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The Domain of Residual Lifetime Attraction for the Classical Distributions of the Reliability Theory

Vladimir Rusev *,† and Alexander Skorikov *,†

Department of Higher Mathematics, National University of Oil and Gas (Gubkin University), Moscow 19991, Russia
* Correspondence: rusev.v@gubkin.ru (V.R.); skorikov.a@gubkin.ru (A.S.)
† These authors contributed equally to this work.

Abstract: The asymptotic behavior of the residual lifetime of the system and its characteristics are studied for the main distributions of reliability theory. Sufficiently precise and simple conditions for the domain of attraction of the exponential distribution are proposed, which are applicable for a wide class of distributions. This approach allows us to take into account important information about modeling the failure-free operation of equipment that has worked reliably for a long time. An analysis of the domain of attraction for popular distributions with "heavy tails" is given.

Keywords: Weibull–Gnedenko distribution; residual lifetime; domain of attraction; GG distribution; log-normal distribution; Burr distribution; LG distribution

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Citation: Rusev, V.; Skorikov, A. The Domain of Residual Lifetime Attraction for the Classical Distributions of the Reliability Theory. *Mathematics* **2021**, *9*, 2831. https://doi.org/10.3390/math9212831

Academic Editor: János Sztrik

Received: 30 September 2021
Accepted: 4 November 2021
Published: 8 November 2021

1. Introduction

The exponential distribution is widely used in reliability theory. It is well known that exponential distribution and the distribution of the residual lifetime have the property of being memoryless. However, usually the memoryless property (which is equivalent to the absence of ageing or wearing) does not agree with physical realities. Typical distributions that describe the service life of aging equipment are, for example, the Weibull–Gnedenko distributions and the gamma distribution. If the lifetime of pump submersible equipment [1] lasts long and is modeled by the Weibull–Gnedenko distribution, then it behaves asymptotically as if not ageing. A natural question arises about the set of equipment's lifetime distributions such that the normalized residual lifetime distributions asymptotically have the property of being memoryless. This article provides fairly simple sufficient conditions for such a class of distributions.

Let X be a non-negative random variable with probability distribution F. Consider the random variable $X_t = (X - t|X > t)$, which is called the residual life time (remaining useful life) with distribution F_t and distribution tail

$$P_t(x) = 1 - F_t(x) = P(X - t > x|X > t).$$

The mathematical expectation of random variable X_t, i.e., the function of the average residual time before failures (MRL), is defined as follows:

$$\mu(t) = M(X - t|X > t) = \frac{\int_t^{+\infty} P(x)dx}{P(t)},$$

where $P(x) = 1 - F(x)$ is the reliability function.

Consider the problem of asymptotic behavior of the residual lifetime distribution (tail distribution). For each limit distribution G, de Haan [2] determines the domain of residual lifetime attraction. The domain consists of all distribution functions F for which there is a normalizing function a such that a weak limit of the following:

$$\lim_{t\to\infty} F_t(xa(t)) = G(x) \tag{1}$$

exists. The possible limit types (see [3]) are as follows:

$$\Pi(x) = 1 - e^{-x},\ x \geq 0$$

and the following is the case.

$$\Gamma_\alpha(x) = 1 - (1+x)^{-\alpha},\ x \geq 0.$$

What is domain of residual lifetime attraction of exponential tail $\Pi(x)$? Thus, it is necessary to find conditions for the distribution for which there is a continuous function $a(t)$ such that a weak convergence of the distribution tails takes place.

$$P\left(\frac{X-t}{a(t)} > x | X > t\right) \to e^{-x}, \quad t \to +\infty.$$

The function $a(t)$ was obtained by de Haan [2], who proved that one can take the mean residual time of $\mu(t)$ as $a(t)$ for distribution $F(x) < 1$ for all x.

Remark that the notion of the domain of residual lifetime attraction is closely related to the theory of the classical extreme values [3]. Gnedenko (1943) determined the concept of "the domain of attraction" in extreme value theory. Since then, there was a real boom in the investigation of this field (see the Introduction by R. L. Smith in [4]).

The paper is organized as follows. In Section 2, we formulate the necessary and sufficient conditions by Balkema de Haan and the corresponding result for the Weibull–Gnedenko distribution. In Section 3, we formulate and prove the main results . We derive asymptotics for the coefficient of variation and prove the sufficient condition for describing the domain of residual lifetime attraction. In Section 4, we consider the problem of domain of attraction for GG distribution, log-normal distribution, Burr distribution and LG distribution. We prove also asymptotics for mean residual lifetime and residual variance for LG distribution.

2. Necessary and Sufficient Conditions of the Domain of Residual Life Time Attraction

There is a complete description of the structure of domain of residual lifetime attraction. The criterion Balkema de Haan (see [3] , Theorem 8) is as follows:

$$\lim_{t\to+\infty} c_v(t) = \lim_{t\to+\infty} \frac{\sigma(t)}{\mu(t)} = 1 \tag{2}$$

which provides the necessary and sufficient conditions for determining the domain of the residual lifetime attraction of the limiting exponential distribution. Earlier, the relationship between the asymptotic behavior of the distribution tails and the convergence of the first moments has been studied by Meilijson [5].

The authors of [6] obtained asymptotics of the mean residual life, residual variance and coefficient of variation for a Weibull–Gnedenko distribution with scale α and shape β parameters.

$$c_v(t) = \frac{\sigma(t)}{\mu(t)} = 1 + \frac{1-\beta}{\beta(\alpha t)^\beta} + o\left(\frac{1}{t^\beta}\right), \quad \alpha > 0, \beta > 0, \quad t \to +\infty.$$

These formulas and criterion (2) provide the necessary and sufficient conditions to belong to the domain of residual lifetime attraction for Weibull–Gnedenko distribution for all $\alpha > 0, \beta > 0$. Thus, it is established that the Weibull–Gnedenko distribution including those with a heavy tail $0 < \beta < 1$ belongs to the domain of residual lifetime attraction of the exponential distribution.

