i - \alpha_i$, $0 \le \alpha_i \le 1/p_i$ and $\alpha = \sum_{i=1}^m \alpha_i$. For all $x \in V_G$ and $\kappa \ge 1$, apparently, we have

$$\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x) \le \prod_{j=1}^m M^{\kappa}_{\alpha_j,G} f_j(x), \ \ \forall x \in V_G.$$

This together with Hölder's inequality and Theorem 1(i) implies that

$$\|\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})\|_{L^q(V_G)} \le \prod_{j=1}^m \|M^{\kappa}_{\alpha_j,G} f_j\|_{L^{q_j}(V_G)} \le C_{\alpha,p_1,\ldots,p_m,\mathcal{D}(G)} \prod_{j=1}^m \|f_j\|_{L^p(V_G)},$$

which proves part (i).

It remains to prove part (ii). Let $1/q = 1/q_1 + \cdots + 1/q_m$, where $1 < q_i < \infty$, $1/q_i = 1/p_i - \alpha_i/Q$, $0 \le \alpha_i \le Q/p_i$ and $\alpha = \sum_{i=1}^m \alpha_i$. For all $\kappa \ge 1$, it is easy to check that

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) \le \prod_{j=1}^m \widetilde{M}^{\kappa}_{\alpha_j,G} f_j(x), \ \ \forall x \in V_G,$$

which together with Hölder's inequality and Theorem 1(ii) implies the conclusion of part (ii). □

Applying Theorem 2 and (6), we have the following regularity properties for the multilinear maximal operator and its fractional variant.

Corollary 1. *Let $\kappa \ge 1$, $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^{p_j}(V_G)$ for $1 < p_j < \infty$ and $G = (V_G, E_G)$ satisfy $(\mathcal{D})$ and $(\mathcal{UBD})$.*

(i) *Suppose that* $0 \le \alpha \le \sum_{i=1}^m 1/p_i$, $1/q = \sum_{i=1}^m 1/p_i - \alpha \le 1$, *we have*

$$\|\mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})\|_{W^{1,q}(V_G)} \le C_{\alpha,p_1,\dots,p_m,\mathcal{D}} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(V_G)}.$$

(ii) *Let* $Q \ge 1$ *and* $G = (V_G, E_G)$ *satisfy* $(\mathcal{LB} - \mathcal{Q})$. *If* $0 \le \alpha \le \sum_{i=1}^m Q/p_i$ *and* $1/q = \sum_{i=1}^m 1/p_i - \alpha/Q \le 1$, *then*

$$\|\widetilde{\mathfrak{M}}_{\alpha,G}^{\kappa}(\vec{f})\|_{W^{1,q}(V_G)} \le C_{\alpha,p_1,\dots,p_m,Q,\mathcal{B}_{1,Q},\mathcal{D}} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(V_G)}.$$

3. Endpoint Sobolev Regularity of Two Classes of Maximal Operators

Compared with the results of Section 2, this section is devoted to establishing the endpoint Sobolev regularity for the multilinear maximal operator and its fractional variant. Let us firstly introduce the following result.

Theorem 3. *Aussme that* $G = (V_G, E_G)$, $Q \ge 1$, $\kappa \ge 1$, $0 < \delta \le 1$, $0 \le \alpha < m$ *and* $0 < \xi < Q(m-\alpha) + \delta - 1$. *If the graph G satisfies* $(\mathcal{D})$, $(\mathcal{LB} - \mathcal{Q})$, $(\mathcal{ADP} - \delta)$ *and* $(\mathcal{UBS} - \xi)$, *we have*

$$\||\nabla \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})|\|_{L^1(V_G)} \le C \prod_{j=1}^m \|f_j\|_{L^1(V_G)}$$

which holds for all $\vec{f} = (f_1, \dots, f_m)$ *with each* $f_j \in L^1(V_G)$, *and C depends on* $\alpha, Q, \delta, m, \xi, \mathcal{D}, \mathcal{B}_{1,Q}$, $\mathcal{B}_{3,\delta}$, $\mathcal{B}_{4,\xi}$.

Proof. From the definotion of $\mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})$, we know that there must be a positve integer r_u, such that

$$\mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(u) = |B_G(u, r_u)|^{\alpha} \prod_{j=1}^m \frac{1}{|B_G(u, \kappa r_u)|} \sum_{w \in B_G(u, r_u)} |f_j(v)|$$

for $f_j \in L^1(V_G)$ and any $v \in V_G$. Apparently by the definiton of $|\nabla f|$, for fixed $x \in V_G$, we can write

$$\begin{aligned}
|\nabla \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x)| &\le \sum_{y \in N_G(x)} |\mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x) - \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y)| \\
&= \sum_{y \in I_1(x)} \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x) - \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y)| + \sum_{y \in I_2(x)} \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y) - \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x)| \\
&=: II_1 + II_2,
\end{aligned}$$

where we set

$$\begin{aligned}
I_1(x) &:= \{y \in N_G(x) : \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x) > \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y)\}, \\
I_2(x) &:= \{y \in N_G(x) : \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x) < \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y)\}.
\end{aligned}$$

We first analyze II_1. Fixing $y \in I_1(x)$, we have

$$\begin{aligned}
\mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(y) &\ge |B_G(y, r_x + 1)|^{\alpha} \prod_{j=1}^m \frac{1}{|B_G(y, \kappa(r_x + 1))|} \sum_{w \in B_G(y, r_x + 1)} |f_j(w)| \\
&\ge |B_G(x, r_x)|^{\alpha} \frac{|B_G(x, \kappa r_x)|^m}{|B_G(y, \kappa(r_x + 1))|^m} \prod_{j=1}^m \frac{1}{|B_G(x, \kappa r_x)|} \sum_{w \in B_G(x, r_x)} |f_j(w)| \\
&\ge \frac{|B_G(x, \kappa r_x)|^m}{|B_G(y, \kappa(r_x + 1))|^m} \mathfrak{M}_{\alpha,G}^{\kappa}(\vec{f})(x),
\end{aligned}$$

which gives

$$
\begin{aligned}
&\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(y)\\
&\le \Big(1 - \frac{|B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\Big)\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x)\\
&\le \frac{|B_G(y,\kappa(r_x+1))|^m - |B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\frac{|B_G(x,r_x)|^\alpha}{|B_G(x,\kappa r_x)|^m}\prod_{j=1}^m \sum_{w\in B_G(x,r_x)} |f_j(w)|.
\end{aligned}
$$

