mathematics MDPI

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Doss ρ-Almost Periodic Type Functions in $\mathbb{R}^n$

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Abstract: In this paper, we investigate various classes of multi-dimensional Doss ρ-almost periodic type functions of the form $F : \Lambda \times X \to Y$, where $n \in \mathbb{N}$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, X and Y are complex Banach spaces, and ρ is a binary relation on Y. We work in the general setting of Lebesgue spaces with variable exponents. The main structural properties of multi-dimensional Doss ρ-almost periodic type functions, like the translation invariance, the convolution invariance and the invariance under the actions of convolution products, are clarified. We examine connections of Doss ρ-almost periodic type functions with (ω, c)-periodic functions and Weyl-ρ-almost periodic type functions in the multi-dimensional setting. Certain applications of our results to the abstract Volterra integro-differential equations and the partial differential equations are given.

Keywords: Doss ρ-almost periodic type functions in $\mathbb{R}^n$; Lebesgue spaces with variable exponents; abstract Volterra integro-differential equations

MSC: 42A75; 43A60; 47D99

Citation: Kostić, M.; Du, W.-S.; Fedorov, V.E. Doss ρ-Almost Periodic Type Functions in $\mathbb{R}^n$. *Mathematics* **2021**, *9*, 2825. https://doi.org/10.3390/math9212825

Academic Editor: Christopher Goodrich

Received: 12 October 2021
Accepted: 5 November 2021
Published: 7 November 2021

1. Introduction and Preliminaries

The notion of almost periodicity was introduced by H. Bohr around 1925 and later generalized by many others (see the research monographs [1–10] for more details about the subject). Suppose that $F : \mathbb{R}^n \to X$ is a continuous function, where X is a complex Banach space equipped with the norm $\| \cdot \|$. It is said that $F(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in \mathbb{R}^n$ there exists $\tau \in B(\mathbf{t}_0, l) \equiv \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$ such that:

$$\left\| F(\mathbf{t} + \tau) - F(\mathbf{t}) \right\| \leq \epsilon, \quad \mathbf{t} \in \mathbb{R}^n;$$

here, $| \cdot - \cdot |$ denotes the Euclidean distance in $\mathbb{R}^n$. Any almost periodic function $F : \mathbb{R}^n \to X$ is bounded and uniformly continuous, any trigonometric polynomial in $\mathbb{R}^n$ is almost periodic, and a continuous function $F(\cdot)$ is almost periodic if and only if there exists a sequence of trigonometric polynomials in $\mathbb{R}^n$, which converges uniformly to $F(\cdot)$; see the monographs [7,9] for more details about multi-dimensional almost periodic functions.

Concerning Stepanov, Weyl and Besicovitch classes of almost periodic functions, we will only recall a few well known definitions and results for the functions of one real variable. Let $1 \leq p < \infty$, and let $f,\ g \in L^p_{loc}(\mathbb{R} : X)$. We define the Stepanov metric by:

$$D_{S^p}\left[f(\cdot), g(\cdot)\right] := \sup_{x \in \mathbb{R}} \left[\int_x^{x+1} \left\| f(t) - g(t) \right\|^p dt \right]^{1/p}.$$

It is said that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Stepanov p-bounded if and only if

$$\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

The space $L^p_S(\mathbb{R} : X)$ consisting of all S^p-bounded functions becomes a Banach space equipped with the above norm. A function $f \in L^p_S(\mathbb{R} : X)$ is said to be Stepanov p-almost periodic if and only if the Bochner transform $\hat{f} : \mathbb{R} \to L^p([0,1] : X)$, defined by $\hat{f}(t)(s) := f(t+s)$, $t \in \mathbb{R}$, $s \in [0,1]$ is almost periodic. It is well known that if $f(\cdot)$ is an almost periodic, then the function $f(\cdot)$ is Stepanov p-almost periodic for any finite exponent $p \in [1, \infty)$. The converse statement is false, however, but we know that any uniformly continuous Stepanov p-almost periodic function $f : \mathbb{R} \to X$ is almost periodic ($p \in [1, \infty)$). Further on, suppose that $f \in L^p_{loc}(\mathbb{R} : X)$. Then, we say that the function $f(\cdot)$ is:

(i) equi-Weyl-p-almost periodic, if and only if for each $\epsilon > 0$ we can find two real numbers $l > 0$ and $L > 0$ such that any interval $I \subseteq \mathbb{R}$ of length L contains a point $\tau \in I$ such that:

$$\sup_{x \in \mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon.$$

(ii) Weyl-p-almost periodic, if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $I \subseteq \mathbb{R}$ of length L contains a point $\tau \in I$ such that:

$$\lim_{l \to \infty} \sup_{x \in \mathbb{R}} \left[\frac{1}{l} \int_x^{x+l} \|f(t+\tau) - f(t)\|^p \, dt \right]^{1/p} \leq \epsilon.$$

Let us recall that any Stepanov p-almost periodic function is equi-Weyl-p-almost periodic, as well as that any equi-Weyl-p-almost periodic function is Weyl-p-almost periodic ($p \in [1, \infty)$). The class of Besicovitch p-almost periodic functions can be also considered, and we will only note here that any equi-Weyl-p-almost periodic function is Besicovitch p-almost periodic as well as that there exists a Weyl-p-almost periodic function which is not Besicovitch p-almost periodic ($p \in [1, \infty)$); see [7]. For further information in this direction, we may also refer the reader to the excellent survey article [11] by J. Andres, A. M. Bersani and R. F. Grande. Regarding multi-dimensional Stepanov, Weyl and Besicovitch classes of almost periodic functions, the reader may consult the above-mentioned monographs [7,9] and references cited therein.

On the other hand, the notion of c-almost periodicity was recently introduced by M. T. Khalladi et al. in [12] and later extended to the multi-dimensional setting in [13]. A further generalization of the concept c-almost periodicity presents the concept ρ-almost periodicity, which has recently been introduced and analyzed in [14]; here, ρ denotes a general binary relation acting on a corresponding pivot space (see also M. Fečkan et. al [15] for the first steps made in the investigation of general classes of ρ-almost periodic type functions; the main assumption used in [15] is that $\rho = \mathbb{T}$ is a linear isomorphism). The Stepanov and Weyl classes of multi-dimensional ρ-almost periodic functions have recently been studied in [16]; it is also worth noting that multi-dimensional $(S, \mathbb{D}, \mathcal{B})$-asymptotically (ω, ρ)-periodic type functions, multi-dimensional quasi-asymptotically ρ-almost periodic type functions and multi-dimensional ρ-slowly oscillating type functions have recently been analyzed in [17].

The main aim of this paper is to analyze various classes of multi-dimensional Doss ρ-almost periodic functions in the Lebesgue spaces with variable exponent (concerning one-dimensional classes of Doss uniformly recurrent functions and Doss almost periodic functions, we may refer to our recent research study [18]). To the best knowledge of the authors, this is the first research study of multi-dimensional Doss almost periodic type functions of any type (this was a strong motivational factor for the genesis of paper); more to the point, the notion of Doss ρ-almost periodicity and the notion of Doss c-almost periodicity seem to be new even in the one-dimensional setting ($c \in \mathbb{C}$). We wish here, in fact, to present a rather general concept which extends the concepts Stepanov and Weyl ρ-almost periodicity in the multi-dimensional setting as well as the usual concept of Besicovitch almost periodicity in the multi-dimensional setting (it is worth noting that the classes of Besicovitch c-almost periodic functions and Besicovitch ρ-almost periodic functions have not been analyzed so far, even for the functions depending on one real variable; this could be a very interesting topic of ongoing investigations). The introduced class of functions retains, in a certain sense, many important structural properties of the corresponding classes of Stepanov, Weyl and Besicovitch almost periodic functions.

