Article

New Bounds for the Modified Bessel Function of the First Kind and Toader-Qi Mean

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Abstract: Let $I_p(x)$ be the modified Bessel function of the first kind of order p . The upper and lower bounds in the form of simple rational functions about $\cosh t$ and $(\sinh t)/t$ for the function $I_0(x)$ are obtained. The corresponding inequalities for the Toader-Qi mean do not match those in the existing literature.

Keywords: bounds; modified Bessel function of the first kind; hyperbolic function; mean; inequality

1. Introduction

We know that the modified Bessel function of the first kind of order p is denoted as $I_p(x)$ ([1] p. 77), ([2] p. 374)) and has important applications in many fields of natural science. The function $I_p(x)$ denotes a class of solutions of the second-order differential equation ([2] p. 374)

$$x^2 y''(x) + xy'(x) - (x^2 + p^2)y(x) = 0, \quad (1)$$

which can be written as the infinite series

$$I_p(x) = \left(\frac{x}{2}\right)^p \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(p+n+1)}, \quad x \in \mathbb{R}, \quad p \in \mathbb{R} \setminus \{-1, -2, \dots\}, \quad (2)$$

where $\Gamma(x)$ is the gamma function ([2] p. 255). Moreover, the normalized modified Bessel function defined by $\mathcal{I}_p : \mathbb{R} \rightarrow [1, +\infty)$ as $\mathcal{I}_p(x) = 2^p \Gamma(p+1) x^{-p} I_p(x)$ was considered in [3]. Then, we have

$$\mathcal{I}_{p+1/2}(x) = 2^{p+1/2} \Gamma(p+3/2) x^{-(p+1/2)} I_{p+1/2}(x). \quad (3)$$

Particularly, for $p = 0$, $p = 1$ in the above, the function $\mathcal{I}_{p+1/2}(x)$ reduces to some elementary functions, as follows ([1] p. 54):

$$\mathcal{I}_{1/2}(x) = \sqrt{\frac{\pi}{2x}} I_{1/2}(x) = \frac{\sinh x}{x}, \quad \mathcal{I}_{3/2}(x) = \frac{3}{x} \sqrt{\frac{\pi}{2x}} I_{3/2}(x) = -3 \left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2} \right). \quad (4)$$

The properties and various inequalities of this special function $\mathcal{I}_p(x)$ are widely discussed in References [3–11].

Throughout this article, we assume that a and b are two different positive real numbers. This paper focuses on

$$I_0(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n! \Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} (n!)^2}, \quad (5)$$

which is the modified Bessel function of the first kind of order 0, which is related to the Toader-Qi mean (see [12,13]) defined by

$$TQ(a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} d\theta = \sqrt{ab} I_0 \left(\ln \sqrt{\frac{a}{b}} \right). \quad (6)$$

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Let

$$\begin{aligned} A &\equiv A(a, b) = \frac{a+b}{2}, & G &\equiv G(a, b) = \sqrt{ab}, \\ L &\equiv L(a, b) = \frac{a-b}{\ln a - \ln b}, & I &\equiv I(a, b) = e^{-1} \left(\frac{b^b}{a^a} \right)^{1/(b-a)} \end{aligned} \quad (7)$$

and

$$A_\nu \equiv A_\nu(a, b) = \left(\frac{a^\nu + b^\nu}{2} \right)^{1/\nu} \text{ if } \nu \neq 0$$

be the arithmetic mean, geometric mean, logarithmic mean, exponential mean, and power mean of order ν , respectively. Then, an inequality chain (see [14–16])

$$G < L < A_{1/3} < \frac{A+2G}{3} < A_{1/2} < \frac{2A+G}{3} < A_{2/3} < I < A_{\ln 2} < A_1 \quad (8)$$

holds. Alzer [17] also provided the following result:

$$\sqrt{AG} < \sqrt{LI} < \frac{L+I}{2} < \frac{A+G}{2}. \quad (9)$$

From now on, let us focus on the Toader–Qi mean of $I_0(x)$. In 2017, Qi, Shi, Liu, and Yang [13] proved that the inequalities

$$L < TQ < \frac{A+G}{2} < \frac{2A+G}{3} < I \quad (10)$$

hold Yang and Chu ([18] Theorem 3.4) established the following results:

$$L^{3/4} A^{1/4} < TQ < \sqrt{LA_{2/3}} < \frac{3}{4}L + \frac{1}{4}A, \quad (11)$$

$$\sqrt{\frac{2AL}{\pi}} < TQ < \sqrt{AL}, \quad (12)$$

$$\sqrt{\left(\frac{2}{\pi}A + 1 - \frac{2}{\pi} \right)L} < TQ < \sqrt{(\lambda_0 A + 1 - \lambda_0)L}, \quad (13)$$

where $\lambda_0 = 0.6766\dots$ Yang, Chu, and Song [19] and Yang, Tian, and Zhu ([20] Remark 4) obtained

$$\sqrt{\frac{e}{\pi}} \sqrt{LI} < \frac{2^{4/3}}{\sqrt{\pi}} \sqrt{LA_{2/3}} < TQ < \sqrt{LA_{2/3}} < \sqrt{LI}. \quad (14)$$