Note that the application of criterion of Balkema de Haan (2) is complicated by difficult calculations of the asymptotics of the moments. In extreme value theory, there exist known sufficient conditions that were obtained by von Mises [7]. It is of considerable interest to have convenient sufficient conditions for distribution functions that belong to the domain of residual lifetime attraction of the limit distribution. A convenient sufficient condition for a general class of distributions is presented in the next section. We will prove that the region of attraction of the residual lifetime is characterized by the derivatives of the logarithms of the distribution tail and distribution density. These conditions facilitate the distribution's examination.

3. Main Results

Theorem 1. *Let X be a random variable with a distribution function F, positive density $f(t) = F'(t),\ t \geq t_0$ and hazard rate function*

$$h(t) = f(t)/(1 - F(t)),$$

then the asymptotics for the coefficient of variation exists and the following is the case.

$$\lim_{t \to +\infty} (c_v(t))^2 = \lim_{t \to +\infty} h(t) \frac{\int\limits_t^{+\infty} dx \int\limits_x^{+\infty} (1 - F(\xi))d\xi}{\int\limits_t^{+\infty} (1 - F(x))\, dx}. \tag{3}$$

Proof. By using the following formulas:

$$\mu(t) = \frac{\int\limits_t^{+\infty} (1 - F(x))dx}{1 - F(t)}, \tag{4}$$

$$\sigma^2(t) = \frac{2}{1 - F(t)} \int\limits_t^{+\infty} (1 - F(x)) \cdot \mu(x)dx - \mu^2(t) \tag{5}$$

we have the following case.

$$\begin{aligned}
(c_v(t))^2 = \frac{\sigma^2(t)}{\mu^2(t)} &= 2 \cdot \frac{\int\limits_t^{+\infty} (1 - F(t))\mu(x)dx}{(1 - F(t)) \cdot \mu^2(t)} - 1 \\
&= 2 \cdot \int\limits_t^{+\infty} (1 - F(x))\mu(x)dx \Bigg/ \left((1 - F(t)) \cdot \frac{\left(\int\limits_t^{+\infty} (1 - F(x))dx \right)^2}{(1 - F(t))^2} \right) - 1 \\
&= 2 \cdot \frac{(1 - F(t)) \cdot \int\limits_t^{+\infty} (1 - F(x))\mu(x)dx}{\left(\int\limits_t^{+\infty} (1 - F(x))dx \right)^2} - 1.
\end{aligned}$$

By L'Hopital's formula, we calculate the following limit.

$$\lim_{t\to+\infty} \frac{(1-F(t))\cdot \int\limits_t^{+\infty}(1-F(x))\mu(x)dx}{\left(\int\limits_t^{+\infty}(1-F(x))dx\right)^2} =$$

$$\begin{aligned}
&= \lim_{t\to+\infty} \frac{-F'(t)\cdot \int\limits_t^{+\infty}(1-F(x))\mu(x)\,dx + (1-F(t))\cdot(-(1-F(t))\mu(t))}{-2\int\limits_t^{+\infty}(1-F(x))dx\cdot(1-F(t))} \\
&= \lim_{t\to+\infty} \frac{F'(t)\int\limits_t^{+\infty}(1-F(x))\mu(x)dx + (1-F(t))^2\cdot\mu(t)}{2\cdot(1-F(t))\int\limits_t^{+\infty}(1-F(x))dx} \\
&= \lim_{t\to+\infty} \frac{1}{2}\cdot \frac{F'(t)\int\limits_t^{+\infty}(1-F(x))\mu(x)dx + (1-F(t))\int\limits_t^{+\infty}(1-F(x))dx}{(1-F(t))\int\limits_t^{+\infty}(1-F(x))dx}.
\end{aligned}$$

The statement follows from above and equation $(1-F(x))\mu(x) = \int\limits_x^{+\infty}(1-F(\xi))\,d\xi$.

$$\lim_{t\to+\infty}(c_v(t))^2 = 2\cdot\frac{1}{2}\cdot\lim_{t\to+\infty}\left[\frac{f(t)}{1-F(t)}\cdot\frac{\int\limits_t^{+\infty}(1-F(x))\mu(x)dx}{\int\limits_t^{+\infty}(1-F(x))dx}+1\right]-1$$

$$= \lim_{t\to+\infty} h(t)\cdot\frac{\int\limits_t^{+\infty}(1-F(x))\mu(x)dx}{\int\limits_t^{+\infty}(1-F(x))dx} = \lim_{t\to+\infty} h(t)\frac{\int\limits_t^{+\infty}dx\int\limits_x^{+\infty}(1-F(\xi))d\xi}{\int\limits_t^{+\infty}(1-F(x))dx}.$$

□

Next, we use the asymptotic of integrals with a variable lower limit (see Theorem 2.6 [8]):

$$\int\limits_t^{+\infty} f(x)e^{-S(x)}dx \sim \frac{f(t)}{S'(t)}e^{-S(t)}, \quad (t\to+\infty), \tag{6}$$

where functions $f(x)$ and $S(x)$ satisfy the following conditions:

$$A_1. \quad f(x)\in C^1,\ S(x)\in C^2,\ S'(x)>0,\ S(+\infty)=+\infty; \tag{7}$$

$$A_2. \quad S''(x) = o(S'^2(x)), \quad x\to+\infty; \tag{8}$$

$$A_3. \quad f(x)>0,\ x\geq 0;\ \frac{f'(x)}{f(x)} = o(S'(x)),\ x\to+\infty, \tag{9}$$

and C^1 and C^2 are classes of the continuously differentiable and twice continuously differentiable functions, respectively. We will derive the domain of attraction for exponential distribution by Theorem 1 and asymptotics (6).