We continue, in such a way, our previous research studies [14,17–20] and revisit some known structural characterizations of one-dimensional Doss almost periodic functions [6]. We introduce several new classes of multi-dimensional Doss almost periodic functions following a combination of methods already established in [14,18,19]; basically, multi-dimensional Doss almost periodic functions retain almost all structural properties of one-dimensional Doss almost periodic functions. However, some important peculiarities appear in the multi-dimensional setting: in our definitions we require, for example, that the Doss ϵ-periods of functions under our consideration belong to a non-empty subset Λ' of $\mathbb{R}^n$, roughly speaking. This can be also required in the one-dimensional setting but the real beauty and importance of such notion is clearly manifested in the (still very unexplored) multi-dimensional setting, when the set Λ' can possess various geometrical features. We investigate the main structural properties of Doss ρ-almost periodic functions; in particular, we analyze the convolution invariance of Doss ρ-almost periodicity, the invariance of Doss ρ-almost periodicity under the actions of convolution products, and provide certain applications to the abstract Volterra integro-differential equations and the partial differential equations (unfortunately, it would be really difficult and almost impossible to fully compare here the results and similarities/differences of this work with the results of papers mentioned in the former three paragraphs). It is also worth noting that some similar classes of almost periodic functions have been introduced and analyzed by D. M. Umbetzhanov [21], M. Akhmet, M. O. Fen [22] and M. Akhmet [23]. In [21], the author has investigated the class of Stepanov almost periodic functions with the Bessel-Mackdonald kernels and provided some applications to the higher-order elliptic equations, while the authors of [22] have introduced the class of unpredictable functions and provided some applications in the chaos theory and the theory of neural networks. In this research, we have provided some different applications of Doss ρ-almost periodic functions; for example, we have considered the fractional Poisson heat equations, a class of abstract fractional semilinear Cauchy inclusions, and revisit the famous d'Alembert formula, the Poisson formula and the Kirchhoff formula in our context. We have also described how the considered classes of Doss ρ-almost periodic functions can be further generalized and applied in the study of second-order partial differential equations whose solutions are governed by the Newtonian potential. To the best knowledge of the authors, these applications are completely new in the subject area.

The organization and main ideas of this paper can be briefly described as follows. Section 1 recalls the basic definitions and results about the Lebesgue spaces with variable exponents $L^{p(x)}$. In Section 2, we introduce and analyze various classes of multi-dimensional Doss ρ-almost periodic type functions of the form $F : \Lambda \times X \to Y$, where Y is a Banach space equipped with the norm $\| \cdot \|_Y$, $\rho \subseteq Y \times Y$ is a binary relation, Λ is a general non-empty subset of $\mathbb{R}^n$, and $p \in \mathcal{P}(\Lambda)$; see Section 1 for the notion. In Definition 1, we introduce the notions of Besicovitch-$(p, \phi, \mathrm{F}, \mathcal{B})$-boundedness, Besicovitch-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-continuity, Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodicity, and Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-uniform recurrence. After that, we clarify the main structural characterizations of the introduced function spaces (see e.g., Propositions 1, 2–4, 6 and 7 below), providing also some illustrations in Examples 1, 3, 5 and 6. Of particular importance is to stress that the class of multi-dimensional Weyl-p-almost periodic functions, taken in the generalized approach of A. S. Kovanko [24], is contained in the class of multi-dimensional Doss-p-almost periodic functions for any finite exponent $p \geq 1$ (see Section 2.1 for more details; especially, Proposition 8 and Example 7, where we propose some open problems and issues for further analyses). In Section 2.2, we investigate the invariance of Doss ρ-almost periodicity under the actions of convolution products; see also [6] for the first results in this direction. The main aim of Section 3 is to provide certain applications of our results to the abstract Volterra integro-differential equations and the partial differential equations. In the final section of paper, we present some conclusions, remarks and proposals for further research studies.

Notation and terminology. Suppose that X and Y are given non-empty sets. Let us recall that a binary relation between X into Y is any subset $\rho \subseteq X \times Y$. The domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x,y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x,y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x,y) \in \rho\}$ $(x \in X)$, $x\,\rho\,y \Leftrightarrow (x,y) \in \rho$. Define $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$.

We assume henceforth that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are three complex Banach spaces, $n \in \mathbb{N}$, as well as that $\mathcal{B}$ is a certain collection of subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X,Y)$ we denote the Banach space of all linear continuous functions from X into Y; $L(X) \equiv L(X,X)$. If $\mathbf{t_0} \in \mathbb{R}^n$ and $\epsilon > 0$, then we set $B(\mathbf{t}_0, \epsilon) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t_0}| \leq \epsilon\}$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. Define $\Lambda_M := \{\mathbf{t} \in \Lambda : |\mathbf{t}| \leq M\}$; $diam(\Lambda)$ denotes the diameter of set Λ and $m(\Lambda)$ denotes its Lebesgue measure ($\Lambda \subseteq \mathbb{R}^n$; $M > 0$). The symbol $\chi_A(\cdot)$ stands for the characteristic function of a set A; $\lfloor s \rfloor := \sup\{k \in \mathbb{Z} : k \leq s\}$ $(s \in \mathbb{R})$. By $D_{t,+}^{\gamma}$ we denote the Weyl-Liouville fractional derivative of order $\gamma > 0$ ([6]); I stands for the identity operator on Y. Define $S_1 \equiv \{z \in \mathbb{C} : |z| = 1\}$.

Lebesgue Spaces with Variable Exponents $L^{p(x)}$

Let $\emptyset \neq \Omega \subseteq R^n$ be a nonempty Lebesgue measurable subset and let $M(\Omega : X)$ denote the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Further on, by $\mathcal{P}(\Omega)$ we denote the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we set

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, & t \geq 0, \ 1 \leq p(x) < \infty, \\ 0, & 0 \leq t \leq 1, \ p(x) = \infty, \\ \infty, & t > 1, \ p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_{\Omega} \varphi_{p(x)}(\|f(x)\|)\,dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent by

$$L^{p(x)}(\Omega : X) := \Big\{ f \in M(\Omega : X) : \lim_{\lambda \to 0+} \rho(\lambda f) = 0 \Big\}.$$

It is well known that

$$L^{p(x)}(\Omega : X) = \Big\{ f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty \Big\};$$

see, for example, [25] (p. 73). For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{ \lambda > 0 : \rho(u/\lambda) \leq 1 \Big\}.$$

Equipped with this norm, $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g., [25] (Theorem 3.2.7) for the scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function. Further on, for any $p \in M(\Omega)$, we define

$$p^- := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \text{esssup}_{x\in\Omega} p(x).$$

Set

$$D_+(\Omega) := \{p \in M(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega\}.$$

If $p \in D_+(\Omega)$, then we know

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X) ; \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\Big\}.$$

We will use the following lemma (cf. [25] for the scalar-valued case):

Lemma 1. *(i)* *(The Hölder inequality) Let p, q, $r \in \mathcal{P}(\Omega)$ such that*

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) *Let Ω be of a finite Lebesgue's measure and let p, $q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$, and the constant of embedding is less than or equal to $2(1 + m(\Omega))$.*

(iii) *Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.*

For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to the monograph [25] by L. Diening, P. Harjulehto, P. Hästüso, M. Ruzicka and the lists of references quoted in this monograph and the forthcoming monograph [7].