Let $b > a > 0$ and $t = \ln \sqrt{a/b}$. Then, those means mentioned above can be represented in terms of hyperbolic functions:

$$\begin{aligned} \frac{A}{G} &= \cosh t, \quad \frac{L}{G} = \frac{\sinh t}{t}, \quad \frac{I}{G} = \exp\left(\frac{t}{\tanh t} - 1\right), \\ \frac{TQ}{G} &= I_0(t), \quad \frac{A_p}{G} = \cosh^{1/p}(pt) \text{ for } p \neq 0. \end{aligned}$$

Correspondingly, the inequalities (10)–(14) mentioned above are equivalent to

$$\frac{\sinh t}{t} < I_0(t) < \frac{\cosh t + 1}{2} < \frac{2 \cosh t + 1}{3} < \exp\left(\frac{t}{\tanh t} - 1\right), \quad (15)$$

$$\left(\frac{\sinh t}{t} \right)^{3/4} (\cosh t)^{1/4} < I_0(t) < \sqrt{\frac{\sinh t}{t} \cosh^{3/2}\left(\frac{2t}{3}\right)} < \frac{3}{4} \frac{\sinh t}{t} + \frac{1}{4} \cosh t \quad (16)$$

$$\sqrt{\frac{\sinh(2t)}{\pi t}} < I_0(t) < \sqrt{\frac{\sinh(2t)}{2t}}, \quad (17)$$

$$\sqrt{\left(\frac{2}{\pi} \cosh t + 1 - \frac{2}{\pi}\right) \frac{\sinh t}{t}} < I_0(t) < \sqrt{(\lambda_0 \cosh t + 1 - \lambda_0) \frac{\sinh t}{t}}, \quad (18)$$

and

$$\begin{aligned} & \sqrt{\frac{e}{\pi}} \sqrt{\frac{\sinh t}{t} \exp\left(\frac{t}{\tanh t} - 1\right)} \\ & < \frac{2^{3/4}}{\sqrt{\pi}} \sqrt{\frac{\sinh t}{t} \cosh^{3/2}\left(\frac{2t}{3}\right)} \\ & < I_0(t) < \sqrt{\frac{\sinh t}{t} \cosh^{3/2}\left(\frac{2t}{3}\right)} < \sqrt{\frac{\sinh t}{t} \exp\left(\frac{t}{\tanh t} - 1\right)} \end{aligned} \quad (19)$$

hold for $t > 0$.

In this paper, we first consider finding the bound of $TQ(a, b)$ in the following form:

$$\frac{x_1 A^2 + y_1 AL + z_1 G^2}{A + w_1 G},$$

which is equivalent to searching for a bound for $TQ(a, b)/G = I_0(t)$ in the form of

$$\begin{aligned} \frac{x_1 A^2 + y_1 AL + z_1 G^2}{(A + w_1 G)G} &= \frac{x_1 A^2 + y_1 AL + z_1 G^2}{AG + w_1 G^2} \\ &= \frac{x_1 (A/G)^2 + y_1 (A/G)(L/G) + z_1}{A/G + w_1} \\ &= \frac{x_1 (\cosh t)^2 + y_1 (\cosh t)((\sinh t)/t) + z_1}{\cosh t + w_1}. \end{aligned}$$

In this way, we can consider the power series expansion

$$\begin{aligned} & (\cosh t + w_1)I_0(t) - \left[x_1 (\cosh t)^2 + y_1 (\cosh t)((\sinh t)/t) + z_1 \right] \\ &= (-x_1 - y_1 - z_1 + w_1 + 1) - t^2 \left(x_1 + \frac{2}{3}y_1 - \frac{1}{4}w_1 - \frac{3}{4} \right) \\ & \quad - t^4 \left(\frac{1}{3}x_1 + \frac{2}{15}y_1 - \frac{1}{64}w_1 - \frac{35}{192} \right) - t^6 \left(\frac{2}{45}x_1 + \frac{4}{315}y_1 - \frac{1}{2304}w_1 - \frac{77}{3840} \right) \\ & \quad - t^8 \left(\frac{1}{315}x_1 + \frac{2}{2835}y_1 - \frac{1}{147,456}w_1 - \frac{143}{114,688} \right) \\ & \quad - t^{10} \left(\frac{2}{14,175}x_1 + \frac{4}{155,925}y_1 - \frac{1}{14,745,600}w_1 - \frac{46,189}{928,972,800} \right) + O(t^{12}). \end{aligned}$$

When letting

$$\begin{cases} -x_1 - y_1 - z_1 + w_1 + 1 = 0 \\ x_1 + \frac{2}{3}y_1 - \frac{1}{4}w_1 - \frac{3}{4} = 0 \\ \frac{1}{3}x_1 + \frac{2}{15}y_1 - \frac{1}{64}w_1 - \frac{35}{192} = 0 \\ \frac{2}{45}x_1 + \frac{4}{315}y_1 - \frac{1}{2304}w_1 - \frac{77}{3840} = 0 \end{cases}$$

we can obtain that $x_1 = 2263/11,456$, $y_1 = 20,475/22,912$, $z_1 = 1879/22,912$ and $w_1 = 31/179$ and find that

$$\begin{aligned} & \left(\cosh t + \frac{31}{179} \right) I_0(t) - \left[\frac{2263}{11,456} (\cosh t)^2 + \frac{20,475}{22,912} \frac{\sinh t \cosh t}{t} + \frac{1879}{22,912} \right] \\ &= -\frac{209}{21,995,520} t^8 - \frac{16,217}{15,242,895,360} t^{10} + O(t^{12}), \end{aligned}$$

which motivates us to prove the inequality

$$\begin{aligned} I_0(t) &< \frac{\frac{2263}{11,456} (\cosh t)^2 + \frac{20,475}{22,912} \frac{\sinh t \cosh t}{t} + \frac{1879}{22,912}}{\cosh t + \frac{31}{179}} \\ &= \frac{8284t + 20,475 \sinh 2t + 4526t \cosh 2t}{256t(179 \cosh t + 31)}, \quad t > 0. \end{aligned}$$

Similarly, we can find the bound for $TQ(a, b)$ in the form of

$$\frac{x_2 L^2 + y_2 LG + z_2 G^2}{A + w_2 G}$$

and obtain the exact bound by the approximation method used just now. The main conclusions of this paper are as follows.