Theorem 2. *Let X be a random variable with a distribution function F, $F(0) = 0$, positive density $f(t) = F'(t)$, $t \geq t_0$, $f(t) \in C^1$ and hazard rate function $h(t) = f(t)/(1 - F(t))$, $h(t) \in C^1$, $\int\limits_0^{+\infty} h(x)dx = +\infty$.*

Distribution function F belongs to the domain of residual life time attraction of the exponential distribution if the following is the case.

$$(\ln f(t))' \sim (\ln(1 - F(t)))', \quad t \to +\infty. \tag{10}$$

Proof. Condition $F(0) = 0$ implies the following.

$$F(t) = 1 - e^{-\Lambda(t)}, \quad \Lambda(t) = \int\limits_0^t h(x)dx. \tag{11}$$

By Theorem 1 and asymptotics (6), we have the following.

$$\lim_{t\to+\infty} (c_v(t))^2 = \lim_{t\to+\infty} h(t) \cdot \frac{\int\limits_t^{+\infty} dx \int\limits_x^{+\infty} (1 - F(\xi))d\xi}{\int\limits_t^{+\infty} (1 - F(x))dx}$$

$$= \lim_{t\to+\infty} h(t) \cdot \frac{\int\limits_t^{+\infty} dx \int\limits_x^{+\infty} e^{-\Lambda(\xi)}d\xi}{\int\limits_t^{+\infty} e^{-\Lambda(x)}dx}$$

$$= \lim_{t\to+\infty} h(t) \cdot \frac{\int\limits_t^{+\infty} \frac{e^{-\Lambda(x)}}{h(x)}dx}{\frac{e^{-\Lambda(t)}}{h(t)}} = \lim_{t\to+\infty} h(t) \cdot \frac{\frac{1}{(h(t))^2}e^{-\Lambda(t)}}{\frac{e^{-\Lambda(t)}}{h(t)}} = 1.$$

We examine condition A2. From (10), the following is the case.

$$1 = \lim_{t\to+\infty} \frac{(\ln f(t))'}{(\ln P(t))'} = \lim_{t\to+\infty} \frac{f'(t)}{f(t)} \cdot \frac{P(t)}{P'(t)} = -\lim_{t\to+\infty} \frac{f'(t)}{f(t)} \cdot \frac{P(t)}{f(t)}.$$

Hence, from the following:

$$\frac{h'(t)}{h^2(t)} = \frac{f'(t)(1 - F(t)) + f^2(t)}{f^2(t)} = 1 + \frac{f'(t)(1 - F(t))}{f^2(t)} \to 0, \ t \to +\infty,$$

we obtain the following case:

$$h'(t) = o(h^2(t)), \quad t \to +\infty. \tag{12}$$

It is easy to see that the verification of condition (9) reduces to the verification of (12). □

4. Examples

4.1. Exponential Distribution

The probability density function is $f(t) = \lambda e^{-\lambda t}$, and the cumulative distribution function is $F(t) = 1 - e^{-\lambda t}$. By Theorem 2, the following is the case.

$$\lim_{t\to+\infty} \frac{(\ln f(t))'}{(\ln(1 - F(t)))'} = \lim_{t\to+\infty} \frac{-\lambda}{-\lambda} = 1.$$

The exponential distribution belongs to its own domain of residual lifetime attraction for $\lambda > 0$.

4.2. Generalized Gamma Distribution (GG Distribution)

The probability density function (PDF) and the corresponding cumulative distribution function (CDF) are given by the following:

$$f(x) = \frac{\left(\frac{p}{a^d}\right) x^{d-1} e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}, \qquad F(x) = \frac{\gamma\left(\frac{d}{p}, \left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}, \ a > 0, \ \ p > 0, \quad d > 0, \tag{13}$$

where $\gamma(s,x) = \int\limits_o^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function, and $\Gamma(\cdot)$ denotes complete gamma function. The generalized gamma distribution is extremely useful for parametric models in survival analysis. This distribution is introduced by Stacy [9] as containing both the gamma distribution and the Weibull–Gnedenko distribution $(d = p)$. We use the sufficient condition (10) of Theorem 2. Let $a = 1$. It is easy to derive by using asymptotic (6).

$$(\ln(1 - F(x)))'$$

$$= \frac{-e^{-x^p} \cdot x^{p\left(\frac{d}{p}-1\right)} \cdot px^{p-1}}{\int\limits_{x^p}^{+\infty} e^{-t} \cdot t^{\frac{d}{p}-1} dt} \sim \frac{-e^{-x^p} \cdot x^{d-p} \cdot px^{p-1}}{e^{-x^p} \cdot x^{d-p}} = -px^{p-1}, \quad x \to +\infty.$$

Thus, the following is the case.

$$\lim_{x \to +\infty} \frac{(\ln f(x))'}{(\ln(1 - F(x)))'}$$

$$= \lim_{x \to +\infty} \frac{\left(\ln p + (d-1)\ln x - x^p - \ln \Gamma\left(\frac{d}{p}\right)\right)'}{\left(\ln \int\limits_{x^p}^{+\infty} e^{-t} \cdot t^{\frac{d}{p}-1} dt - \ln \Gamma\left(\frac{d}{p}\right)\right)'}$$

$$= \lim_{x \to +\infty} \frac{\frac{d-1}{x} - px^{p-1}}{-px^{p-1}} = 1.$$

Then, the condition of Theorem 2 for the generalized gamma distribution is fulfilled. Theorem 2 yields that the GG distribution belongs to the domain of residual lifetime attraction of the exponential distribution.