2. Multi-Dimensional Doss ρ-Almost Periodic Type Functions

In this paper, we will always assume that $\rho \subseteq Y \times Y$ is a binary relation, Λ is a general non-empty subset of $\mathbb{R}^n$ as well as that $p \in \mathcal{P}(\Lambda)$ and the following condition holds:

$$\phi : [0, \infty) \to [0, \infty) \text{ is measurable, } \mathrm{F} : (0, \infty) \to (0, \infty) \text{ and } p \in \mathcal{P}(\Lambda).$$

Set

$$\Lambda'' := \{\tau \in \mathbb{R}^n : \tau + \Lambda \subseteq \Lambda\}.$$

In the remainder of paper, we will always assume that $\emptyset \neq \Lambda' \subseteq \Lambda''$ so that $\Lambda + \Lambda' \subseteq \Lambda$.

In the following definition, we will extend the notion introduced in [6] (Definition 2.13.2(i)-(iii)) and [18] (Definition 7) (we can similarly extend the notion considered in [18] (Definition 8; Definition 9); we will skip all details concerning such classes of Doss almost periodic type functions for brevity):

Definition 1. *(i)* *Suppose that the function $F : \Lambda \times X \to Y$ satisfies that $\phi(\|F(\cdot; x)\|) \in L^{p(\cdot)}(\Lambda_t)$ for all $t > 0$ and $x \in X$. Then we say that the function $F(\cdot; \cdot)$ is Besicovitch-*

$(p, \phi, \mathrm{F}, \mathcal{B})$-bounded if and only if, for every $B \in \mathcal{B}$, there exists a finite real number $M_B > 0$ such that

$$\limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\big(\|F(\cdot; x)\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} \leq M_B.$$

(ii) *Suppose that the function $F : \Lambda \times X \to Y$ satisfies that $\phi(\|F(\cdot + \tau; x) - y_{\cdot;x}\|) \in L^{p(\cdot)}(\Lambda_t)$ for all $t > 0$, $x \in X$, $\tau \in \Lambda'$ and $y_{\cdot;x} \in \rho(F(\cdot; x))$.*

(a) *We say that the function $F : \Lambda \times X \to Y$ is Besicovitch-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-continuous if and only if, for every $B \in \mathcal{B}$ as well as for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho(F(\cdot; x))$ such that*

$$\lim_{\tau \to 0, \tau \in \Lambda'} \limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\big(\|F(\cdot + \tau; x) - y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} = 0.$$

(b) *We say that the function $F(\cdot; \cdot)$ is Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodic if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists a point $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho(F(\cdot; x))$ such that*

$$\limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\big(\|F(\cdot + \tau; x) - y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon. \tag{1}$$

(c) *We say that the function $F(\cdot; \cdot)$ is Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-uniformly recurrent if and only if, for every $B \in \mathcal{B}$, there exists a sequence $(\tau_k) \in \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, we have the existence of an element $y_{\cdot;x} \in \rho(F(\cdot; x))$ such that*

$$\lim_{k \to +\infty} \limsup_{t \to +\infty} \mathrm{F}(t) \sup_{x \in B} \Big[\phi\big(\|F(\cdot + \tau_k; x) - y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} = 0.$$

If $F : \Lambda \to Y$, then we omit the term "$\mathcal{B}$" from the notation. Further on, if $\rho = c\mathrm{I}$ for some $c \in \mathbb{C}$, then we also write "c" in place of "ρ"; we omit "c" from the notation if $c = 1$. We also omit the term "Λ'" from the notation if $\Lambda' = \Lambda$.

Let $F : \Lambda \to Y$. We would like to note that the notion introduced in Definition 1 is rather general as well as that the classical concept of Doss-p-almost periodicity (Doss-p-uniform recurrence) of function $F(\cdot)$ is obtained by plugging $\rho = \mathrm{I}$, $\Lambda' = \Lambda = \mathbb{R}^n$ or $[0, \infty)^n$, $\phi(x) \equiv x$, $x \geq 0$, $p(\cdot) \equiv p \in [1, \infty)$ and $\mathrm{F}(t) \equiv t^{-(n/p)}$, $t > 0$. The classical concept of Besicovitch-p-boundedess of function $F(\cdot)$ is obtained by plugging the same values of p, ϕ, Λ, F; a function $F(\cdot)$ is said to be Besicovitch bounded if and only if $F(\cdot)$ is Besicovitch-1-bounded.

Remark 1. *Let $1 \leq p < \infty$. The class of Besicovitch-p-almost periodic functions (see e.g., [1,6] for the notion) has been analyzed by numerous mathematicians by now. It is worth noticing that R. Doss has clarified, in [26,27], some equivalent conditions for a locally integrable function $f : \mathbb{R} \to \mathbb{C}$ to be Besicovitch-p-almost periodic. In the one-dimensional setting, with the same values of parameters p, ϕ and F as above, he employed conditions (a)–(c) from Definition 1, and the following non-trivial conditions ($\Lambda' = \Lambda = \mathbb{R}$):*

(A) *([26]) For every real number $a \in \mathbb{R}$, there exists a p-locally integrable function $f^{(a)} : \mathbb{R} \to \mathbb{C}$ of period a such that*

$$\lim_{k \to +\infty} \limsup_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} \left| k^{-1} \sum_{l=0}^{k-1} f(x + la) - f^{(a)}(x) \right|^p dx = 0.$$

(B) ([27]; $p = 1$) *For every* $\lambda \in \mathbb{R}$, *we have:*

$$\lim_{l\to+\infty} \limsup_{t\to+\infty} \frac{1}{l}\left[\frac{1}{2t}\int_{-t}^{t}\left|\left(\int_{x}^{x+l} - \int_{0}^{l}\right)e^{i\lambda s}f(s)\,ds\right|dx\right] = 0.$$

As emphasized in [6], the results established in [26,27] cannot be so simply extended to the vector-valued functions.

We continue by providing the following illustrative example:

Example 1 (J. de Vries [28] (point 6., p. 208), [7])**.** *Let* $(p_i)_{i\in\mathbb{N}}$ *be a strictly increasing sequence in* $\mathbb{N}$ *satisfying that* $p_i|p_{i+1}$, $i \in \mathbb{N}$ *and* $\lim_{i\to\infty}(p_i/p_{i+1}) = 0$. *Define the function* $f_i : [-p_i, p_i] \to [0,1]$ *by* $f_i(t) := |t|/p_i$, $t \in [-p_i, p_i]$ *and extend the function* $f_i(\cdot)$ *periodically to the whole real axis; the obtained function, denoted likewise by* $f_i(\cdot)$, *is of period* $2p_i$ ($i \in \mathbb{N}$). *Define now*

$$f(t) := \sup\{f_i(t) : i \in \mathbb{N}\}, \quad t \in \mathbb{R}.$$

Then we know that (see e.g., [7]):

$$\limsup_{t\to+\infty} \frac{1}{t}\int_{-t}^{t}|f(x) - 1|\,dx \geq \frac{1}{4}.$$

The last estimate simply implies that, for every real number $\tau \in \mathbb{R}$, *we have:*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_{-t}^{t}|f(x+\tau) - 1|\,dx \geq \frac{1}{4}.$$

Therefore, there does not exist a non-empty subset Λ' *of* $\mathbb{R}$ *such that the function* $f(\cdot)$ *is Doss-*$(1, x, t^{-(1/p)}, \Lambda', \rho)$*-uniformly recurrent with* $\rho(z) := 1$ ($z \in \mathbb{C}$) *and the meaning clear.*

Before proceeding any further, we would like to emphasize that the notion introduced in Definition 1 has not been considered elsewhere if $\rho = c\mathrm{I}$ for some $c \in \mathbb{C} \setminus \{1\}$, even in the one-dimensional setting with $\Lambda = \Lambda' = \mathbb{R}$ or $[0,\infty)$, $\phi(x) \equiv x$, $x \geq 0$, $p(\cdot) \equiv p \in [1,\infty)$ and $F(t) \equiv t^{-(1/p)}$, $t > 0$; if this is the case, then we simply say that the function $F(\cdot)$ is Doss-(p,c)-almost periodic (Doss-(p,c)-uniformly recurrent). We accept the same notation if $p(\cdot) \in \mathcal{P}(\Lambda)$ has not a constant value. Although we formulate almost all structural results of ours with the general function $\phi(\cdot)$, the dominant case in our analysis is that one in which we have $\phi(x) \equiv x$, $x \geq 0$; observe also that any Doss-(p,c)-almost periodic (Doss-(p,c)-uniformly recurrent) function is automatically Doss-$(1, x^p, t^{-1}, \Lambda, c)$-almost periodic (Doss-$(1, x^p, t^{-1}, \Lambda, c)$-uniformly recurrent), with the meaning clear.