Theorem 1. *The inequality*

$$\begin{aligned} I_0(t) &< \frac{\frac{2263}{11,456} (\cosh t)^2 + \frac{20,475}{22,912} \frac{\sinh t \cosh t}{t} + \frac{1879}{22,912}}{\cosh t + \frac{31}{179}} \\ &= \frac{8284t + 20,475 \sinh 2t + 4526t \cosh 2t}{256t(179 \cosh t + 31)} \end{aligned} \quad (20)$$

holds for $t > 0$, or equivalently,

$$\begin{aligned} TQ(a, b) &< \frac{\frac{2263}{11,456} A^2 + \frac{20,475}{22,912} AL + \frac{1879}{22,912} G^2}{A + \frac{31}{179} G} \\ &= \frac{20,475 AL + 4526 A^2 + 1879 G^2}{128(179A + 31G)} \end{aligned} \quad (21)$$

holds for $a, b > 0$ with $a \neq b$.

Theorem 2. *The inequality*

$$\begin{aligned} I_0(t) &> \frac{\frac{4659}{568} \left(\frac{\sinh t}{t} \right)^2 + \frac{7941}{284} \frac{\sinh t}{t} - \frac{4877}{568}}{\cosh t + \frac{1887}{71}} \\ &= \frac{4659 \cosh 2t + 31,764t \sinh t - 9754t^2 - 4659}{16t^2(71 \cosh t + 1887)} \end{aligned} \quad (22)$$

holds for $t > 0$, or equivalently,

$$\begin{aligned} TQ(a, b) &> \frac{\frac{4659}{568} L^2 + \frac{7941}{284} LG - \frac{4877}{568} G^2}{A + \frac{1887}{71} G} \\ &= \frac{4659 L^2 + 15,882 GL - 4877 G^2}{8(71A + 1887G)} \end{aligned} \quad (23)$$

holds for $a, b > 0$ with $a \neq b$.

2. Lemmas

The precise power series representation of $(\cosh t)I_0(t)$ is required to establish two theorems in this article. We thank Prof. Dr. H. W. Gould for compiling an important result on the finite sum of combinatorial numbers in his monograph [21], which we consider as the substance of Lemma 1.

Lemma 1 ([21] (3.175)). Let $\binom{n}{k}$ be the number of different ways to choose k elements from a given set with n distinct elements, that is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then, for all $x \in \mathbb{R}$,

$$\sum_{k=0}^n \binom{x}{2k} \binom{x-2k}{n-k} 2^{2k} = \binom{2x}{2n}. \quad (24)$$

Lemma 2. We have

$$I_0(t) \cosh t = \sum_{n=0}^{\infty} \frac{(4n-1)!!}{[(2n)!]^2} t^{2n} =: \sum_{n=0}^{\infty} \kappa_n t^{2n}, \quad (25)$$

where

$$\kappa_n = \frac{(4n-1)!!}{[(2n)!]^2}. \quad (26)$$

Proof. Using the Cauchy product formula, we have

$$\begin{aligned} I_0(t) \cosh t &= \left(\sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} t^{2n} \right) \left(\sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{1}{2^{2k}(k!)^2} \frac{1}{(2n-2k)!} t^{2n} \\ &=: \sum_{n=0}^{\infty} \kappa_n t^{2n}, \end{aligned}$$

where

$$\kappa_n = \sum_{k=0}^n \frac{1}{2^{2k}(k!)^2} \frac{1}{(2n-2k)!}.$$

Since

$$\begin{aligned} \binom{2n}{2k} &= \frac{(2n)!}{(2k)!(2n-2k)!}, \\ (2k)! &= (k!)^2 \binom{2k}{k}, \end{aligned}$$

we have

$$\begin{aligned} \kappa_n &= \sum_{k=0}^n \frac{1}{2^{2k}(k!)^2(2n-2k)!} = \frac{1}{(2n)!} \sum_{k=0}^n \frac{(2k)!}{2^{2k}(k!)^2} \binom{2n}{2k} \\ &= \frac{1}{(2n)!} \sum_{k=0}^n \frac{(k!)^2 \binom{2k}{k}}{2^{2k}(k!)^2} \binom{2n}{2k} = \frac{1}{(2n)!} \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2n}{2k} \binom{2k}{k}. \end{aligned}$$