By $(\mathcal{ADP}-\delta)$ and $(\mathcal{D})$, we have

$$
\begin{aligned}
&\frac{|B_G(y,\kappa(r_x+1))|^m - |B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\\
&\le \frac{|B_G(x,\kappa(r_x+2))|^m - |B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\\
&= \frac{|B_G(x,\kappa(r_x+2))| - |B_G(x,\kappa r_x)|}{|B_G(y,\kappa(r_x+1))|^m}\\
&\quad\times(|B_G(x,\kappa(r_x+2))|^{m-1} + |B_G(x,\kappa(r_x+2))|^{m-2}|B_G(x,\kappa r_x)| + \cdots + |B_G(x,\kappa r_x)|^{m-1})\\
&\le m\mathcal{B}_{3,\delta}\Big(\frac{2}{r_x+2}\Big)^\delta \frac{|B_G(x,\kappa(r_x+2))|^m}{|B_G(y,\kappa(r_x+1))|^m}\\
&\le m\mathcal{B}_{3,\delta}\Big(\frac{2}{r_x+2}\Big)^\delta \frac{|B_G(y,\kappa(r_x+3))|^m}{|B_G(y,\kappa(r_x+1))|^m}\\
&\le m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2\Big(\frac{2}{r_x+2}\Big)^\delta.
\end{aligned}
\tag{7}
$$

Applying $(\mathcal{LB}-\mathcal{Q})$, one has

$$
|B_G(x,\kappa r_x)| \ge \max\{\mathcal{B}_{1,Q}(\kappa r_x)^Q, 1\} \ge \mathcal{B}_{1,Q}(2+\mathcal{B}_{1,Q})^{-Q}(\kappa r_x+1)^Q. \tag{8}
$$

In view of (7) and (8), we have that for any $y \in I_1(x)$,

$$
\begin{aligned}
&\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(y)\\
&\le m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2\Big(\frac{2}{r_x+2}\Big)^\delta\Big(\mathcal{B}_{1,Q}(2+\mathcal{B}_{1,Q})^{-Q}(\kappa r_x+1)^Q\Big)^{\alpha-m}\prod_{j=1}^m\sum_{w\in B_G(x,r_x)}|f_j(w)|\\
&\le m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 2^\delta \mathcal{B}_{1,Q}^{\alpha-m}(2+\mathcal{B}_{1,Q})^{Q(m-\alpha)}(r_x+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^m\sum_{w\in B_G(x,r_x)}|f_j(w)|.
\end{aligned}
\tag{9}
$$

For II_2 similar to II_1, we obtain

$$
\begin{aligned}
&\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(y) - \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x)\\
&\le m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 2^\delta \mathcal{B}_{1,Q}^{\alpha-m}(2+\mathcal{B}_{1,Q})^{Q(m-\alpha)}(r_y+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^m\sum_{w\in B_G(y,r_y)}|f_j(w)|.
\end{aligned}
\tag{10}
$$

From (9), (10) and the inclusion relation $I_i(x) \subset N_G(x), i = 1,2$ and Remark 1 (i), we have

$$
\begin{aligned}
|\nabla \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x)| \le\ & C_{\alpha,Q,\delta,m,\mathcal{D},\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta}}\Big(\sum_{y\in I_1(x)}(r_x+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_x)}|f_j(w)| \\
&+\sum_{y\in I_2(x)}(r_y+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(y,r_y)}|f_j(w)|\Big) \\
\le\ & C_{\alpha,Q,\delta,m,\mathcal{D}(G),\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta}}\Big((r_x+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_x)}|f_j(w)| \\
&+(r_y+1)^{Q(\alpha-m)-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_y+1)}|f_j(w)|\Big) \\
\le\ & C_{\alpha,Q,\delta,m,\mathcal{D}(G),\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta}}\prod_{j=2}^{m}\|f_j\|_{L^1(V_G)}\Big((r_x+1)^{Q(\alpha-m)-\delta}\sum_{w\in B_G(x,r_x)}|f_1(w)| \\
&+(r_y+1)^{Q(\alpha-m)-\delta}\sum_{w\in B_G(x,r_y+1)}|f_1(w)|\Big).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\||\nabla \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)} \le\ & C_{\alpha,Q,\delta,m,\mathcal{D}(G)),\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta}}\prod_{j=2}^{m}\|f_j\|_{L^1(V_G)} \\
&\times\Big(\sum_{x\in V_G}\sum_{w\in B_G(x,r_x)}|f_1(w)|(r_x+1)^{Q(\alpha-m)-\delta} \\
&+\sum_{x\in V_G}\sum_{w\in B_G(x,r_y+1)}|f_1(w)|(r_y+1)^{Q(\alpha-m)-\delta}\Big).
\end{aligned} \tag{11}
$$

Notice that

$$
\begin{aligned}
&\sum_{x\in V_G}\sum_{w\in B_G(x,r_x)}|f_1(w)|(r_x+1)^{Q(\alpha-m)-\delta} \\
&\le \sum_{x\in V_G}\sum_{w\in V_G}|f_1(w)|\chi_{d_G(w,x)\le r_x}(w)(r_x+1)^{Q(\alpha-m)-\delta} \\
&\le \sum_{w\in V_G}|f_1(w)|\sum_{x\in V_G}(d_G(w,x)+1)^{Q(\alpha-m)-\delta} \\
&\le \|f_1\|_{L^1(V_G)}\sup_{w\in V_G}\sum_{x\in V_G}(d_G(w,x)+1)^{Q(\alpha-m)-\delta}.
\end{aligned}
$$