It is worth noting that [19] (Example 2.13, Example 2.15) can be formulated for multi-dimensional Doss almost periodic type functions. In particular, Ref. [19] (Example 2.15(ii)) shows that the assumption in which $\Lambda' \neq \Lambda$ can occur (cf. also [7] (Example 6.1.15)):

Example 2. *Suppose that* $1 \leq p < \infty$, *as well as that the mapping*

$$t \mapsto \left(\int_0^t f_1(s)\,ds, \ldots, \int_0^t f_n(s)\,ds\right) \in X^n, \quad t \in \mathbb{R},$$

is bounded and Doss-p-almost periodic, resp. bounded and Doss-p-uniformly recurrent, as well as that the strongly continuous operator families $(T_j(t))_{t\in\mathbb{R}} \subseteq L(X,Y)$ *are uniformly bounded* ($1 \leq j \leq n$). *Define*

$$F(t_1, \cdots, t_{2n}) := \sum_{j=1}^{n} T_j(t_j - t_{j+n}) \int_{t_j}^{t_{j+n}} f_j(s)\,ds \text{ for all } t_j \in \mathbb{R},\ 1 \leq j \leq 2n.$$

Then, for every $t_i,\ \tau_i \in \mathbb{R}$ $(1 \leq j \leq 2n)$ with $\tau_j = \tau_{j+n}$ $(1 \leq j \leq n)$, we have:

$$\begin{aligned}&\big\|F\big(t_1+\tau_1,\cdot\cdot\cdot,t_{2n}+\tau_{2n}\big)-F\big(t_1,\cdot\cdot\cdot,t_{2n}\big)\big\|_Y\\&\leq M\sum_{j=1}^{n}\left\{\left\|\int_0^{t_j+\tau_j}f_j(s)\,ds-\int_0^{t_j}f_j(s)\,ds\right\|+\left\|\int_0^{t_{j+n}+\tau_j}f_j(s)\,ds-\int_0^{t_{j+n}}f_j(s)\,ds\right\|\right\},\end{aligned}$$

where $M = \sup_{1\leq j\leq n}\sup_{t\in\mathbb{R}}\|T_j(t)\|$. This simply implies that the mapping $F : \mathbb{R}^{2n} \to Y$ is Doss-(p, Λ')-almost periodic, resp. Doss-(p, Λ')-uniformly recurrent, with the meaning clear, where $\Lambda' = \{(\tau, \tau) : \tau \in \mathbb{R}^n\}$.

In the following result, we continue our analysis from [14] (Proposition 2.2):

Proposition 1. *Suppose that $p(\cdot) \equiv p \in [1, \infty)$, $\phi(\cdot)$ is monotonically increasing and there exists a finite real constant $d > 0$ such that*

$$\phi(x+y) \leq d[\phi(x)+\phi(y)], \quad x,\ y \geq 0. \tag{2}$$

Suppose, further, that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda \pm \Lambda' \subseteq \Lambda$, and the function $F : \Lambda \times X \to Y$ is Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodic (Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-uniformly recurrent), where ρ is a binary relation on Y satisfying $R(F) \subseteq D(\rho)$ and $\rho(y)$ is a singleton for any $y \in R(F)$. If for each $\tau \in \Lambda'$ we have $\tau + \Lambda = \Lambda$, then $\Lambda + (\Lambda' - \Lambda') \subseteq \Lambda$ and for each $b \in (0, 1)$ the function $F(\cdot;\cdot)$ is Doss-$(p, \phi, \mathrm{F}_b, \mathcal{B}, \Lambda' - \Lambda', \mathrm{I})$-almost periodic (Doss-$(p, \phi, \mathrm{F}_b, \mathcal{B}, \Lambda' - \Lambda', \mathrm{I})$-uniformly recurrent), where $\mathrm{F}_b(t) := \mathrm{F}(bt)$, $t > 0$.

Proof. The inclusion $\Lambda + (\Lambda' - \Lambda') \subseteq \Lambda$ can be deduced as in the proof of the above-mentioned proposition. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. If $\tau_1,\ \tau_2 \in \Lambda'$ satisfy (1), then there exists a sufficiently large real number $t_0(\epsilon, \tau_1, \tau_2) > 0$ such that

$$\mathrm{F}(t)\sup_{x\in B}\Big[\phi\big(\|F(\cdot+\tau_1;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon/2d, \quad t \geq t_0(\epsilon,\tau_1,\tau_2)$$

and

$$\mathrm{F}(t)\sup_{x\in B}\Big[\phi\big(\|F(\cdot+\tau_2;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon/2d, \quad t \geq t_0(\epsilon,\tau_1,\tau_2).$$

Since $\phi(\cdot)$ is monotonically increasing and there exists a finite real number $d > 0$ such that (2) holds, the last three estimates in combination with Lemma 1(iii) simply imply that:

$$\begin{aligned}&\Big[\phi\big(\|F(\cdot+\tau_1;x)-F(\cdot+\tau_2;x)\|_Y\big)\Big]_{L^p(\Lambda_t)}\\&\leq \Big[\phi\big(\|F(\cdot+\tau_1;x)-\rho(F(\cdot;x))\|_Y+\|F(\cdot+\tau_2;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)}\\&\leq d\left\{\Big[\phi\big(\|F(\cdot+\tau_1;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)}+\Big[\phi\big(\|F(\cdot+\tau_2;x)-\rho(F(\cdot;x))\|_Y\big)\Big]_{L^p(\Lambda_t)}\right\}\\&\leq \epsilon/\mathrm{F}(t), \quad t \geq t_0(\epsilon,\tau_1,\tau_2),\ x \in B.\end{aligned}$$

Let a number $b \in (0, 1)$ be fixed. Since we have assumed that $\Lambda - \Lambda' \subseteq \Lambda$, it is clear that there exists a sufficiently large number $t_1(\epsilon, \tau_1, \tau_2) \geq t_0(\epsilon, \tau_1, \tau_2)$ such that $\Lambda_{bt} \subseteq \tau_2 + \Lambda_t$ for all $t \geq t_1(\epsilon, \tau_1, \tau_2)$. The last estimate in the above computation therefore yields

$$\Big[\phi\big(\|F(\cdot+[\tau_2-\tau_1];x)-F(\cdot;x)\|_Y\big)\Big]_{L^p(\Lambda_t+\tau_2)} \leq \epsilon/\mathrm{F}(t), \quad t \geq t_0(\epsilon,\tau_1,\tau_2),\ x \in B,$$

and the final conclusion simply follows from the estimate

$$\Big[\phi\big(\|F(\cdot+[\tau_2-\tau_1];x)-F(\cdot;x)\|_Y\big)\Big]_{L^p(\Lambda_t+\tau_2)}$$
$$\geq \Big[\phi\big(\|F(\cdot+[\tau_2-\tau_1];x)-F(\cdot;x)\|_Y\big)\Big]_{L^p(\Lambda_{bt})}, \quad t\geq t_1(\epsilon,\tau_1,\tau_2),\ x\in B,$$