Letting $x = 2n$ in (24) gives

$$\sum_{k=0}^n \binom{2n}{2k} \binom{2n-2k}{n-k} 2^{2k} = \binom{4n}{2n}.$$

On the one hand, let $n - k = l$. Then

$$\begin{aligned} \sum_{k=0}^n \binom{2n}{2k} \binom{2n-2k}{n-k} 2^{2k} &= \sum_{l=0}^n \binom{2n}{2n-2l} \binom{2l}{l} 2^{2(n-l)} \\ &= 2^{2n} \sum_{l=0}^n \frac{1}{2^{2l}} \binom{2n}{2n-2l} \binom{2l}{l} \\ &= 2^{2n} \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2n}{2n-2k} \binom{2k}{k} \\ &= 2^{2n} \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2n}{2k} \binom{2k}{k}. \end{aligned}$$

On the other hand, we have

$$\binom{4n}{2n} = \frac{(4n)!}{[(2n)!]^2}.$$

Therefore,

$$2^{2n} \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2n}{2k} \binom{2k}{k} = \frac{(4n)!}{[(2n)!]^2} \iff \sum_{k=0}^n \frac{1}{2^{2k}} \binom{2n}{2k} \binom{2k}{k} = \frac{(4n)!}{2^{2n}[(2n)!]^2}$$

and

$$\begin{aligned} \kappa_n &= \frac{1}{(2n)!} \frac{(4n)!}{2^{2n}[(2n)!]^2} = \frac{1}{(2n)!} \frac{(4n)(4n-1)(4n-2)(4n-3) \cdots 4 \cdot 3 \cdot 2 \cdot 1}{2^{2n}[(2n)!]^2} \\ &= \frac{1}{(2n)!} \frac{(4n)(4n-2) \cdots 4 \cdot 2(4n-1)!!}{2^{2n}[(2n)!]^2} = \frac{1}{(2n)!} \frac{2^{2n}(2n)!(4n-1)!!}{2^{2n}[(2n)!]^2} \\ &= \frac{(4n-1)!!}{[(2n)!]^2}. \end{aligned}$$

□

3. Proofs of Theorems 1 and 2

3.1. Proof of Theorem 1

The inequality to be proved is

$$\begin{aligned} f_1(t) &= : 8284 + 20,475 \frac{1}{t} \sinh 2t + 4526 \cosh 2t \\ &> 256(179 \cosh t + 31)I_0(t) =: g_1(t), \quad t > 0. \end{aligned} \quad (27)$$

Using the power series expansion of hyperbolic sine and hyperbolic cosine functions, we have

$$\begin{aligned} f_1(t) &= 8284 + 20,475 \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} t^{2n} + 4526 \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n} \\ &= 8284 + \sum_{n=0}^{\infty} \left[\frac{20,475 \cdot 2^{2n+1}}{(2n+1)!} + \frac{4526 \cdot 2^{2n}}{(2n)!} \right] t^{2n} \\ &= 53,760 + 36,352t^2 + \frac{25,432}{3}t^4 + \frac{41,504}{45}t^6 \\ &\quad + \sum_{n=4}^{\infty} \left[\frac{20,475 \cdot 2^{2n+1}}{(2n+1)!} + \frac{4526 \cdot 2^{2n}}{(2n)!} \right] t^{2n}. \end{aligned}$$

Then, using Formulas (5) and (25), we obtain

$$\begin{aligned} g_1(t) &= 45,824I_0(t) \cosh t + 7936I_0(t) \\ &= 45,824 \sum_{n=0}^{\infty} \kappa_n t^{2n} + 7936 \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} t^{2n} \\ &= \sum_{n=0}^{\infty} \left[45,824\kappa_n + \frac{7936}{2^{2n}(n!)^2} \right] t^{2n} \\ &= 53,760 + 36,352t^2 + \frac{25,432}{3}t^4 + \frac{41,504}{45}t^6 \\ &\quad + \sum_{n=4}^{\infty} \left[45,824\kappa_n + \frac{7936}{2^{2n}(n!)^2} \right] t^{2n}. \end{aligned}$$

The inequality (27) is proved when the following inequality holds:

$$\begin{aligned} 45,824\kappa_n + \frac{7936}{2^{2n}(n!)^2} &< \frac{20,475 \cdot 2^{2n+1}}{(2n+1)!} + \frac{4526 \cdot 2^{2n}}{(2n)!}, \quad n \geq 4 \\ \Leftrightarrow 45,824 \frac{(4n-1)!!}{((2n)!)^2} + \frac{7936}{2^{2n}(n!)^2} &< \frac{20,475 \cdot 2^{2n+1}}{(2n+1)!} + \frac{4526 \cdot 2^{2n}}{(2n)!}, \quad n \geq 4 \\ \Leftrightarrow 45,824 \frac{(4n-1)!!}{((2n)!)^2} &< \frac{20,475 \cdot 2^{2n+1}}{(2n+1)!} + \frac{4526 \cdot 2^{2n}}{(2n)!} - \frac{7936}{2^{2n}(n!)^2}, \quad n \geq 4 \end{aligned} \quad (28)$$

Next, we prove (28) by mathematical induction. It is not difficult to verify that the above Formula (28) is true for $n = 4$. Suppose (28) holds for m , that is

$$45,824 \frac{(4m-1)!!}{((2m)!)^2} < \frac{20,475 \cdot 2^{2m+1}}{(2m+1)!} + \frac{4526 \cdot 2^{2m}}{(2m)!} - \frac{7936}{2^{2m}(m!)^2}, \quad m \geq 4. \quad (29)$$