4.3. Log-Normal Distribution

The log-normal distribution is commonly and widely used in the description of natural growth processes in biology, medicine, chemistry, social sciences and demographics and scientometrics. With respect to the reliability theory, this distribution is applied for modelling times to repair a maintainance system and internet traffic. The PDF of the log-normal distribution can be written as follows.

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}.$$

Let $\mu = 0, \sigma = 1$. Then, the following is the case.

$$f(x) = \frac{\alpha}{x} e^{-\frac{(\ln x)^2}{2}}, \quad 1 - F(x) = \alpha \cdot \int\limits_{\ln x}^{+\infty} e^{-\frac{t^2}{2}} dt, \quad \alpha = \frac{1}{\sqrt{2\pi}}.$$

We use the well-known relation for the tail of the standard normal distribution function.

$$1 - \Phi(u) \sim \frac{\varphi(u)}{u}, \; u \to +\infty.$$

This implies the following.

$$\int\limits_{\ln x}^{+\infty} e^{-\frac{t^2}{2}} dt \sim \frac{-e^{-\frac{(\ln x)^2}{2}}}{\ln x}, \qquad x \to +\infty;$$

Thus, the following is the case.

$$(\ln(1 - F(x)))' = \frac{-e^{-\frac{(\ln x)^2}{2}}}{x \cdot \int\limits_{\ln x}^{+\infty} e^{-\frac{t^2}{2}} dt} \sim \frac{-e^{-\frac{(\ln x)^2}{2}}}{x \cdot e^{-\frac{(\ln x)^2}{2}}} \cdot \ln x = -\frac{\ln x}{x}, \quad x \to +\infty.$$

Then, one can join the equations of the following:

$$(\ln f(x))' = \left(\ln \alpha - \ln x - \frac{1}{2} (\ln x)^2 \right)' = -\frac{1}{x} - \frac{\ln x}{x}$$

and we obtain sufficient condition (10).

$$(\ln f(x))' \sim (\ln(1 - F(x)))', \; x \to +\infty.$$

4.4. Burr Distribution

We consider the Burr type XII distribution (see [10]). The random variable X is said to have the Burr-distribution if its probability density function and cumulative distribution function are given by the following.

$$f(t) = \frac{ck \cdot t^{c-1}}{(1 + t^c)^{k+1}}, \quad F(t) = 1 - \frac{1}{(1 + t^c)^k}, \quad c > 0, \quad k > 0, \quad t \geq 0.$$

This distribution is used in many areas such as reliability, biology, engineering, applied statistics and econometrics.

Let us check condition of Theorem 2 . Since the following is the case:

$$\ln f(t) = \ln ck + (c - 1) \ln t - (k + 1) \ln(1 + t^c);$$

$$(\ln f(t))' = (c - 1)/t - (k + 1) ct^{c-1}/(1 + t^c);$$

$$\ln(1 - F(t)) = -k \ln(1 + t^c);$$

$$(\ln(1 - F(t)))' = -kct^{c-1}/(1 + t^c);$$

then, the following obtains.

$$\frac{(\ln f(t))'}{(\ln(1 - F(t)))'} = \frac{k + 1}{k} - \frac{c - 1}{kc} \cdot \frac{1 + t^c}{t^c} \to 1 + \frac{1}{kc} \neq 1, \; t \to +\infty.$$

The sufficient condition is not fulfilled.

Let us check the criterion of Balkema de Haan (2) for the coefficient of variation (3).

We have the following case.

$$h(t) = \frac{f(t)}{1-F(t)} = \frac{ck \cdot t^{c-1} \cdot (1+t^c)^k}{(1+t^c)^{k+1}} = \frac{ck \cdot t^{c-1}}{1+t^c}.$$

It is obvious that the following is the case:

$$\int_t^{+\infty} (1-F(x))dx = \int_t^{+\infty} \frac{dx}{(1+x^c)^k} \sim \int_t^{+\infty} \frac{dx}{x^{ck}}, \quad t \to +\infty, \quad ck > 1;$$

$$\int_t^{+\infty} x^{-ck}dx = \frac{x^{-ck+1}}{-ck+1}\bigg|_t^{+\infty} = \frac{-1}{1-ck} \cdot \frac{1}{t^{ck-1}} = \frac{1}{ck-1} \cdot \frac{1}{t^{ck-1}};$$

consequently, the following obtains:

$$\int_t^{+\infty} dx \int_x^{+\infty} (1-F(\xi))d\xi \sim \int_t^{+\infty} \frac{dx}{(ck-1)x^{ck-1}} = \frac{1}{ck-1} \int_t^{+\infty} x^{1-ck}dx =$$

$$= \frac{1}{(ck-1)} \cdot \frac{x^{-ck+2}}{-ck+2}\bigg|_t^{+\infty} = \frac{1}{(ck-1)(ck-2)} \cdot \frac{1}{t^{ck-2}}, \quad t \to +\infty$$

for $ck-2>0$. Whence by Theorem 1, the following is the case.

$$\lim_{t\to+\infty} (c_v(t))^2 = \lim_{t\to+\infty} \frac{h(t) \cdot \int_t^{+\infty} dx \int_x^{+\infty} (1-F(\xi))d\xi}{\int_t^{+\infty} (1-F(x))dx}$$

$$= \lim_{t\to+\infty} \left(\frac{ck \cdot t^{c-1}}{1+t^c} \cdot \frac{1}{(ck-1)(ck-2)t^{ck-2}} \right) \div \left(\frac{1}{ck-1} \cdot \frac{1}{t^{ck-1}} \right)$$

$$= \lim_{t\to+\infty} \frac{ck \cdot t^{c-1}}{t^c} \cdot \frac{t^{ck-1}}{(ck-2)t^{ck-2-1}} = \lim_{t\to+\infty} \frac{ck \cdot t^{-1}}{(ck-2) \cdot t^{-1}} = \frac{ck}{ck-2} \neq 1.$$

Thus, the Burr distribution does not belong to the domain of residual lifetime attraction of the exponential distribution and is contained in the region of residual attraction lifetime of the power-law distribution $\Gamma_\alpha(\cdot)$ (see [3]).