and the substitution $bt \mapsto t$, $t>0$. $\square$

Corollary 1. *Suppose that $p(\cdot)\equiv p\in[1,\infty)$, $\phi(\cdot)$ is monotonically increasing and there exists a finite real constant $d>0$ such that (2) holds. Suppose, further, that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, and the function $F:\mathbb{R}^n\times X\to Y$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-almost periodic (Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-uniformly recurrent), where ρ is a binary relation on Y satisfying $R(F)\subseteq D(\rho)$ and $\rho(y)$ is a singleton for any $y\in R(F)$. Then for each $b\in(0,1)$ the function $F(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F}_b,\mathcal{B},\Lambda'-\Lambda',\mathrm{I})$-almost periodic (Doss-$(p,\phi,\mathrm{F}_b,\mathcal{B},\Lambda'-\Lambda',\mathrm{I})$-uniformly recurrent), where $\mathrm{F}_b(t):=\mathrm{F}(bt)$, $t>0$.*

Keeping in mind Lemma 1, we can simply reformulate the conclusions established in [18] (Remark 1, Proposition 5(i)) in our new framework. Concerning [18] (Example 5), we would like to emphasize the following:

Example 3. *Let $p\in[1,\infty)$ and $\phi(x)\equiv x$, $x\geq 0$.*

(i) Suppose that $\Lambda=\mathbb{R}$ and $F(t):=\chi_{[0,1/2]}(t)$, $t\in\mathbb{R}$. Then $F(\cdot)$ is Doss-$(p,\phi,t^{-\sigma},c)$-almost periodic for every $\sigma>0$ and $c\in\mathbb{C}$. This simply follows from the estimate

$$\int_{-t}^{t}\big|F(x+\tau)-cF(x)\big|^p\,dx\leq\frac{1}{2}\big(1+|c|\big)^p+2\int_{-\infty}^{\infty}\big|F(x)\big|^p\,dx,\quad t\geq 0,\ \tau\in\mathbb{R},\ c\in\mathbb{C}.$$

(ii) Suppose now that $\Lambda=\Lambda'=\mathbb{R}^n$ and $F(\mathbf{t}):=\chi_K(\mathbf{t})$, $\mathbf{t}\in\mathbb{R}^n$, where K is a compact subset of $\mathbb{R}^n$. Arguing as above (cf. also [7] (Example 6.3.8)), we may conclude that $F(\cdot)$ is Doss-$(p,\phi,t^{-\sigma},c)$-almost periodic for every $\sigma>0$ and $c\in\mathbb{C}$.

(iii) Suppose that $0<\sigma\leq 1/p$ and $F(t):=\chi_{[0,\infty)}(t)$, $t\in\mathbb{R}$. Then $F(\cdot)$ cannot be Doss-$(p,\phi,t^{-\sigma},c)$-almost periodic if $c\in\mathbb{C}\setminus\{1\}$. This simply follows from the estimate

$$\int_{-t}^{t}\big|F(x+\tau)-cF(x)\big|^p\,dx$$
$$\geq\int_{0}^{t}\big|F(x+\tau)-cF(x)\big|^p\,dx=\big|1-c\big|^p t,\quad \tau>0,\ t\geq 0,\ c\in\mathbb{C}.$$

(iv) Suppose now that $\Lambda=\Lambda'=\mathbb{R}^n$ and $F(\mathbf{t}):=\chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t}\in\mathbb{R}^n$. In the analysis carried out in [7] (Example 6.3.9), we have proved that for each $\tau\in\mathbb{R}^n$ and $t\in(|\tau|,+\infty)$ we have

$$\int_{\Lambda_t}\big|F(\mathbf{t}+\tau)-F(\mathbf{t})\big|^p\,d\mathbf{t}\leq 2^n t^{n-1}|\tau|.$$

This simply implies that the function $F(\cdot)$ is Doss-$(p,\phi,t^{-\sigma})$-almost periodic for every $\sigma>(n-1)/p$; this is also the optimal result we can obtain here, as easily approved. On the other hand, arguing as in part (iii) it follows that for each $0<\sigma\leq n/p$ the function $F(\cdot)$ cannot be Doss-$(p,\phi,t^{-\sigma},c)$-almost periodic if $c\in\mathbb{C}\setminus\{1\}$.

Concerning the pointwise products of multi-dimensional Doss almost periodic type functions, we will present the following illustrative example, only:

Example 4. *Suppose that $p,\ q,\ r\in[1,\infty)$ and $1/p+1/r=1/q$, there exists a sequence $(\tau_k)\in\mathbb{R}^n$ such that $\lim_{k\to+\infty}|\tau_k|=\infty$ as well as that a scalar-valued function $f(\cdot)$ is Doss-*

$(p, x, \mathrm{F}_1, \Lambda', \mathrm{I})$-uniformly recurrent, a Y-valued function $F(\cdot)$ is Doss-$(r, x, \mathrm{F}_2, \Lambda', \mathrm{I})$-uniformly recurrent with $\Lambda' := \{\tau_k : k \in \mathbb{N}\}$,

$$\lim_{k\to+\infty} \limsup_{t\to+\infty} \mathrm{F}_1(t)\big\|f(\cdot+\tau_k) - f(\cdot)\big\|_{L^p(\Lambda_t)} = 0$$

and

$$\lim_{k\to+\infty} \limsup_{t\to+\infty} \mathrm{F}_2(t)\big\|F(\cdot+\tau_k) - F(\cdot)\big\|_{L^r(\Lambda_t:Y)} = 0. \tag{3}$$

Suppose, further, that there exist a positive integer $k \in \mathbb{N}$ and two positive real numbers $t_0 > 0$ and $M > 0$ such that

$$\frac{\mathrm{F}(t)}{\mathrm{F}_2(t)}\left(\int_{\Lambda_t} \big|f(s+\tau_k)\big|^p\, ds\right)^{1/p} + \frac{\mathrm{F}(t)}{\mathrm{F}_1(t)}\left(\int_{\Lambda_t} \big\|F(s)\big\|_Y^r\, ds\right)^{1/r} \leq M, \quad t \geq t_0,\ k \geq k_0. \tag{4}$$

Then the function $[fF](\cdot)$ is Doss-$(q, x, \mathrm{F}, \Lambda', \mathrm{I})$-uniformly recurrent and (3) holds with the functions $\mathrm{F}_2(\cdot)$ and $F(\cdot)$ replaced therein with the functions $\mathrm{F}(\cdot)$ and $[fF](\cdot)$, respectively. This simply follows from the decomposition

$$\begin{aligned}&\big\|f(s+\tau)F(s+\tau) - f(s)F(s)\big\|_Y \\ &\quad \leq |f(s+\tau)| \cdot \big\|F(s+\tau) - F(s)\big\|_Y + |f(s+\tau) - f(s)| \cdot \big\|F(s)\big\|_Y,\ s \in \Lambda,\ \tau \in \Lambda',\end{aligned}$$

an application of the Hölder inequality (see Lemma 1(i)) and the imposed condition (4). This enables one to simply construct many examples of multi-dimensional Doss-$(q, x, \mathrm{F}, \Lambda', \mathrm{I})$-uniformly recurrent functions of the form $F(t_1, \cdot \cdot \cdot, t_n) = f_1(t_1) \cdot \cdot \cdot f_n(t_n)$; see [7] for more details.