By (29) we have

$$\begin{aligned} &45,824 \frac{(4m+3)!!}{[(2m+2)!]^2} \\ &= \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \left[45,824 \frac{(4m-1)!!}{((2m)!)^2} \right] \\ &< \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \left[\frac{20,475 \cdot 2^{2m+1}}{(2m+1)!} + \frac{4526 \cdot 2^{2m}}{(2m)!} - \frac{7936}{2^{2m}(m!)^2} \right]. \end{aligned}$$

The inequality (28) is proved when proving

$$\begin{aligned} & \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \left[\frac{20,475 \cdot 2^{2m+1}}{(2m+1)!} + \frac{4526 \cdot 2^{2m}}{(2m)!} - \frac{7936}{2^{2m}(m!)^2} \right] \\ & < \frac{20,475 \cdot 2^{2m+3}}{(2m+3)!} + \frac{4526 \cdot 2^{2m+2}}{(2m+2)!} - \frac{7936}{2^{2m+2}((m+1)!)^2}, \end{aligned}$$

that is

$$\begin{aligned} & \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \frac{20,475 \cdot 2^{2m+1}}{(2m+1)!} + \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \frac{4526 \cdot 2^{2m}}{(2m)!} \\ & - \left(\frac{20,475 \cdot 2^{2m+3}}{(2m+3)!} + \frac{4526 \cdot 2^{2m+2}}{(2m+2)!} \right) \\ & < \frac{(4m+3)(4m+1)}{(2m+2)^2(2m+1)^2} \frac{7936}{2^{2m}(m!)^2} - \frac{7936}{2^{2m+2}((m+1)!)^2} \end{aligned}$$

or

$$-2 \times 2^{2m} \frac{-70,909m - 68,754m^2 + 36,208m^3 + 6735}{(2m+3)!(m+1)(2m+1)^2} < \frac{3968(6m+6m^2+1)}{(m+1)!^2 2^{2m}(2m+1)^2}.$$

Since $36,208m^3 - 68,754m^2 - 70,909m + 6735 > 0$ for $m \geq 4$, the above inequality is obviously true.

The proof of the Theorem 1 has been completed.

3.2. Proof of Theorem 2

The conclusion to be proved is equivalent to

$$\begin{aligned} f_2(t) &= : \left(1136t^2 \cosh t + 30,192t^2 \right) I_0(t) \\ &> 4659 \cosh 2t + 31,764t \sinh t - 9754t^2 - 4659 =: g_2(t), \quad t > 0. \end{aligned} \quad (30)$$

Using Formulas (5) and (25), we obtain

$$\begin{aligned} f_2(t) &= 1136t^2 I_0(t) \cosh t + 30,192t^2 I_0(t) \\ &= 1136 \sum_{n=0}^{\infty} \kappa_n t^{2n+2} + 30,192 \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n!)^2} t^{2n+2} \\ &= \sum_{n=0}^{\infty} \left[1136\kappa_n + \frac{30,192}{2^{2n}(n!)^2} \right] t^{2n+2} \\ &= 31,328t^2 + 8400t^4 + \frac{4073}{6}t^6 + \frac{2153}{60}t^8 + \sum_{n=4}^{\infty} \left[1136\kappa_n + \frac{30,192}{2^{2n}(n!)^2} \right] t^{2n+2}. \end{aligned}$$

Using the power series expansion of hyperbolic sine and hyperbolic cosine functions, we have

$$\begin{aligned}
 g_2(t) &= 4659 \sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} t^{2n} + 31,764 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+2} - 9754t^2 - 4659 \\
 &= 4659 \sum_{n=-1}^{\infty} \frac{2^{2n+2}}{(2n+2)!} t^{2n+2} + 31,764 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+2} - 9754t^2 - 4659 \\
 &= 31,328t^2 + 8400t^4 + \frac{4073}{6}t^6 + \frac{2153}{60}t^8 \\
 &\quad + 4659 \sum_{n=4}^{\infty} \frac{2^{2n+2}}{(2n+2)!} t^{2n+2} + 31,764 \sum_{n=4}^{\infty} \frac{1}{(2n+1)!} t^{2n+2} \\
 &= 31,328t^2 + 8400t^4 + \frac{4073}{6}t^6 + \frac{2153}{60}t^8 \\
 &\quad + \sum_{n=4}^{\infty} \left[\frac{4659 \cdot 2^{2n+2}}{(2n+2)!} + \frac{31,764}{(2n+1)!} \right] t^{2n+2}.
 \end{aligned}$$

The inequality (27) is proved when the following inequality

$$1136\kappa_n + \frac{30,192}{2^{2n}(n!)^2} > \frac{4659 \cdot 2^{2n+2}}{(2n+2)!} + \frac{31,764}{(2n+1)!}$$

holds for $n \geq 4$, that is,

$$\frac{1136(4n-1)!!}{((2n)!)^2} + \frac{30,192}{2^{2n}(n!)^2} > \frac{4659 \cdot 2^{2n+2}}{(2n+2)!} + \frac{31,764}{(2n+1)!}, \quad n \geq 4$$

or

$$\frac{1136(4n-1)!!}{((2n)!)^2} > \frac{4659}{(2n+2)!} 2^{2n+2} + \frac{31,764}{(2n+1)!} - \frac{30,192}{2^{2n}(n!)^2}, \quad n \geq 4. \quad (31)$$