On the other hand, one has

$$
\begin{aligned}
&\sum_{x\in V_G}\sum_{w\in B_G(x,r_y+1)}|f_1(w)|(r_y+1)^{Q(\alpha-m)-\delta} \\
&\le \sum_{w\in V_G}\sum_{x\in V_G}|f_1(w)|\chi_{d_G(w,x)\le r_y+1}(w)(r_y+1)^{Q(\alpha-m)-\delta} \\
&\le \sum_{w\in V_G}|f_1(w)|\sum_{x\in V_G}d_G(w,x)^{Q(\alpha-m)-\delta} \\
&\le \|f_1\|_{L^1(V_G)}\sup_{w\in V_G}\sum_{x\in V_G}d_G(w,x)^{Q(\alpha-m)-\delta}.
\end{aligned}
$$

Hence, we get from (11) that

$$
\begin{aligned}
&\||\nabla \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)} \\
&\le C_{\mathcal{D}(G),\alpha,\delta,Q,m,\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta}}\prod_{j=1}^{m}\|f_j\|_{L^1(V_G)}\sup_{w\in V_G}\sum_{x\in V_G}d_G(w,x)^{Q(\alpha-m)-\delta}.
\end{aligned} \tag{12}
$$

Fixing $w \in V_G$, by $(\mathcal{UBS} - \xi)$ and the fact that $Q(m-\alpha)+\delta > \xi+1$,

$$\begin{aligned}&\sum_{x\in V_G} d_G(w,x)^{Q(\alpha-m)-\delta}\\ &\leq \sum_{k=0}^{\infty} \sum_{x\in V_G, d_G(w,x)=k} k^{Q(\alpha-m)-\delta}\\ &\leq \mathcal{B}_{4,\tau} \sum_{k=0}^{\infty} k^{Q(\alpha-m)-\delta+\xi} \leq C_{\alpha,m,Q,\delta,\xi,\mathcal{B}_{4,\xi}}.\end{aligned} \tag{13}$$

Combining (13) with (12) implies that

$$\| |\nabla \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})| \|_{L^1(V_G)} \leq C_{\mathcal{D}(G),\alpha,\delta,Q,m,\xi,\mathcal{B}_{1,Q},\mathcal{B}_{3,\delta},\mathcal{B}_{4,\xi}} \prod_{j=1}^{m} \|f_j\|_{L^1(V_G)}.$$

Theorem 3 is complete. □

We then give the following theorem.

Theorem 4. *Assume that $\kappa \geq 1$, $Q \geq 1$, $0 < \delta \leq 1$, $0 \leq \alpha < m$ and $0 < \tau < Qm + \delta - \alpha - 1$. Suppose that $G = (V_G, E_G)$ satisfies $(\mathcal{D})$, $(\mathcal{LB} - \mathcal{Q})$, $(\mathcal{ADP} - \delta)$ and $(\mathcal{UBS} - \tau)$. Then,*

$$\| |\nabla \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})| \|_{L^1(V_G)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^1(V_G)}$$

holds for all $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^1(V_G)$, and C depends on $\alpha, Q, \delta, m, \tau, \mathcal{D}, \mathcal{B}_{1,Q}, \mathcal{B}_{3,\delta}$, $\mathcal{B}_{4,\tau}$.

Proof. The proof of Theorem 4 is similar to Thereom 3. Here, we just give a partial derivation for completeness. From the definotion of $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})$, we know that there must be a positve integer r_u, such that

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(u) = r_u^{\alpha} \prod_{j=1}^{m} \frac{1}{|B_G(u,\kappa r_u)|} \sum_{v\in B_G(u,r_u)} |f_j(v)|,$$

for $f_j \in L^1(V_G)$ and any $v \in V_G$. We can write by definition

$$\begin{aligned}|\nabla \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)| &\leq \sum_{y\in N_G(x)} |\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)|\\ &= \sum_{y\in J_1(x)} (\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)) + \sum_{y\in J_2(x)} (\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x))\\ &=: JJ_1 + JJ_2,\end{aligned}$$

where for fixed $x \in V_G$, we denote

$$J_1(x) := \{y \in N_G(x) : \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) > \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)\},$$

$$J_2(x) := \{y \in N_G(x) : \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) < \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)\}.$$

We first analyze JJ_1, and for fixed $y \in J_1(x)$, we have

$$\begin{aligned}\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y) &\geq (r_x+1)^{\alpha}\prod_{j=1}^{m}\frac{1}{|B_G(y,\kappa(r_x+1))|}\sum_{w\in B_G(y,r_x+1)}|f_j(w)|\\ &\geq r_x^{\alpha}\frac{|B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\prod_{j=1}^{m}\frac{1}{|B_G(x,\kappa r_x)|}\sum_{w\in B_G(x,r_x)}|f_j(w)|\\ &\geq \frac{|B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x),\end{aligned}$$

which leads to

$$\begin{aligned}&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)-\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)\\ &\leq \Big(1-\frac{|B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\Big)\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x)\\ &\leq \frac{|B_G(y,\kappa(r_x+1))|^m-|B_G(x,\kappa r_x)|^m}{|B_G(y,\kappa(r_x+1))|^m}\frac{r_x^{\alpha}}{|B_G(x,\kappa r_x)|^m}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_x)}|f_j(w)|.\end{aligned}$$

In view of (7) and (8), we have that for $y \in J_1(x)$,

$$\begin{aligned}&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)-\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)\\ &\leq m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2\Big(\frac{2}{r_x+2}\Big)^{\delta}\Big(\mathcal{B}_{1,Q}(2+\mathcal{B}_{1,Q})^{-Q}(\kappa r_x+1)^Q\Big)^{-m}r_x^{\alpha}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_x)}|f_j(w)|\\ &\leq m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 2^{\delta}\mathcal{B}_{1,Q}^{-m}(2+\mathcal{B}_{1,Q})^{Qm}(r_x+1)^{\alpha-Qm-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(x,r_x)}|f_j(w)|.\end{aligned}\tag{14}$$