4.5. Log-Gamma Distribution (LG Distribution)

The logarithmic gamma distribution is widely used in hydrology and in the analysis of system reliability. For example, according to the recommendation of the United States Water Resources Council (W.R.C.), this distribution is the official method for predicting flood intervals. The two-parameter gamma distribution has the following PDF and reliability function.

$$f(x) = \frac{\alpha^\beta}{\Gamma(\beta)} (\ln x)^{\beta-1} \cdot x^{-\alpha-1}$$

$$1-F(x) = \frac{\alpha^\beta}{\Gamma(\beta)} \int_x^{+\infty} (\ln t)^{\beta-1} \cdot t^{-\alpha-1}dt,$$

$$\alpha > 0, \ \beta > 0, \ x > 1.$$

We use sufficient condition (10) of Theorem 2. It is evident that the following is the case.

$$(\ln f(x))' = \left(\ln\left(\frac{\alpha^\beta}{\Gamma(\beta)}\right) + (\beta-1)\ln(\ln x) - (\alpha+1)\ln x\right)' = \frac{\beta-1}{x\ln x} - \frac{\alpha+1}{x};$$

$$(\ln(1-F(x)))' = \left(\ln\left(\frac{\alpha^\beta}{\Gamma(\beta)}\right) + \ln\int_x^{+\infty}(\ln t)^{\beta-1}\cdot t^{-\alpha-1}dt\right)' = \frac{-(\ln x)^{\beta-1}\cdot x^{-\alpha-1}}{\int_x^{+\infty}(\ln t)^{\beta-1}\cdot t^{-\alpha-1}dt}.$$

We obtain the following by using the transformation $z = \ln t$ and asymptotics (6):

$$\int_x^{+\infty}(\ln t)^{\beta-1}\cdot t^{-\alpha-1}dt = \int_{\ln x}^{+\infty} z^{\beta-1}e^{-\alpha z}dz \sim \frac{(\ln x)^{\beta-1}}{\alpha}x^{-\alpha},\ x \to +\infty$$

and the following is the case.

$$(\ln(1-F(x)))' \sim -\frac{\alpha}{x},\ x \to +\infty.$$

In this case, the following is obtained:

$$\frac{(\ln f(x))'}{(\ln(1-F(x)))'} \sim \frac{\alpha+1}{\alpha} - \frac{\beta-1}{\alpha\ln x} \to \frac{\alpha+1}{\alpha} \neq 1,\ x \to +\infty,$$

and we cannot apply Theorem 2.

Theorem 3. *Let X be a random variable with the two-parameter log-gamma distribution, then* $\forall\alpha > 2$, $\forall\beta > 0$ *and the asymptotic expansion of MRL function is as follows.*

$$\mu(t) = \frac{t}{\alpha-1}\cdot\left[1 + \frac{(\alpha-1)(\beta-1)}{\alpha t} + O\left(\frac{1}{t^2}\right)\right],\ t \to +\infty; \tag{14}$$

The asymptotic expansion of residual variance is as follows.

$$\sigma^2(t) = \frac{t^2}{(\alpha-1)^2}\cdot\left[\frac{\alpha}{\alpha-2} + \frac{2(\beta-1)}{t} + O\left(\frac{1}{t^2}\right)\right],\quad t \to +\infty. \tag{15}$$

Proof. By using the transformation $z = \alpha\ln t$ and asymptotic expansion 8.357 (see [11]) of the upper incomplete gamma function, we obtain the following:

$$1 - F(x)$$
$$= \frac{\alpha^\beta}{\Gamma(\beta)}\cdot\frac{1}{\alpha}(\ln x)^{\beta-1}\cdot x^{-\alpha}\cdot\left[1 + \frac{\beta-1}{x} + \frac{(\beta-1)(\beta-2)}{x^2} + O\left(\frac{1}{x^3}\right)\right],\ x \to \infty; \tag{16}$$

and the following is the case.

$$\int_t^{+\infty}(1-F(x))dx$$
$$= \frac{\alpha^\beta}{\Gamma(\beta)}\cdot\left[\int_t^{+\infty}(\ln x)^{\beta-1}\cdot x^{-\alpha}dx + (\beta-1)\int_t^{+\infty}(\ln x)^{\beta-1}\cdot x^{-\alpha-1}dx + \cdots\right].$$

We may denote the following.

$$I_1 = \int_t^{+\infty} (\ln x)^{\beta-1} \cdot x^{-\alpha} dx, \; I_2 = (\beta-1) \int_t^{+\infty} (\ln x)^{\beta-1} \cdot x^{-\alpha-1} dx, \ldots .$$

In addition, by expansion 8.357 from Reference [11], we have the following.