The following results are motivated by some observations made in [18] (Example 3):

Proposition 2. *Suppose that $\omega \in \mathbb{R}^n \setminus \{0\}$, $c \in \mathbb{C} \setminus \{0\}$, $|c| = 1$, $\omega + \Lambda \subseteq \Lambda$ and $F : \Lambda \to Y$ is a continuous function. Suppose that $F(\cdot)$ is (ω, c)-periodic, that is, $F(\mathbf{t} + \omega) = cF(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$; set $\Lambda' := \{k\omega : k \in \mathbb{N}\}$. Suppose, further, that $\phi(\cdot)$ is monotonically increasing, $F(\cdot)$ is essentially bounded as well as that for each $\epsilon > 0$ there exists a finite real number $t_\epsilon > 0$ such that*

$$\mathrm{F}(t)\Big[\phi\big(\epsilon\|F\|_\infty\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon, \quad t \geq t_\epsilon. \tag{5}$$

Then the function $F(\cdot)$ is Doss-$(p, \phi, \mathrm{F}, \Lambda', c')$-almost periodic for each $c' \in \{c^k : k \in \mathbb{N}\}$, provided that $c = e^{i\pi\varphi}$ for some rational number $\varphi \in (-\pi, \pi]$, resp. for each $c' \in S_1$, provided that $c = e^{i\pi\varphi}$ for some irrational number $\varphi \in (-\pi, \pi]$.

Proof. We will consider the case in which $c = e^{i\pi\varphi}$ for some irrational number $\varphi \in (-\pi, \pi]$, only. Let $c' \in S_1$ be arbitrary, and let $\epsilon > 0$ be given. Then there exists a strictly increasing sequence (n_k) of positive integers such that $|c^{n_k} - c'| < \epsilon$ for all $k \in \mathbb{N}$ and $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) < +\infty$; see for example [7]. Inductively one easily proves that $\Lambda' \subseteq \Lambda''$ as well as that $F(\mathbf{t} + n_k\omega) = c^{n_k}F(\mathbf{t})$ for all $\mathbf{t} \in \Lambda$ and $k \in \mathbb{N}$. Since $\phi(\cdot)$ is monotonically increasing and (5) holds, we get

$$\begin{aligned}&\limsup_{t\to+\infty}\mathrm{F}(t)\Big[\phi\big(\|F(\cdot+n_k\omega) - c'F(\cdot)\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} \\ &\quad \leq \limsup_{t\to+\infty} \mathrm{F}(t)\Big[\phi\big(|c^{n_k} - c'| \cdot \|F\|_\infty\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} \\ &\quad \leq \limsup_{t\to+\infty} \mathrm{F}(t)\Big[\phi\big(\epsilon\|F\|_\infty\big)\Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon,\end{aligned}$$

which simply implies the required. □

Proposition 3. *Suppose that $F : \Lambda \times X \to Y$ is a continuous function, $c \in \mathbb{C} \setminus \{0\}$ and $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, \Lambda', c)$-almost periodic, that is, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that*

$$\left\|F(\mathbf{t}+\tau;x) - cF(\mathbf{t};x)\right\|_Y \le \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B.$$

Then the function $F(\cdot;\cdot)$ is Doss-$(p, \phi, \mathrm{F}, \Lambda', c)$-almost periodic provided that for each $\epsilon > 0$ there exists a finite real number $t_\epsilon > 0$ such that

$$\mathrm{F}(t)\left[\phi(\epsilon)\right]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon, \quad t \ge t_\epsilon.$$

Proof. The proof is very similar to the proof of Proposition 2 and therefore omitted. □

Now we would like to provide some illustrative applications of Propositions 2 and 3:

Example 5. *(i) Suppose that $\Lambda = [0, \infty)$ and a continuous function $F : [0, \infty) \to Y$ is Bloch (ω, k)-periodic, that is, $F(t + \omega) = e^{i\omega k}F(t)$ for all $t \ge 0$ ($\omega > 0$, $k \in \mathbb{R}$). Let $\omega k/\pi$ be an irrational number. Applying Proposition 2, we get that, for every number $c' \in S_1$, the function $F(\cdot)$ is Doss-$(p, \phi, \mathrm{F}, c')$-almost periodic with $\phi(x) \equiv x$, $x \ge 0$, $p(\cdot) \equiv p \in [1, \infty)$ and $\mathrm{F}(t) \equiv t^{-(1/p)}$, $t > 0$. We can extend the established conclusions for the Bloch (ω, k)-periodic functions defined on the whole real line, because then we can take $\Lambda' = \{k\omega : k \in \mathbb{Z}\}$.*

(ii) The question whether a trigonometric polynomial is Doss-$(p(\cdot), c)$-almost periodic, where $p \in D_+(\Lambda)$ and $c \in \mathbb{C}$ are given in advance, is not simple. For example, using Proposition 3 and our conclusions from [12] (Example 2.15), we can simply prove that the function $f_\varphi(t) := e^{it\varphi}$, $t \in \mathbb{R}$ is Doss-$(p(\cdot), c)$-almost periodic if and only if:

(a) $c \in S_1$, provided that $\varphi \in (-\pi, \pi] \setminus \{0\}$;

(b) $c = 1$, provided that $\varphi = 0$,

as well as that the function $f(t) := \cos t$, $t \in \mathbb{R}$ is Doss-$(p(\cdot), c)$-almost periodic if $c = \pm 1$. Let us show that $f(\cdot)$ is not Doss-$(p(\cdot), c)$-almost periodic if $c \in \mathbb{C} \setminus \{-1, 1\}$. This is clear for $c = 0$; for the remainder, it suffices to show that $f(\cdot)$ is not Doss-$(1, c)$-almost periodic if $c \in \mathbb{C} \setminus \{-1, 0, 1\}$. Suppose that $c = re^{i\varphi}$ for some $r > 0$ and $\varphi \in (-\pi, \pi]$. Suppose further that $\cos\varphi = \pm 1$; then $r \ne 1$ and $\sin\varphi = 0$ so that

$$\int_{-t}^{t} \left|\cos(s+\tau) - c\cos s\right|^p ds = \int_{-t}^{t} \left|\cos(s+\tau) - r\cos s\right|^p ds,$$

for any $\tau \in \mathbb{R}$. Since the function $s \mapsto |\cos(s+\tau) - r\cos s|$, $s \in \mathbb{R}$ is periodic and not identically equal to zero, the last estimate yields the existence of a finite real number $c_\tau > 0$ such that ($\tau \in \mathbb{R}$ is given in advance):

$$\int_{-t}^{t} \left|\cos(s+\tau) - c\cos s\right|^p ds \ge c_\tau \lfloor (t/2\pi) \rfloor, \quad t > 0,$$

which simply yields a contradiction. Therefore, $\cos\varphi \ne \pm 1$ and there exists a constant $d \in (0, 1)$ such that, for every $\tau \in \mathbb{R}$, we have:

$$\begin{aligned}
&\int_{-t}^{t} \left|\cos(s+\tau) - c\cos s\right|^p ds \\
&= \int_{-t}^{t} \sqrt{\cos^2(s+\tau) - 2r\cos\varphi\cos s \cdot \cos(s+\tau) + r^2\cos^2 s}^{\,p} ds \\
&\ge (dr)^{1/p} \int_{-t}^{t} \left|\cos s \cdot \cos(s+\tau)\right|^p ds.
\end{aligned}$$

Since the function $s \mapsto |\cos s \cdot \cos(s+\tau)|$, $s \in \mathbb{R}$ is periodic and not identically equal to zero, the last estimate yields the existence of a finite real number $d_\tau > 0$ such that ($\tau \in \mathbb{R}$ is given in advance):

$$\int_{-t}^{t} \left|\cos(s+\tau) - c\cos s\right|^p \, ds \geq d_\tau (cr)^{1/p} \lfloor (t/2\pi) \rfloor, \quad t > 0,$$

which implies the required.