It can be verified that the above formula holds when $n = 4$. Suppose that Formula (31) holds for m , i.e.,

$$\frac{1136(4m-1)!!}{((2m)!)^2} > \frac{4659}{(2m+2)!} 2^{2m+2} + \frac{31,764}{(2m+1)!} - \frac{30,192}{2^{2m}(m!)^2}, \quad m \geq 4. \quad (32)$$

Next, we prove that Formula (31) is also true for $m+1$. By (32), we have

$$\begin{aligned}
 &\frac{1136(4m+3)!!}{((2m+2)!)^2} \\
 &= \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{1136(4m-1)!!}{((2m)!)^2} \\
 &> \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \left[\frac{4659}{(2m+2)!} 2^{2m+2} + \frac{31,764}{(2m+1)!} - \frac{30,192}{2^{2m}(m!)^2} \right].
 \end{aligned}$$

In this way, we complete the proof of (31) when proving

$$\begin{aligned}
 &\frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \left[\frac{4659}{(2m+2)!} 2^{2m+2} + \frac{31,764}{(2m+1)!} - \frac{30,192}{2^{2m}(m!)^2} \right] \\
 &> \frac{4659}{(2m+4)!} 2^{2m+4} + \frac{31,764}{(2m+3)!} - \frac{30,192}{2^{2m+2}((m+1)!)^2}
 \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{4659}{(2m+2)!} 2^{2m+2} + \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{31,764}{(2m+1)!} \\
&\quad - \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{30,192}{2^{2m}(m!)^2} \\
&> \frac{4659}{(2m+4)!} 2^{2m+4} + \frac{31,764}{(2m+3)!} - \frac{30,192}{2^{2m+2}((m+1)!)^2} \\
&\Leftrightarrow \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{4659}{(2m+2)!} 2^{2m+2} + \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{31,764}{(2m+1)!} \\
&\quad - \left(\frac{4659}{(2m+4)!} 2^{2m+4} + \frac{31,764}{(2m+3)!} \right) \\
&> \frac{(4m+3)(4m+1)}{((2m+2)(2m+1))^2} \frac{30,192}{2^{2m}(m!)^2} - \frac{30,192}{2^{2m+2}((m+1)!)^2} \\
&\Leftrightarrow \frac{6 \left[\begin{aligned} &(74,544m^3 + 170,830m^2 + 107,157m + 15,530)2^{2m} \\ &+ 127,056m^5 + 719,984m^4 + 1,503,496m^3 \\ &+ 1,413,498m^2 + 577,046m + 74,116 \end{aligned} \right]}{(2m+4)!(m+1)^2(2m+1)^2} \\
&> \frac{15,096(6m+6m^2+1)}{[(m+1)!]^2 2^{2m}(2m+1)^2} \\
&\Leftrightarrow \frac{\left[\begin{aligned} &(74,544m^3 + 170,830m^2 + 107,157m + 15,530)2^{2m} \\ &+ 127,056m^5 + 719,984m^4 + 1,503,496m^3 \\ &+ 1,413,498m^2 + 577,046m + 74,116 \end{aligned} \right]}{(2m+4)!} \\
&> \frac{2516(6m+6m^2+1)}{(m!)^2 2^{2m}},
\end{aligned}$$

that is,

$$\frac{(m!)^2 2^{2m}}{(2m+4)!} > \frac{2516(6m+6m^2+1)}{\left[\begin{aligned} &(74,544m^3 + 170,830m^2 + 107,157m + 15,530)2^{2m} \\ &+ 127,056m^5 + 719,984m^4 + 1,503,496m^3 \\ &+ 1,413,498m^2 + 577,046m + 74,116 \end{aligned} \right]} \quad (33)$$

holds for $m \geq 4$.

Then, we use mathematical induction to prove (33). It is not difficult to verify that the above Formula (33) is true for $m = 4$. Suppose that (33) holds for $m = n$, i.e.,

$$\begin{aligned}
f(n) &= : \frac{(n!)^2 2^{2n}}{(2n+4)!} \\
&> \frac{2516(6n+6n^2+1)}{\left[\begin{aligned} &(74,544n^3 + 170,830n^2 + 107,157n + 15,530)2^{2n} \\ &+ 127,056n^5 + 719,984n^4 + 1,503,496n^3 \\ &+ 1,413,498n^2 + 577,046n + 74,116 \end{aligned} \right]} =: g(n), \quad n \geq 4. \quad (34)
\end{aligned}$$

By (34), we have

$$\begin{aligned} f(n+1) &= \frac{((n+1)!)^2 2^{2n+2}}{(2n+6)!} = \frac{4(n+1)^2}{(2n+6)(2n+5)} \frac{n!^2 2^{2n}}{(2n+4)!} \\ &> \frac{4(n+1)^2}{(2n+6)(2n+5)} \frac{2516(6n+6n^2+1)}{\left[\begin{aligned} &(74,544n^3 + 170,830n^2 + 107,157n + 15,530)2^{2n} \\ &+ 127,056n^5 + 719,984n^4 + 1,503,496n^3 \\ &+ 1,413,498n^2 + 577,046n + 74,116 \end{aligned} \right]}, \end{aligned}$$

so the proof of (33) is complete when we can prove that

$$\begin{aligned} \frac{A}{B} &= : \frac{2516 \cdot 4(n+1)^2(6n+6n^2+1)}{(2n+6)(2n+5) \left[\begin{aligned} &(74,544n^3 + 170,830n^2 + 107,157n + 15,530)2^{2n} \\ &+ 127,056n^5 + 719,984n^4 + 1,503,496n^3 \\ &+ 1,413,498n^2 + 577,046n + 74,116 \end{aligned} \right]} \\ &> g(n+1) \\ &= \frac{2516(6(n+1) + 6(n+1)^2 + 1)}{\left[\begin{aligned} &(74,544(n+1)^3 + 170,830(n+1)^2 + 107,157(n+1) + 15,530)2^{2n+2} \\ &+ 127,056(n+1)^5 + 719,984(n+1)^4 + 1,503,496(n+1)^3 \\ &+ 1,413,498(n+1)^2 + 577,046(n+1) + 74,116 \end{aligned} \right]} \\ &= : \frac{C}{D}. \end{aligned}$$