$$I_1 = \frac{1}{\alpha-1} (\ln t)^{\beta-1} \cdot t^{-\alpha+1} \cdot \left[1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + O\left(\frac{1}{t^3}\right) \right], \quad t \to +\infty, \tag{17}$$

$$I_2 = \frac{1}{\alpha} (\ln t)^{\beta-1} \cdot t^{-\alpha} \cdot \left[1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + O\left(\frac{1}{t^3}\right) \right], \quad t \to +\infty. \tag{18}$$

Therefore, the following obtains.

$$\begin{aligned}
&\int_t^{+\infty} (1 - F(x)) dx \\
= \;& \frac{\alpha^\beta}{\Gamma(\beta)} \cdot (\ln t)^{\beta-1} \cdot t^{-\alpha} \cdot \left[\frac{1}{\alpha-1} t \left(1 + \frac{A}{t} + \frac{B}{t^2} + \cdots \right) \right. \\
&\left. + \frac{\beta-1}{\alpha} \left(\frac{1}{\alpha-1} t \left(1 + \frac{A}{t} + \frac{B}{t^2} \cdots \right) \right) + \frac{(\beta-1)(\beta-2)}{(\alpha+1)t} \left(1 + \frac{A}{t} + \frac{B}{t^2} \cdots \right) \right] \\
&= \frac{\alpha^\beta}{\Gamma(\beta)} (\ln t)^{\beta-1} \cdot t^{-\alpha} \cdot \left[\frac{t}{\alpha-1} + \frac{\beta-1}{\alpha-1} + \frac{(\beta-1)(\beta-2)}{(\alpha-1) \cdot t} + \cdots \right. \\
&+ \frac{\beta-1}{\alpha} + \frac{(\beta-1)^2}{\alpha \cdot t} + \frac{(\beta-1)^2 \cdot (\beta-2)}{\alpha \cdot t^2} + \cdots \\
&\left. + \frac{(\beta-1)(\beta-2)}{(\alpha+1) \cdot t} + \frac{(\beta-1)^2 (\beta-2)}{(\alpha+1) \cdot t^2} + \frac{(\beta-1)^2 (\beta-2)^2}{(\alpha+1) \cdot t^3} + \cdots \right] \\
&= \frac{\alpha^\beta}{\Gamma(\beta)} \cdot (\ln t)^{\beta-1} \cdot t^{-\alpha} \cdot \left[\frac{t}{\alpha-1} + (\beta-1) \left(\frac{1}{\alpha} + \frac{1}{\alpha-1} \right) \right. \\
&+ (\beta-1) \left(\frac{\beta-2}{\alpha-1} + \frac{\beta-1}{\alpha} + \frac{\beta-2}{\alpha+1} \right) \cdot \frac{1}{t} \\
&\left. + \frac{1}{t^2} (\beta-1)^2 (\beta-2) \left(\frac{1}{\alpha} + \frac{1}{\alpha+1} \right) + \cdots \right].
\end{aligned}$$

Then, Equation (4) and the relation of the following:

$$\begin{aligned}
&\frac{1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots}{1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots} \\
&= 1 + (a_1 - b_1) x + (a_2 - (b_1 c_1 + b_2)) x^2 + \left(a_3 - \left(b_1 c_2 + b_2 c_1 + b^3 \right) \right) x^3 + \cdots
\end{aligned}$$

imply asymptotics $\mu(t)$:

$$\mu(t) = \frac{t}{\alpha-1} \cdot \left[1 + \frac{\widetilde{A}}{t} + \frac{\widetilde{B}}{t^2} + O\left(\frac{1}{t^3}\right) \right], \quad t \to \infty, \tag{19}$$

where the following is the case.

$$\widetilde{A} = \frac{(\alpha-1)(\beta-1)}{\alpha}, \quad \widetilde{B} = (\beta-1)(\beta-2)\frac{2-(\alpha-1)^2}{\alpha^2-1}. \tag{20}$$

Following these procedures, one is able to prove (15) (see Appendix A). $\square$

Similarly, from (14) and (15), the following is the case.

Corollary 1. *For the two-parameter log-gamma distribution $\alpha > 2,\ \beta > 0$, the following holds.*

$$\lim_{t\to\infty} (c_v(t))^2 = \lim_{t\to\infty} \frac{\sigma^2(t)}{\mu^2(t)} = \frac{\alpha}{\alpha-2} \neq 1. \tag{21}$$

Using this asymptotic, we find that necessary condition (2) for the log-gamma distribution is not satisfied. Consequently, the log-gamma distribution does not belong to the domain of residual life attraction of exponential distributions.

Remark 1. *We have asymptotics for the hazard rate function of log-gamma distribution.*

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} \cdot x^{-\alpha-1} \div \frac{\alpha^\beta}{\Gamma(\beta)} \int_x^{+\infty} (\ln t)^{\beta-1} \cdot t^{-\alpha-1} dt$$

$$\sim \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} \cdot x^{-\alpha-1} \div \frac{\alpha^\beta}{\alpha\Gamma(\beta)}(\ln x)^{\beta-1} \cdot x^{-\alpha} = \frac{\alpha}{x}, \quad x \to +\infty.$$

Thus, there is the asymptotic of the following.

$$h(x) \cdot \mu(x) \to \frac{\alpha}{\alpha-1} \neq 1,\ x \to +\infty.$$

This implies that the MRL function is not asymptotically inverse to the hazard rate function (discussion in [12]).

5. Conclusions

The residual lifetime distribution is described in terms of the asymptotic behavior residual mean and residual variance. We have proposed new sufficient conditions for determining the domain of residual lifetime attraction. The usefulness of the proposed conditions is illustrated by five examples of distributions. Explicit asymptotic expressions are found for the mean and variance of residual lifetime log-gamma distribution.