Similar to JJ_2, we obtain that

$$\begin{aligned}&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)-\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)\\ &\leq m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 2^{\delta}\mathcal{B}_{1,Q}^{-m}(2+\mathcal{B}_{1,Q})^{Qm}(r_y+1)^{\alpha-Qm-\delta}\prod_{j=1}^{m}\sum_{w\in B_G(y,r_y)}|f_j(w)|.\end{aligned}\tag{15}$$

By using (14) and (15) and the arguments similar to those used for the proof of Theorem 3, we can obtain the conclusion of Theorem 4. The details are omitted. □

From (iv) of Remark 1 together with the above two theorems, we have

Corollary 2. *Assume that $G=(V_G,E_G)$, $\kappa \geq 1$, $Q \geq 1$, $0 \leq \alpha < m$, $0 < \delta \leq 1$, and assume that G satisfies $(\mathcal{LB}-\mathcal{Q})$, $(\mathcal{UBS}-\tau)$ and $(\mathcal{ADP}-\delta)$ with $\mathcal{B}_{3,\delta} < 2^{\delta}$. Then,*

(i) When $0<\tau<Q(m-\alpha)+\delta-1$, for all $\vec{f}=(f_1,\dots,f_m)$ with each $f_j \in L^1(V_G)$, we have

$$\||\nabla\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)} \leq C\prod_{j=1}^{m}\|f_j\|_{L^1(V_G)},$$

where C depends on $\alpha, Q, \delta, m, \tau, \mathcal{D}, \mathcal{B}_{1,Q}, \mathcal{B}_{3,\delta}, \mathcal{B}_{4,\tau}$.

(ii) When $0<\tau<Qm+\delta-\alpha-1$, for all $\vec{f}=(f_1,\dots,f_m)$ with each $f_j \in L^1(V_G)$, then

$$\||\nabla\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})|\|_{L^1(V_G)} \leq C\prod_{j=1}^{m}\|f_j\|_{L^1(V_G)},$$

where C depends on $\alpha, Q, \delta, m, \tau, \mathcal{D}, \mathcal{B}_{1,Q}, \mathcal{B}_{3,\delta}, \mathcal{B}_{4,\tau}$.

4. Boundedness on Hajłasz–Sobolev Spaces

In this section, we want to study whether there are certain smoothing properties about the multilinear fractional maximal operators on Hajłasz–Sobolev spaces defined on graph. Let us now introduce the definition of the spaces.

Definition 5. *Assume the function g defined on V_G and $s \geq 0$. The set $\mathbb{D}^s(g)$ consists of all generalized s-Hajłasz gradients of g. A nonnegative function h is said to be $h \in \mathbb{D}^s(g)$ if*

$$|g(x) - g(y)| \leq d_G(x,y)^s(h(x) + h(y)), \quad \forall x, y \in V_G.$$

For $1 \leq p < \infty$, we say that a function $g \in L^p(V_G)$ belongs to Hajłasz–Sobolev space $M^{s,p}(V_G)$ if there exist functions $h \in L^p(V_G) \cap \mathbb{D}^s(g)$ and their norms satisfy

$$\|g\|_{M^{s,p}(V_G)} = \Big(\|g\|^p_{L^p(V_G)} + \inf_{h\in\mathbb{D}^s(u)} \|h\|^p_{L^p(V_G)}\Big)^{1/p} < \infty.$$

We establish the following theorem.

Theorem 5. *Let $\kappa \geq 1$, $Q \geq 1$ and $0 < \delta \leq 1$. Let G satisfy $(\mathcal{D})$, $(\mathcal{UB} - \mathcal{Q})$ and $(\mathcal{ADP} - \delta)$. Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f \in L^{p_j}(V_G)$ for $1 < p_j < \infty$ and $\delta/Q \leq \alpha < \sum_{i=1}^m 1/p_i$. Then, $\mathcal{B}^{\delta/Q}_{2,Q}(m2^\delta\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 + 1)\mathfrak{M}^\kappa_{\alpha-\delta/Q,G}(\vec{f})$ is a generalized δ-gradient of $\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})$. That is,*

$$\mathcal{B}^{\delta/Q}_{2,Q}(m2^\delta\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 + 1)\mathfrak{M}^\kappa_{\alpha-\delta/Q,G}(\vec{f}) \in \mathbb{D}^\delta(\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})). \tag{16}$$

Proof. We first choose two fixed and unequal points $x, y \in V_G$, and set $d_G(x,y) = a$. To prove (16), we only need to prove that

$$\begin{aligned} &|\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(y)| \\ &\leq \mathcal{B}^{\delta/Q}_{2,Q}(m2^\delta\mathcal{B}_{3,\delta}\mathcal{D}(G)^2 + 1)a^\delta(\mathfrak{M}^\kappa_{\alpha-\delta/Q,G}(\vec{f})(x) + \mathfrak{M}^\kappa_{\alpha-\delta/Q,G}(\vec{f})(y)). \end{aligned} \tag{17}$$

In general, one can suppose that inequality $\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(x) \geq \mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(y)$ holds. By the definition of $\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})$, for given $\epsilon > 0$, there must be positive integer r such that

$$\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(x) \leq \frac{|B_G(x,r)|^\alpha}{|B_G(x,\kappa r)|^m} \prod_{l=1}^m \sum_{w\in B_G(x,r)} |f_l(w)| + \epsilon. \tag{18}$$