In fact,

$$\begin{aligned} &(AD - BC)/2516 \\ &= 9318 \times 2^{2n} \left(\begin{aligned} &349n + 4227n^2 + 13,036n^3 + 17,962n^4 \\ &+ 12,512n^5 + 4296n^6 + 576n^7 - 18 \end{aligned} \right) \\ &\quad - 10,588(n+3)(n+2)(n+1) \left(\begin{aligned} &1238n + 2702n^2 + 2496n^3 \\ &+ 1008n^4 + 144n^5 + 177 \end{aligned} \right) \quad (35) \\ &> 0 \end{aligned}$$

holds for all $n \geq 4$. The above formula is obviously equivalent to

$$2^{2n} > \frac{\left[\begin{aligned} &5294(n+1)(n+2)(n+3) \\ &(1238n + 2702n^2 + 2496n^3 + 1008n^4 + 144n^5 + 177) \end{aligned} \right]}{4659 \left[\begin{aligned} &349n + 4227n^2 + 13,036n^3 + 17,962n^4 \\ &+ 12,512n^5 + 4296n^6 + 576n^7 - 18 \end{aligned} \right]}, \quad n \geq 4. \quad (36)$$

We use mathematical induction for the third time in the process of proving Theorem 2. The Formula (36) is obviously true for $n = 4$. We assume that (36) holds for $n = m$, that is,

$$2^{2m} > \frac{\left[\begin{aligned} &5294(m+1)(m+2)(m+3) \\ &(1238m + 2702m^2 + 2496m^3 + 1008m^4 + 144m^5 + 177) \end{aligned} \right]}{4659 \left[\begin{aligned} &349m + 4227m^2 + 13,036m^3 + 17,962m^4 \\ &+ 12,512m^5 + 4296m^6 + 576m^7 - 18 \end{aligned} \right]}. \quad (37)$$

It is proved below that (36) holds for $n = m + 1$. By (37), we have

$$2^{2m+2} = 4 \cdot 2^{2m} > 4 \cdot \frac{\left[\frac{5294(m+1)(m+2)(m+3)}{(1238m + 2702m^2 + 2496m^3 + 1008m^4 + 144m^5 + 177)} \right]}{4659 \left[\frac{349m + 4227m^2 + 13,036m^3 + 17,962m^4}{+12,512m^5 + 4296m^6 + 576m^7 - 18} \right]},$$

so the proof of (36) is complete when proving

$$\begin{aligned} & 4 \cdot \frac{5294}{4659} \frac{\left[\frac{(m+1)(m+2)(m+3)}{(1238m + 2702m^2 + 2496m^3 + 1008m^4 + 144m^5 + 177)} \right]}{\left[\frac{349m + 4227m^2 + 13,036m^3 + 17,962m^4}{+12,512m^5 + 4296m^6 + 576m^7 - 18} \right]} \\ & > \frac{5294}{4659} \frac{\left[\frac{(m+2)(m+3)(m+4)}{\left(\frac{1238(m+1) + 2702(m+1)^2 + 2496(m+1)^3}{+1008(m+1)^4 + 144(m+1)^5 + 177} \right)} \right]}{\left[\frac{349(m+1) + 4227(m+1)^2 + 13,036(m+1)^3 + 17,962(m+1)^4}{+12,512(m+1)^5 + 4296(m+1)^6 + 576(m+1)^7 - 18} \right]} \\ & \Leftrightarrow \frac{a}{b} =: \frac{4(m+1) \left[\frac{1238m + 2702m^2 + 2496m^3}{+1008m^4 + 144m^5 + 177} \right]}{\left[\frac{349m + 4227m^2 + 13,036m^3 + 17,962m^4}{+12,512m^5 + 4296m^6 + 576m^7 - 18} \right]} \\ & > \frac{(m+4) \left[\frac{1238(m+1) + 2702(m+1)^2 + 2496(m+1)^3}{+1008(m+1)^4 + 144(m+1)^5 + 177} \right]}{\left[\frac{349(m+1) + 4227(m+1)^2 + 13,036(m+1)^3 + 17,962(m+1)^4}{+12,512(m+1)^5 + 4296(m+1)^6 + 576(m+1)^7 - 18} \right]} \\ & = : \frac{c}{d}. \end{aligned}$$

We calculated that

$$\begin{aligned} ad - bc &= 248,832m^{13} + 5,505,408m^{12} + 55,199,232m^{11} + 332,101,728m^{10} \\ &+ 1,337,449,104m^9 + 3,805,389,696m^8 + 7,862,263,692m^7 \\ &+ 11,923,195,508m^6 + 13,236,235,612m^5 + 10,578,592,104m^4 \\ &+ 5,876,921,975m^3 + 2,125,982,239m^2 + 440,485,650m + 38,040,600 \\ &> 0. \end{aligned}$$

4. Comparisons of New and Old Results

Through observations, in the literature, the more accurate upper and lower bounds of $I_0(t)$ are the second inequality in (11)

$$I_0(t) < \sqrt{\frac{\sinh t}{t} \cosh^3 t} \left(\frac{2t}{3} \right) \quad (38)$$

and the left-hand side of the inequality in (11)

$$\left(\frac{\sinh t}{t} \right)^{3/4} (\cosh t)^{1/4} < I_0(t). \quad (39)$$

Next, we compare them with the results of this paper.