Author Contributions: Conceptualization, methodology, writing, A.S.; formal analysis, validation, writing—review and editing, V.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MRL	Mean residual life;
PDF	Probability density function;
CDF	Cumulative distribution function;
LG	Two-parameter log-gamma distribution;
GG	Generalized gamma distribution.

Appendix A. Proof of the Expansion (15)

Proof. We shall derive expansion (15) by using the following known relationship.

$$\sigma^2(t) = \frac{2}{1-F(t)} \int_t^{+\infty} (1-F(x)) \cdot \mu(x)dx - \mu^2(t).$$

In accordance with (14), by computing the coefficients of expansion $\mu^2(t)$, we obtain the following:

$$\mu^2(t) = \frac{t^2}{(\alpha-1)^2}\left[1 + \frac{2\widetilde{A}}{t} + \frac{2\widetilde{B} + \widetilde{A}^2}{t} + O\left(\frac{1}{t^3}\right)\right], \quad t \to +\infty,$$

where coefficients $\widetilde{A}$ and $\widetilde{B}$ are defined by Formula (20). At the next juncture, we would like to obtain the expansion of the first term of residual variance. Using (16) and (19), it can be easily established for $\alpha > 2$.

$$\int_t^{+\infty} (1-F(x))\mu(x)dx =$$

$$= \frac{\alpha^{\beta-1}}{\Gamma(\beta)(\alpha-1)}\left[\int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha+1} dx + \left(\beta - 1 + \widetilde{A}\right)\int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha} dx + \right.$$

$$\left. + \left((\beta-1)\widetilde{A} + (\beta-1)(\beta-2) + \widetilde{B}\right)\int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha-1} dx + \cdots\right].$$

It is clearly similar to (17) and (18).

$$J_1 = \int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha+1} dx = \frac{(\ln t)^{\beta-1} t^{-\alpha+2}}{\alpha-2}\left[1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + \cdots\right];$$

$$J_2 = \int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha} dx = \frac{(\ln t)^{\beta-1} t^{-\alpha+1}}{\alpha-1}\left[1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + \cdots\right];$$

$$J_3 = \int_t^{+\infty} (\ln x)^{\beta-1} x^{-\alpha-1} dx = \frac{(\ln t)^{\beta-1} t^{-\alpha}}{\alpha}\left[1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + \cdots\right].$$

Then, the above expansions provide the following after some cumbrous algebra:

$$\int_t^{+\infty} (1-F(x))\mu(x)dx$$

$$= \frac{\alpha^{\beta-1}}{\Gamma(\beta)\cdot(\alpha-1)}\left[J_1 + \left(\beta-1+\widetilde{A}\right)J_2 + \left((\beta-1)\widetilde{A} + (\beta-1)(\beta-2) + \widetilde{B}\right)J_3\right]$$

$$= \frac{\alpha^{\beta-1}(\ln t)^{\beta-1}\cdot t^{-\alpha}}{\Gamma(\beta)\cdot(\alpha-1)}\left\{\frac{t^2}{\alpha-2}\cdot\left(1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + \cdots\right)\right.$$

$$+ \left((\beta-1) + \widetilde{A}\right)\frac{t}{\alpha-1}\left(1 + \frac{\beta-1}{t} + \frac{(\beta-1)(\beta-2)}{t^2} + \cdots\right)$$

$$+\Big((\beta-1)\widetilde{A}+(\beta-1)(\beta-2)+\widetilde{B}\Big)\frac{1}{\alpha}\left(1+\frac{\beta-1}{t}+\frac{(\beta-1)(\beta-2)}{t^2}+\cdots\right)\Bigg\}$$

$$=\frac{\alpha^{\beta-1}}{\Gamma(\beta)\cdot(\alpha-1)}(\ln t)^{\beta-1}\cdot t^{-\alpha}\left[\frac{t^2}{\alpha-2}+\left(\frac{\beta-1}{\alpha-2}+\frac{(\beta-1+\widetilde{B})}{\alpha-1}\right)t\right.$$

$$\left.+\frac{(\beta-1)(\beta-2)}{\alpha-2}+\frac{(\beta-1+\widetilde{A})}{\alpha-1}(\beta-1)+\frac{(\beta-1)\widetilde{A}+(\beta-1)(\beta-2)+\widetilde{B}}{\alpha}+\cdots\right].$$

$$\sigma^2(t)=\frac{2}{1-F(t)}\int_t^{+\infty}(1-F(x))\mu(x)dx-\mu^2(t)$$

$$=2\cdot\frac{\alpha^{\beta-1}}{\Gamma(\beta)(\alpha-1)}(\ln t)^{\beta-1}\cdot t^{-\alpha+2}\left[\frac{1}{\alpha-2}+\left(\frac{\beta-1}{\alpha-2}+\frac{\beta-1+\widetilde{A}}{\alpha-1}\right)\frac{1}{t}\right.$$

$$+\left[\frac{(\beta-1)(\beta-2)}{\alpha-2}+\frac{\beta-1+\widetilde{A}}{\alpha-1}(\beta-1)\right.$$

$$\left.\left.+\frac{(\beta-1)\widetilde{A}+\Big((\beta-1)(\beta-2)+\widetilde{B}\Big)}{\alpha}\right]\frac{1}{t^2}+\cdots\right]$$

$$\div\frac{\alpha^{\beta-1}}{\Gamma(\beta)}(\ln t)^{\beta-1}\cdot t^{-\alpha}\cdot\left[1+\frac{\beta-1}{t}+\frac{(\beta-1)(\beta-2)}{t^2}+\cdots\right]$$