We consider two cases:
Case 1: ($r > a$). In view of (18) and $B_G(y, r+a) \supset B_G(x,r)$, we have

$$\begin{aligned} &\mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^\kappa_{\alpha,G}(\vec{f})(y) \\ &\leq \frac{|B_G(x,r)|^\alpha}{|B_G(x,\kappa r)|^m} \prod_{l=1}^m \sum_{w\in B_G(x,r)} |f_l(w)| - \frac{|B_G(y,r+a)|^\alpha}{|B_G(y,\kappa(r+a))|^m} \prod_{l=1}^m \sum_{w\in B_G(y,r+a))} |f_l(w)| + \epsilon \\ &\leq |B_G(x,r)|^\alpha\Big(\frac{1}{|B_G(x,\kappa r)|^m} - \frac{1}{|B_G(y,\kappa(r+a))|^m}\Big) \prod_{l=1}^m \sum_{w\in B_G(x,r)} |f_l(w)| + \epsilon. \end{aligned} \tag{19}$$

A computation similar to (7) shows that

$$\frac{|B_G(y,\kappa(r+a))|^m - |B_G(x,\kappa r)|^m}{|B_G(y,\kappa(r+a))|^m} \leq m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2\Big(\frac{2a}{r+2a}\Big)^\delta. \tag{20}$$

By $(\mathcal{UB} - \mathcal{Q})$, we see that

$$r \geq (\mathcal{B}^{-1}_{2,Q}|B_G(x,r)|)^{-1/Q}, \quad \forall r \in \mathbb{N} \setminus \{0\}. \tag{21}$$

In view of (19)–(21), one sees that

$$\begin{aligned}
&\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(y) \\
&\leq m2^{\delta}a^{\delta}\mathcal{B}_{3,\delta}\mathcal{D}(G)^2(\mathcal{B}_{2,Q}^{-1}|B_G(x,r)|)^{-\delta/Q}\frac{|B_G(x,r)|^{\alpha}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon \\
&\leq m2^{\delta}a^{\delta}\mathcal{B}_{3,\delta}\mathcal{B}_{2,Q}^{\delta/Q}\mathcal{D}(G)^2\frac{|B_G(x,r)|^{\alpha-\delta/Q}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon \\
&\leq m2^{\delta}a^{\delta}\mathcal{B}_{3,\delta}\mathcal{B}_{2,Q}^{\delta/Q}\mathcal{D}(G)^2\mathfrak{M}^{\kappa}_{\alpha-\delta/Q,G}(\vec{f})(x)+\epsilon.
\end{aligned}$$

We obtain (17) in this case by letting $\epsilon \to 0^+$.
Case 2: ($r \leq a$). In view of (18) and (21), one has

$$\begin{aligned}
&\mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}^{\kappa}_{\alpha,G}(\vec{f})(y) \\
&\leq \frac{|B_G(x,r)|^{\alpha}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon \\
&\leq |B_G(x,r)|^{\delta/Q}\frac{|B_G(x,r)|^{\alpha-\delta/Q}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon \\
&\leq \mathcal{B}_{2,Q}^{\delta/Q}r^{\delta}\frac{|B_G(x,r)|^{\alpha-\delta/Q}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon \\
&\leq \mathcal{B}_{2,Q}^{\delta/Q}a^{\delta}\mathfrak{M}^{\kappa}_{\alpha-\delta/Q,G}(\vec{f})(x)+\epsilon.
\end{aligned}$$

Thus, we obtain (17) in this case by letting $\epsilon \to 0^+$. This completes the proof of Theorem 5. □

Theorem 6. *Let $\kappa \geq 1$, $Q \geq 1$, $0 < \delta \leq 1$ and $\delta \leq \alpha < m$. Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f \in L^{p_j}(V_G)$ for $1 < p_j < \infty$ and G satisfy $(\mathcal{D})$ and $(\mathcal{ADP}-\delta)$. Then, $(1 + 2^{\delta}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2)\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f})$ is a generalized δ-gradient of $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})$. That is,*

$$(1 + 2^{\delta}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2)\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f}) \in \mathbb{D}^{\delta}(\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})). \tag{22}$$

Proof. In order to prove (22), it is enough to obtain

$$|\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)| \leq (1 + 2^{\delta}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2)a^{\delta}(\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f})(x) + \widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f})(y)). \tag{23}$$

In general, one can suppose inequality $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) \geq \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)$ holds. By the definition of $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})$, for given $\epsilon > 0$, there must be positive integer r such that

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) \leq r^{\alpha}\prod_{l=1}^{m}\frac{1}{B_G(x,\kappa r)}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon. \tag{24}$$

We consider two cases:
Case (1): ($r > a$). By (24) and the fact that $B_G(y, r+a) \supset B_G(x,r)$, we have

$$\begin{aligned}
&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y) \\
&\leq \frac{r^{\alpha}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)| - \frac{(r+a)^{\alpha}}{|B_G(y,\kappa(r+a))|^m}\prod_{l=1}^{m}\sum_{w\in B_G(y,r+a))}|f_l(w)|+\epsilon \\
&\leq r^{\alpha}\Big(\frac{1}{|B_G(x,\kappa r)|^m} - \frac{1}{|B_G(y,\kappa(r+a))|^m}\Big)\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon.
\end{aligned} \tag{25}$$

Combining (25) with (20) implies that

$$\begin{aligned}
&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)-\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)\\
&\leq r^{\alpha}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2\Big(\frac{2a}{r+2a}\Big)^{\delta}\frac{1}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon\\
&\leq 2^{\delta}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2a^{\delta}\frac{r^{\alpha-\delta}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon\\
&\leq 2^{\delta}m\mathcal{B}_{3,\delta}\mathcal{D}(G)^2a^{\delta}\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f})(x)+\epsilon.
\end{aligned}$$

This proves (23) by making $\epsilon\to 0^+$ in this case.
Case 2: $(r\leq a)$. In view of (24), one has

$$\begin{aligned}
\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(x)-\widetilde{\mathfrak{M}}^{\kappa}_{\alpha,G}(\vec{f})(y)&\leq\frac{r^{\alpha}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon\\
&\leq a^{\delta}\Big(\frac{r}{a}\Big)^{\delta}\frac{r^{\alpha-\delta}}{|B_G(x,\kappa r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}|f_l(w)|+\epsilon\\
&\leq a^{\delta}\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\delta,G}(\vec{f})(x)+\epsilon.
\end{aligned}$$

Thus, we get (23) by making $\epsilon\to 0^+$ in this case. This theorem is now complete. □

Next, we establish the boundedness of the multilinear fractional maximal operators on the Hajłasz–Sobolev spaces.