(i) The numerical results show that inequality (20) is sharper than the one (38) on $(0, 5.2904)$ while that inequality (38) is sharper than the one (20) on $(5.2904, \infty)$. In other words, the two inequalities (21) and the second inequality in (11) cannot be compared.

(ii) The numerical results show that inequality (22) is sharper than the one (39) on $(0, 4.916)$ while that inequality (39) is sharper than the one (22) on $(4.916, \infty)$. In other words, the two inequalities (23) and the left-hand side of (11) cannot be compared.

5. Conclusions

Let $I_p(x)$ be the modified Bessel function of the first kind of order p . The double inequality

$$\frac{4659 \cosh 2t + 31,764t \sinh t - 9754t^2 - 4659}{16t^2(71 \cosh t + 1887)} < I_0(t) < \frac{8284t + 20,475 \sinh 2t + 4526t \cosh 2t}{256t(179 \cosh t + 31)}$$

was proved to hold for $t > 0$. The corresponding inequalities for a Toader-Qi mean are not in accordance with the conclusions in previous literature.

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References

1. Watson, G.N. *A Treatise on the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1962.
2. Abramowitz, M.; Stegun, I. *Handbook of Mathematical Functions*; U.S. National Bureau of Standards: Washington, DC, USA, 1964.
3. Baricz, Á. Some inequalities involving generalized Bessel functions. *Math. Ineq. Appl.* **2007**, *10*, 827–842. [[CrossRef](#)]
4. Baricz, Á. Generalized Bessel Functions of the First Kind. Ph.D. Thesis, Babes-Bolyai University, Cluj-Napoca, Romania, 2008.
5. Baricz, Á. Jordan-type inequalities for generalized Bessel functions. *J. Inequal. Pure Appl. Math.* **2008**, *9*, 39.
6. Baricz, Á.; Wu, S.H. Sharp Jordan-type inequalities for Bessel functions. *Publ. Math. Debr.* **2009**, *74*, 107–126.
7. Baricz, Á.; Neuman, E. Inequalities involving generalized Bessel functions. *J. Inequal. Pure Appl. Math.* **2005**, *6*, 126. [[CrossRef](#)]
8. Baricz, Á. Functional inequalities involving Bessel and modified Bessel functions of the first kind. *Expo. Math.* **2008**, *26*, 279–293 [[CrossRef](#)]
9. Baricz, Á. Landen-type inequality for Bessel functions. *Comput. Methods Funct. Theory* **2005**, *5*, 373–379. [[CrossRef](#)]
10. Baricz, Á. Functional inequalities involving special functions. *J. Math. Anal. Appl.* **2006**, *319*, 450–459. [[CrossRef](#)]
11. Baricz, Á. Functional inequalities involving special functions II. *J. Math. Anal. Appl.* **2007**, *327*, 1202–1213. [[CrossRef](#)]
12. Toader, G. Some mean values related to the arithmetic-geometric mean. *J. Math. Anal. Appl.* **1998**, *218*, 358–368. [[CrossRef](#)]
13. Qi, F.; Shi, X.-T.; Liu, F.-F.; Yang, Z.-H. A double inequality for an integral mean in terms of the exponential and logarithmic means. *Period Math. Hung.* **2017**, *75*, 180–189. [[CrossRef](#)]
14. Lin, T.-P. The power mean and the logarithmic mean. *Am. Math. Mon.* **1974**, *81*, 879–883. [[CrossRef](#)]
15. Stolarsky, K.B. Generalizations of the Logarithmic Mean. *Math. Mag.* **1975**, *48*, 87–92. [[CrossRef](#)]
16. Páles, Z. Inequalities for differences of powers. *J. Math. Anal. Appl.* **1988**, *131*, 271–281. [[CrossRef](#)]
17. Alzer, H. Ungleichungen für Mittelwerte. *Arch. Math.* **1986**, *47*, 422–426. [[CrossRef](#)]
18. Yang, Z.-H.; Chu, Y.-M. On approximating the modified Bessel function of the first kind and Toader-Qi mean. *J. Inequal. Appl.* **2016**, *2016*, 40. [[CrossRef](#)]
19. Yang, Z.-H.; Chu, Y.-M.; Song, Y.-Q. Sharp bounds for Toader-Qi mean in terms of logarithmic and identric means. *Math. Inequal. Appl.* **2016**, *19*, 721–730. [[CrossRef](#)]
20. Yang, Z.-H.; Tian, J.-F.; Zhu, Y.-R. New sharp bounds for the modified Bessel function of the first kind and Toader-Qi mean. *Mathematics* **2020**, *8*, 901. [[CrossRef](#)]
21. Gould, H.W. *Combinatorial Identities*; Morgantown Printing and Binding: Morgantown, WV, USA, 1972.