$$-\frac{t^2}{(\alpha-1)^2}\left(1+\frac{2\widetilde{A}}{t}+\frac{2\widetilde{B}+\widetilde{A}^2}{t^2}+\cdots\right)$$

$$=\frac{2\cdot t^2}{(\alpha-1)(\alpha-2)}\left(1+(\alpha-2)\left(\frac{\beta-1}{\alpha-2}+\frac{\beta-1+\tilde{A}}{\alpha-1}\right)\frac{1}{t}+\gamma(\alpha-2)\cdot\frac{1}{t^2}+\cdots\right)$$

$$\div\left(1+\frac{\beta-1}{t}+\frac{(\beta-1)(\beta-2)}{t^2}+\cdots\right)$$

$$-\frac{t^2}{(\alpha-1)^2}\left(1+\frac{2\widetilde{A}}{t}+\frac{2\widetilde{B}+\widetilde{A}^2}{t^2}+\cdots\right)=$$

$$=\frac{2t^2}{(\alpha-1)(\alpha-2)}\cdot\left(1+\left((\alpha-1)\left(\frac{\beta-1}{\alpha-2}+\frac{\beta-1+\widetilde{A}}{\alpha-1}\right)-(\beta-1)\right)\frac{1}{t}\right.$$

$$\left.+\left[\gamma(\alpha-2)-\left((\beta-1)\cdot\frac{\alpha-2}{\alpha-1}\Big(\beta-1+\widetilde{A}\Big)+(\beta-1)(\beta-2)\right)\right]\frac{1}{t^2}+\cdots\right.$$

$$-\frac{t^2}{(\alpha-1)^2}\left(1+\frac{2\widetilde{A}}{t}+\frac{2\widetilde{B}+\widetilde{A}^2}{t^2}+\cdots\right)$$

$$= t^2\left[\frac{2}{(\alpha-1)(\alpha-2)} - \frac{1}{(\alpha-1)^2} + \left(\frac{\alpha-2}{\alpha-1}\cdot\left(\beta-1+\widetilde{A}\right)-2\widetilde{A}\right)\frac{1}{t}\right.$$

$$\left.+\left[\gamma(\alpha-2)-\left((\beta-1)\cdot\frac{\alpha-2}{\alpha-1}\left(\beta-1+\widetilde{A}\right)+(\beta-1)(\beta-2)\right)\right]\frac{1}{t^2}+\cdots\right.$$

$$\left.-\frac{t^2}{(\alpha-1)^2}\left(1+\frac{2\widetilde{A}}{t}+\frac{2\widetilde{B}+\widetilde{A}^2}{t^2}+\cdots\right)\right.$$

$$= t^2\left[\frac{2}{(\alpha-1)(\alpha-2)} - \frac{1}{(\alpha-1)^2} + \left(\frac{\alpha-2}{\alpha-1}\cdot\left(\beta-1+\widetilde{A}\right)-2\widetilde{A}\right)\frac{1}{t}\right.$$

$$+\left(\gamma(\alpha-2)-\frac{(\beta-1)(\alpha-2)}{\alpha-1}(\beta-1+\tilde{A})\right.$$

$$\left.\left.+(\beta-1)(\beta-2)-\left(2\tilde{B}+\tilde{A}^2\right)\right)\frac{1}{t^2}+\cdots\right]$$

$$= t^2\left(a+\frac{b}{t}+\frac{c}{t^2}+O\left(\frac{1}{t^3}\right)\right) \quad t\to+\infty.$$

where the following is the case.

$$\gamma = \frac{(\beta-1)(\beta-2)}{\alpha-2} + \frac{\beta-1+\widetilde{A}}{\alpha-1}(\beta-1) + \frac{(\beta-1)\widetilde{A}+\left((\beta-1)(\beta-2)+\widetilde{B}\right)}{\alpha},$$

By simplifying the above expressions, we obtain the following.

$$a = \frac{2}{(\alpha-1)(\alpha-2)} - \frac{1}{(\alpha-1)^2} = \frac{2(\alpha-1)-(\alpha-2)}{(\alpha-1^2)(\alpha-2)}$$

$$= \frac{2\alpha-2-\alpha+2}{(\alpha-1^2)(\alpha-2)} = \frac{\alpha}{(\alpha-1^2)(\alpha-2)}$$

$$= \frac{2}{(\alpha-1)^2}(\beta-1)\cdot\left(1+\frac{\alpha-1}{\alpha}-\frac{\alpha-1}{\alpha}\right) = \frac{2(\beta-1)}{(\alpha-1)^2};$$

$$b = \frac{\alpha-2}{\alpha-1}\cdot\left(\beta-1+\widetilde{A}\right)-2\widetilde{A}$$

$$= \frac{2}{(\alpha-1)^2}[\beta-1]\cdot\left(1+\frac{\alpha-1}{\alpha}-\frac{\alpha-1}{\alpha}\right) = \frac{2(\beta-1)}{(\alpha-1)^2};$$

$$c = \gamma(\alpha-2) - \frac{(\beta-1)(\alpha-2)}{\alpha-1}\left(\beta-1+\widetilde{A}\right)+(\beta-1)(\beta-2)-\left(2\widetilde{B}+\widetilde{A}^2\right).$$

Thus, the following is obtained.

$$\sigma^2(t) = t^2\left(\frac{\alpha}{(\alpha-1)^2(\alpha-2)}+\frac{2(\beta-1)}{(\alpha-1)^2}\cdot\frac{1}{t}+\cdots\right)$$

$$= \frac{t^2}{(\alpha-1)^2}\left[\frac{\alpha}{\alpha-2}+\frac{2(\beta-1)}{t}+\cdots\right], \quad t\to+\infty.$$

□

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