Theorem 7. *Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^{p_j}(V_G)$ for $1<p_j<\infty$. Let $0\leq\alpha\leq\sum_{i=1}^{m}1/p_i$ and $1/q=\sum_{i=1}^{m}1/p_i-\alpha\leq 1$. If $G=(V_G,E_G)$ satisfies $(\mathcal{ADP}-1)$ and $(\mathcal{D})$ with $\mathcal{D}(G)\in(1,2)$, then*

$$\|\mathfrak{M}_{\alpha,G}(\vec{f})\|_{M^{1,q}(V_G)}\leq C_{\alpha,\mathcal{D}(G),m,p_1,\dots,p_m,\mathcal{B}_{3,1}}\prod_{l=1}^{m}\|f_j\|_{M^{1,p_j}(V_G)}. \tag{26}$$

Proof. Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in M^{1,p_j}(V_G)$ and let $g_j\in L^{p_j}(G)\cap\mathbb{D}(f_j)$. Without loss of generality, we may assume that all $f_j\geq 0$. Let $\alpha=\sum_{j=1}^{m}\alpha_j$ with $\alpha_j\in(0,1)$. It suffices to show that there exists a constant $C>0$ such that

$$C\sum_{l=1}^{m}M_{\alpha_l,G}g_l\prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}}M_{\alpha_\mu,G}f_\mu\in\mathbb{D}(\mathfrak{M}_{\alpha,G}(\vec{f})). \tag{27}$$

In fact, once (27) was proved, then (26) follows easily from (27), Theorem 1 (i) and Theorem 2 (i).

We now prove (27). Let us choose two fixed and unequal points $x,y\in V_G$, and set $d_G(x,y)=a$. In order to prove (27), just prove that there exists a constant $C>0$ such that

$$\begin{aligned}
&|\mathfrak{M}_{\alpha,G}(\vec{f})(x)-\mathfrak{M}_{\alpha,G}(\vec{f})(y)|\\
&\leq C\Big(\sum_{l=1}^{m}M_{\alpha_l,G}g_l(x)\prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}}M_{\alpha_\mu,G}f_\mu(x)+\sum_{l=1}^{m}M_{\alpha_l,G}g_l(y)\prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}}M_{\alpha_\mu,G}f_\mu(y)\Big).
\end{aligned} \tag{28}$$

In general, one can suppose $\mathfrak{M}_{\alpha,G}(\vec{f})(x)\geq\mathfrak{M}_{\alpha,G}(\vec{f})(y)$. Given $\epsilon>0$, there must be a positive integer r such that

$$\mathfrak{M}_{\alpha,G}(\vec{f})(x)\leq\frac{|B_G(x,r)|^{\alpha}}{|B_G(x,r)|^m}\prod_{l=1}^{m}\sum_{w\in B_G(x,r)}f_l(w)+\epsilon. \tag{29}$$

In view of (29) and $B_G(y, r+a) \supset B_G(x, r)$, we have

$$\begin{aligned}
&\mathfrak{M}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}_{\alpha,G}(\vec{f})(y) \\
&\leq \frac{|B_G(x,r)|^{\alpha}}{|B_G(x,r)|^m} \prod_{l=1}^{m} \sum_{w \in B_G(x,r)} f_l(w) - \frac{|B_G(y,r+a)|^{\alpha}}{|B_G(y,r+a)|^m} \prod_{l=1}^{m} \sum_{w \in B_G(y,r+a)} f_l(w) + \epsilon \\
&\leq |B_G(x,r)|^{\alpha} \Big(\prod_{l=1}^{m} (f_l)_{B_G(x,r)} - \prod_{l=1}^{m} (f_l)_{B_G(y,r+a)} \Big) + \epsilon \\
&\leq |B_G(x,r)|^{\alpha} \sum_{l=1}^{m} |(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} (f_\mu)_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon.
\end{aligned} \tag{30}$$

We consider two cases:
Case 1: ($r \leq 3a$). Fix $l \in \{1, 2, \dots, m\}$. Since $g_l \in \mathbb{D}(f_l)$, we have

$$|f_l(u) - f_l(v)| \leq 2d_G(u,v)(g_l(u) + g_l(v)) \leq 4(r+a)(g_l(u) + g_l(v)) \leq 16a(g_l(u) + g_l(v)),$$

for all $u \in B_G(x, r)$ and $v \in B_G(y, r+a)$. This yields that

$$\begin{aligned}
&|(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \\
&\leq \frac{1}{|B_G(x,r)|} \frac{1}{|B_G(y,r+a)|} \sum_{w \in B_G(x,r)} \sum_{v \in B_G(y,r+a)} |f_l(w) - f_l(v)| \\
&\leq 16a((g_l)_{B_G(x,r)} + (g_l)_{B_G(y,r+a)}).
\end{aligned} \tag{31}$$

From $(\mathcal{D})$ and $r \leq 3a$, one has

$$\frac{|B_G(x,r+2a)|}{|B_G(y,r+a)|} \leq \frac{|B_G(y,r+3a)|}{|B_G(y,r+a)|} \leq \frac{|B_G(y,6a)|}{|B_G(y,a)|} \leq \mathcal{D}(G)^3. \tag{32}$$