Example 6 (cf. also [12] (Example 2.22), and [12] (Example 2.23) for the pointwise products of c-almost periodic functions)**.** *Suppose that $c \in S_1$ and $p(\cdot) \in D_+(\mathbb{R})$. Then we have the following:*

(i) *Suppose that $c = 1$. Then the space consisting of all Doss-$p(\cdot)$-uniformly recurrent functions is not a vector space with the usual operations as easily shown. Now we will prove that the space of Doss-1-almost periodic functions is also not a vector space with the usual operations. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := 0$ for $x \leq 0$, $f(x) := \sqrt{n/2}$ if $x \in (n-2, n-1]$ for some $n \in 2\mathbb{N}$ and $f(x) := -\sqrt{n/2}$ if $x \in (n-1, n]$ for some $n \in 2\mathbb{N}$. Then we know that the function $f(\cdot)$ is Weyl-1-almost periodic as well as that for each $n \in 2\mathbb{N}$ we have*

$$\lim_{l \to +\infty} \frac{1}{2l} \sup_{t \in \mathbb{R}} \int_{-l}^{l} \left| f(t+n+x) - f(t+x) \right| dx = 0, \tag{6}$$

and that for each real number $\omega \notin 2\mathbb{Z}$ we have

$$\lim_{l \to +\infty} \frac{1}{2l} \int_{0}^{l} \left| f(x+\omega) - f(x) \right| dx = +\infty; \tag{7}$$

see J. Stryja [29] (pp. 42–47), [11] (Example 4.28) and [7] (Example 8.3.20). Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) := \cos t$, $t \in \mathbb{R}$. Hence, the functions $f(\cdot)$ and $g(\cdot)$ are Doss-1-almost periodic (cf. also Section 2.1 below); but, its sum is not Doss-1-almost periodic. In actual fact, if $\omega \notin 2\mathbb{Z}$, then the consideration from the above example along with the equation (7) indicates that there exists a finite real number $d > 0$ such that

$$\begin{aligned}
&\frac{1}{l} \int_0^l |f(x+\omega) + \cos(x+\omega) - f(x) - \cos x| \, dx \\
&\quad \geq \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx - \frac{1}{l} \int_0^l |\cos(x+\omega) - \cos x| \, dx \\
&\quad = \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx - \frac{|\sin(\omega/2)|}{l} \int_0^l |\sin(x + (\omega/2))| \, dx \\
&\quad \geq \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx - d \to +\infty, \quad l \to +\infty.
\end{aligned}$$

If $\omega \in 2\mathbb{Z}$, then the Equation (6) yields that there exist two finite real numbers $d > 0$ and $l_0 > 0$ such that

$$\begin{aligned}
&\frac{1}{l} \int_0^l |f(x+\omega) + \cos(x+\omega) - f(x) - \cos x| \, dx \\
&\quad \geq \frac{1}{l} \int_0^l |\cos(x+\omega) - \cos x| \, dx - \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx \\
&\quad \geq \frac{1}{2l} |\sin(\omega/2)| \int_0^l |\sin(x + (\omega/2))| \, dx - \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx \\
&\quad \geq d - \frac{1}{l} \int_0^l |f(x+\omega) - f(x)| \, dx \geq d/2, \quad l \geq l_0.
\end{aligned}$$

This implies the required statement; observe also that the above analysis implies that the collection of all Weyl-1-almost periodic functions has not a linear vector structure with the usual operations. We deeply believe that the collection of all Doss-$p(\cdot)$-almost periodic functions and the collection of all Weyl-$p(\cdot)$-almost periodic functions are not vector spaces with the usual operations, as well.

(ii) *Suppose that $c = -1$. Then the space consisting of all Doss-$(p(\cdot), c)$-almost periodic functions is not a vector space with the usual operations since the functions $2^{-1}\cos(4\cdot)$ and $2\cos(2\cdot)$ are Doss-$(p(\cdot), c)$-almost periodic (cf. Example 5(ii)) but its sum is not. To prove the last statement, we argue as follows. For every $\tau \in \mathbb{R}$, we have (cf. also [6] (Example 2.16.5(ii))):*

$$\begin{aligned}
&\int_{-t}^{t} 2^p \Big|2^{-1}\cos(4(s+\tau)) + 2\cos(2(s+\tau)) + 2^{-1}\cos(4s) + 2\cos(2s)\Big|^p \, ds \\
&\quad = \int_{-t}^{t} \Big|8\cos^4(s+\tau) + 8\cos^4 s - 6\Big|^p \, ds \\
&\quad \geq \int_{0}^{t} \Big|8\cos^4(s+\tau) + 8\cos^4 s - 6\Big|^p \, ds \\
&\quad \geq \int_0^{\arccos(4/5)} \Big[8(4/5)^4 - 6\Big] \, ds + \int_{\arccos(4/5)}^{2\pi+\arccos(4/5)} \Big[8(4/5)^4 - 6\Big] \, ds + \cdots \\
&\quad \geq 2^{-1} \lfloor t/(2\pi) \rfloor \arccos(4/5) \Big[8(4/5)^4 - 6\Big],
\end{aligned}$$

which simply implies the required.

(iii) *Suppose that $c \neq \pm 1$. Then the space consisting of all Doss-$(p(\cdot), c)$-almost periodic functions is not a vector space with the usual operations since the functions $f_{\varphi}(\cdot)$ and $f_{-\varphi}(\cdot)$ are Doss-$(p(\cdot), c)$-almost periodic and its sum $2f_0(\cdot) \equiv \cos\cdot$ is not Doss-$(p(\cdot), c)$-almost periodic ($\varphi \in (-\pi, \pi] \setminus \{0\}$). The conclusions in this issue and the issue (ii) remain true for Doss-$(p(\cdot), c)$-uniformly recurrent functions.*

It is worth noting that it is not clear whether we can formulate an analogue of [12] (Proposition 2.11(i)) (cf. also [13] ([Proposition 2.16)) for Doss almost periodic type functions. Further on, let us note that Proposition 3 can be reformulated for the general classes of $(\mathcal{B}, \Lambda', \rho)$-almost periodic functions and $(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent functions (see [14] for more details). For example, the first example of a uniformly anti-recurrent function $F : \mathbb{R} \to \mathbb{R}$ has recently been constructed in [12] (Example 2.20) as follows

$$F(t) := (\sin t) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Big(\frac{t}{3^n}\Big), \quad t \in \mathbb{R}.$$

Let $p \in [1, \infty)$ be fixed; arguing similarly as in the proof of Proposition 3, we get that the function $F(\cdot)$ is Doss-(p, c)-uniformly recurrent if and only if $c = \pm 1$.

The main structural properties of multi-dimensional ρ-almost periodic type functions have been clarified in [14] (Theorem 2.11(i)–(iv)); all these results admit very simple reformulations for multi-dimensional Doss almost periodic type functions introduced in Definition 1 (cf. also [18] (Theorem 2) for the property of translation invariance, where some difficulties obviously occur). Details can be left to the interested readers.