Let $\alpha = \sum_{j=1}^{m} \alpha_j$ with $\alpha_j \in (0,1)$. In view of (30)–(32) as well as $B_G(x, r+2a) \supset B_G(y, r+a)$, we have

$$\begin{aligned}
&\mathfrak{M}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}_{\alpha,G}(\vec{f})(y) \\
&\leq 16a|B_G(x,r)|^{\alpha} \sum_{l=1}^{m} ((g_l)_{B_G(x,r)} + (g_l)_{B_G(y,r+a)}) \Big(\prod_{\mu=1}^{l-1} (f_\mu)_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 16a|B_G(x,r)|^{\alpha} \sum_{l=1}^{m} \Big((g_l)_{B_G(x,r)} + \frac{|B_G(x,r+2a)|}{|B_G(y,r+a)|} (g_l)_{B_G(x,r+2a)} \Big) \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} \frac{|B_G(x,r+2a)|}{|B_G(y,r+a)|} (f_\mu)_{B_G(x,r+2a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 16a|B_G(x,r)|^{\alpha} \sum_{l=1}^{m} \Big((g_l)_{B_G(x,r)} + \mathcal{D}(G)^3 (g_l)_{B_G(x,r+2a)} \Big) \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} \mathcal{D}(G)^3 (f_\mu)_{B_G(x,r+2a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 32a(1 + \mathcal{D}(G)^3)^m \sum_{l=1}^{m} M_{\alpha_l,G} g_l(x) \prod_{\substack{1 \leq \mu \leq m, \\ \mu \neq l}} M_{\alpha_\mu,G} f_\mu(x) + \epsilon.
\end{aligned}$$

This proves (28) in this case by letting $\epsilon \to 0^+$.
Case 2: ($r > 3a$). It was shown in the proof of ([1], [Theorem 4.3]) that

$$|B_G(x,r)|^{\beta} |(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \leq \frac{512a\mathcal{D}(G)^8 \mathcal{B}_{3,1}}{\ln 2 - \ln \mathcal{D}(G)} M_{\beta,G} g_l(y), \tag{33}$$

for any $\beta > 0$ and $l \in \{1, 2, \ldots, m\}$. By $(\mathcal{D})$ and the assumption $r > 3a$, one has

$$\frac{|B_G(y, r+a)|}{|B_G(x, r)|} \leq \frac{|B_G(x, r+2a)|}{|B_G(x, r)|} \leq \frac{|B_G(x, 2r)|}{|B_G(x, r)|} \leq \mathcal{D}(G).$$

This together with $B_G(y, r+a) \supset B_G(x, r)$ implies that

$$\begin{aligned}|B_G(x,r)|^{\alpha_\nu}(f_\nu)_{B_G(x,r)} &\leq \frac{|B_G(y,r+a)|^{1-\alpha_\nu}}{|B_G(x,r)|^{1-\alpha_\nu}}|B_G(y,r+a)|^{\alpha_\nu}(f_\nu)_{B_G(y,r+a)} \\ &\leq \mathcal{D}(G)^{1-\alpha_\nu} M_{\alpha_\nu,G} f_\nu(y).\end{aligned} \tag{34}$$

Combining (34) with (33) and (30) implies

$$\begin{aligned}&\mathfrak{M}_{\alpha,G}(\vec{f})(x) - \mathfrak{M}_{\alpha,G}(\vec{f})(y) \\ &\leq \sum_{l=1}^{m} |B_G(x,r)|^{\alpha_l} |(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \\ &\quad \times \Big(\prod_{\mu=1}^{l-1} |B_G(x,r)|^{\alpha_\mu} (f_\mu)_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} |B_G(x,r)|^{\alpha_\nu} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\ &\leq \frac{512a\mathcal{D}(G)^8 \mathcal{B}_{3,1}}{\ln 2 - \ln \mathcal{D}(G)} \sum_{l=1}^{m} M_{\alpha_l,G} g(y) \prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}} (1+\mathcal{D}(G)^{1-\alpha_\mu}) M_{\alpha_\mu,G} f_\mu(y) + \epsilon \\ &\leq \frac{512a\mathcal{D}(G)^8 \mathcal{B}_{3,1}}{\ln 2 - \ln \mathcal{D}(G)} (1+\mathcal{D}(G))^{m-1} \sum_{l=1}^{m} M_{\alpha_l,G} g(y) \prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}} M_{\alpha_\mu,G} f_\mu(y) + \epsilon.\end{aligned}$$

Then, it gives (28) by letting $\epsilon \to 0^+$. $\square$

Theorem 8. *Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^{p_j}(V_G)$ for $1 < p_j < \infty$. Let $Q \geq 1$, $0 \leq \alpha \leq \sum_{i=1}^m Q/p_i$ and $1/q = \sum_{i=1}^m 1/p_i - \alpha/Q \leq 1$. If $G = (V_G, E_G)$ satisfies $(\mathcal{D})$ and $(\mathcal{LB} - \mathcal{Q})$, then*

$$\|\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})\|_{M^{1,q}(V_G)} \leq C_{\alpha,\mathcal{D}(G),p_1,\ldots,p_m,\mathcal{B}_{3,1},\mathcal{B}_{1,Q}} \prod_{l=1}^{m} \|f_j\|_{M^{1,p}(V_G)}. \tag{35}$$

Proof. The proof is similar to that of Theorem 7. Let $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in M^{1,p_j}(V_G)$ and let $g_j \in L^{p_j}(G) \cap \mathbb{D}(f_j)$. Without loss of generality, we may assume that all $f_j \geq 0$. Let $\alpha = \sum_{j=1}^m \alpha_j$ with $\alpha_j \in (0, 1)$. We want to show that there exists a constant $C > 0$, such that

$$C \sum_{l=1}^{m} M_{\alpha_l,G} g_l \prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}} M_{\alpha_\mu,G} f_\mu \in \mathbb{D}(\mathfrak{M}_{\alpha,G}(\vec{f})). \tag{36}$$

In fact, once (36) was proved, then (35) follows easily from (36), Theorem 1 (ii) and Theorem 2 (ii).

We now prove (36). Let us choose two fixed and unequal points $x, y \in V_G$, and set $d_G(x, y) = a$. In order to get (36), we must prove there exists a constant $C > 0$ satisfying

$$\begin{aligned}&|\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(y)| \\ &\leq C\Big(\sum_{l=1}^{m} \widetilde{M}_{\alpha_l,G} g_l(x) \prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}} \widetilde{M}_{\alpha_\mu,G} f_\mu(x) + \sum_{l=1}^{m} \widetilde{M}_{\alpha_l,G} g_l(y) \prod_{\substack{1\leq\mu\leq m,\\ \mu\neq l}} \widetilde{M}_{\alpha_\mu,G} f_\mu(y) \Big).\end{aligned} \tag{37}$$

Without loss of generality, we may assume that $\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) \geq \widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(y)$. Given $\epsilon > 0$, there exists $r > 0$, such that

$$\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) \leq \frac{r^{\alpha}}{|B_G(x,r)|^m} \prod_{l=1}^{m} \sum_{w \in B_G(x,r)} f_l(w) + \epsilon. \tag{38}$$

In view of (38) and the inclusion relation of $B_G(x,r) \subset B_G(y, r+a)$, we have

$$\begin{aligned}
&\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(y) \\
&\leq \frac{r^{\alpha}}{|B_G(x,r)|^m} \prod_{l=1}^{m} \sum_{w \in B_G(x,r)} f_l(w) - \frac{(r+a)^{\alpha}}{|B_G(y,r+a)|^m} \prod_{l=1}^{m} \sum_{w \in B_G(y,r+a)} f_l(w) + \epsilon \\
&\leq r^{\alpha} \Big(\prod_{l=1}^{m} (f_l)_{B_G(x,r)} - \prod_{l=1}^{m} (f_l)_{B_G(y,r+a)} \Big) + \epsilon \\
&\leq r^{\alpha} \sum_{l=1}^{m} |(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \Big(\prod_{\mu=1}^{l-1} (f_{\mu})_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_{\nu})_{B_G(x,r)} \Big) + \epsilon.
\end{aligned} \tag{39}$$

We consider two cases:
Case 1: ($r > 3a$). It was shown in the proof of ([1], [Theorem 4.4]) that

$$r^{\beta} |(f_l)_{B(x,r)} - (f_l)_{B(y,r+a)}| \leq \frac{512 a \mathcal{D}(G)^8 \mathcal{B}_{3,1}}{\ln 2 - \ln \mathcal{D}(G)} \widetilde{M}_{\beta,G} g_l(y) \tag{40}$$

for any $\beta > 0$ and $l \in \{1, 2, \ldots, m\}$. Notice that $\frac{|B_G(y,r+a)|}{|B_G(x,r)|} \leq \mathcal{D}(G)$. This together with $B_G(y, r+a) \supset B_G(x,r)$ implies that

$$r^{\alpha_{\nu}} (f_{\nu})_{B_G(x,r)} \leq r^{\alpha} \frac{|B_G(y, r+a)|}{|B_G(x,r)|} (f_{\nu})_{B_G(y,r+a)} \leq \mathcal{D}(G) \widetilde{M}_{\alpha_{\nu},G} f_{\nu}(y). \tag{41}$$

It follows from (39)–(41)

$$\begin{aligned}
&\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(y) \\
&\leq \sum_{l=1}^{m} r^{\alpha_l} |(f_l)_{B_G(x,r)} - (f_l)_{B_G(y,r+a)}| \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} r^{\alpha_{\mu}} (f_{\mu})_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} r^{\alpha_{\nu}} (f_{\nu})_{B_G(x,r)} \Big) + \epsilon \\
&\leq \frac{512 a \mathcal{D}(G)^8 \mathcal{B}_{3,1}}{\ln 2 - \ln \mathcal{D}(G)} \sum_{l=1}^{m} \widetilde{M}_{\alpha_l,G} g(y) \prod_{\substack{1 \leq \mu \leq m, \\ \mu \neq l}} (1 + \mathcal{D}(G)) \widetilde{M}_{\alpha_{\mu},G} f_{\mu}(y) + \epsilon.
\end{aligned}$$

Then, gives (37) in this case by letting $\epsilon \to 0^+$.

Case 2: ($r \leq 3a$). In view of (39), (40), (42) and the inclusion relation of $B_G(x, r+2a) \supset B_G(y, r+a)$, we have

$$
\begin{aligned}
&\widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(x) - \widetilde{\mathfrak{M}}_{\alpha,G}(\vec{f})(y) \\
&\leq 16ar^{\alpha} \sum_{l=1}^{m} ((g_l)_{B_G(x,r)} + (g_l)_{B_G(y,r+a)}) \Big(\prod_{\mu=1}^{l-1} (f_\mu)_{B_G(y,r+a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 16ar^{\alpha} \sum_{l=1}^{m} \Big((g_l)_{B_G(x,r)} + \frac{|B_G(x, r+2a)|}{|B_G(y, r+a)|} (g_l)_{B_G(x,r+2a)} \Big) \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} \frac{|B_G(x, r+2a)|}{|B_G(y, r+a)|} (f_\mu)_{B_G(x,r+2a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 16ar^{\alpha} \sum_{l=1}^{m} \Big((g_l)_{B_G(x,r)} + \mathcal{D}(G)^3 (g_l)_{B_G(x,r+2a)} \Big) \\
&\quad \times \Big(\prod_{\mu=1}^{l-1} \mathcal{D}(G)^3 (f_\mu)_{B_G(x,r+2a)} \Big) \Big(\prod_{\nu=l+1}^{m} (f_\nu)_{B_G(x,r)} \Big) + \epsilon \\
&\leq 32a(1 + \mathcal{D}(G)^3)^m \sum_{l=1}^{m} \widetilde{M}_{\alpha_l,G} g_l(x) \prod_{\substack{1 \leq \mu \leq m, \\ \mu \neq l}} \widetilde{M}_{\alpha_\mu,G} f_\mu(x).
\end{aligned}
$$

Making $\epsilon \to 0^+$, we prove (37) in this case. Theorem 8 is now proved. □

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