Concerning the statement of [14] (Theorem 2.11(v)), we will state the following result:

Proposition 4. *Suppose that $F_k : \Lambda \times X \to Y$ and $F : \Lambda \times X \to Y$ ($k \in \mathbb{N}$) as well as that $\lim_{k\to+\infty} \sup_{\mathbf{t}\in\Lambda; x\in B} \|F_k(\mathbf{t}; x) - F(\mathbf{t}; x)\|_Y = 0$ for every $B \in \mathcal{B}$. Let $\phi : [0, \infty) \to [0, \infty)$ be monotonically increasing, continuous, and let there exist a finite real constant $d > 0$ such that (2) holds. Let $D(\rho)$ be a closed subset of Y, and let ρ be single-valued and continuous in the following sense:*

(C_ρ) *For each $\epsilon > 0$ there exists $\delta > 0$ such that, for every y_1, $y_2 \in Y$ with $\|y_1 - y_2\|_Y < \delta$, we have $\|z_1 - z_2\|_Y < \epsilon/3$ with $z_1 = \rho(y_1)$ and $z_2 = \rho(y_2)$.*

Assume, further, that there exist two finite real numbers $t_0 > 0$ and $M > 0$ such that

$$\mathrm{F}(t)\,[1]_{L^{p(\cdot)}(\Lambda_t)} \leq M, \quad t \geq t_0. \tag{8}$$

Then we have the following:

(i) *Suppose that the function $F_k(\cdot;\cdot)$ is Besicovitch-$(p,\phi,\mathrm{F},\mathcal{B})$-bounded for all $k\in\mathbb{N}$. Then the function $F(\cdot;\cdot)$ is likewise Besicovitch-$(p,\phi,\mathrm{F},\mathcal{B})$-bounded.*

(ii) *Suppose that the function $F_k(\cdot;\cdot)$ is Besicovitch-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-continuous for all $k\in\mathbb{N}$. Then the function $F(\cdot;\cdot)$ is likewise Besicovitch-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-continuous.*

(iii) *Suppose that the function $F_k(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-almost periodic for all $k\in\mathbb{N}$. Then the function $F(\cdot;\cdot)$ is likewise Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-almost periodic.*

(iv) *Suppose that the function $F_k(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-uniformly recurrent for all $k\in\mathbb{N}$. Then the function $F(\cdot;\cdot)$ is likewise Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-uniformly recurrent.*

Proof. We will prove the issue (iii) only, because the issues (i), (ii) and (iv) can be deduced similarly. It is clear that (2) implies the existence of a finite real constant $d'>0$ such that

$$\phi(x+y+z)\leq d'[\phi(x)+\phi(y)+\phi(z)],\quad x,\ y,\ z\geq 0. \tag{9}$$

Let $B\in\mathcal{B}$ and $\epsilon>0$ be fixed. Since $D(\rho)$ is a closed subset of Y and the sequence $(F_k(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set B, we have that $F(\mathbf{t};x)\in D(\rho)$ for all $\mathbf{t}\in\Lambda$ and $x\in X$. Suppose that a real number $\delta>0$ is chosen in accordance with the continuity of relation ρ and function ϕ (at the point $t=0$). Set $\epsilon_0\equiv\min(\epsilon/3,\delta)$. Then we can find a positive integer $k\in\mathbb{N}$ such that $\sup_{\mathbf{t}\in\Lambda;x\in B}\|F_k(\mathbf{t};x)-F(\mathbf{t};x)\|_Y<\epsilon_0$, and a positive real number $l>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists a point $\tau\in B(\mathbf{t}_0,l)\cap\Lambda'$ such that, for every $t>0$, $x\in B$ and $\cdot\in\Lambda_t$, we have the existence of an element $y^k_{\cdot;x}\in\rho(F_k(\cdot;x))$ such that (1) holds with the number ϵ, the function $F(\cdot;\cdot)$ and the element $y_{\cdot;x}$ replaced respectively with the number ϵ_0, the function $F_k(\cdot;\cdot)$ and the element $y^k_{\cdot;x}$. Fix now a number $t>0$ and an element $x\in B$. Let $y_{\mathbf{t};x}=\rho(F(\mathbf{t};x))$. Then we have (a point $\tau\in\Lambda'$ satisfies the requirements in Definition 1 for the function $F_k(\cdot;\cdot)$):

$$\begin{aligned}\|F(\cdot+\tau;x)-y_{\cdot;x}\|_Y&\leq\|F(\cdot+\tau;x)-F_k(\cdot+\tau;x)\|_Y+\|F_k(\cdot+\tau;x)-y^k_{\cdot;x}\|_Y\\&+\|y_{\cdot;x}-y^k_{\cdot;x}\|_Y,\quad\cdot\in\Lambda_t.\end{aligned}\tag{10}$$

It is clear that $F(\cdot+\tau;x)-y_{\cdot;x}=\lim_{k\to+\infty}[F_k(\cdot+\tau;x)-y^k_{\cdot;x}]$ so that the mapping $\phi(\|F(\cdot+\tau;x)-y_{\cdot;x}\|_Y)$ is measurable due to the continuity of function $\phi(\cdot)$. Furthermore, (9) and (10) together imply

$$\begin{aligned}&\Big[\phi\big(\|F(\cdot+\tau;x)-y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)}\\&\leq d'\Big[\phi\big(\|F(\cdot+\tau;x)-F_k(\cdot+\tau;x)\|_Y\big)\\&+\phi\big(\|F_k(\cdot+\tau;x)-y^k_{\cdot;x}\|_Y\big)+\phi\big(\|y^k_{\cdot;x}-y_{\cdot;x}\|_Y\big)\Big]_{L^{p(\cdot)}(\Lambda_t)}\\&\leq d'3(\epsilon/3)\,[1]_{L^{p(\cdot)}(\Lambda_t)},\quad\cdot\in\Lambda_t.\end{aligned}$$

Then the final conclusion simply follows from (8) and the corresponding definition of Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-almost periodicity. □

Now we would like to state the following analogue of [12] (Proposition 2.9) (cf. also [14] (Example 2.8)):

Proposition 5. *Suppose that $l\in\mathbb{N}$, $\phi:[0,\infty)\to[0,\infty)$ is monotonically increasing, there exists a finite real constant $d>0$ such that (2) holds, and there exists a function $\varphi:[0,\infty)\to[0,\infty)$ such that $\phi(xy)\leq\varphi(x)\phi(y)$ for all $x,\ y\geq 0$. Let $\rho=T\in L(Y)$, let $p(\cdot)\equiv p\in[1,\infty)$, and let a function $F:\Lambda\times X\to Y$ be Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',T)$-almost periodic (Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',T)$-uniformly recurrent). Then the function $F(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},l\Lambda',T^l)$-almost periodic (Doss-$(p,\phi,\mathrm{F},\mathcal{B},l\Lambda',T^l)$-uniformly recurrent); in particular, $F(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},l\Lambda',\mathrm{I})$-almost periodic (Doss-$(p,\phi,\mathrm{F},\mathcal{B},l\Lambda',\mathrm{I})$-uniformly recurrent) if $T^l=\mathrm{I}$.*

Proof. We will outline the main details of the proof for Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', T)$-almost periodic functions. Let τ be as in (1). It is clear that (2) implies the existence of a finite real number $d_l > 0$ such that

$$\phi(x_1 + \cdots + x_l) \leq d_l \Big[\phi(x_1) + \cdots + \phi(x_l)\Big],$$

for all positive real numbers $x_1 \geq 0, \cdot \cdot \cdot, x_l \geq 0$. Using this estimate, the decomposition

$$F(\cdot + l\tau; x) - T^l F(\cdot; x) = \sum_{j=0}^{l-1} T^j \Big[F(\cdot + (l-j)\tau; x) - TF(\cdot + (l-j-1)\tau; x)\Big]$$

and the existence of a function $\varphi(\cdot)$ with the prescribed properties, we easily get that for each $t > 0$ and $x \in X$ we have:

$$\begin{aligned} &\Big[\phi\big(\|F(\cdot + l\tau; x) - T^l F(\cdot; x)\|_Y\big)\Big]_{L^{p}(\Lambda_t)} \\ &\leq d_l \sum_{j=0}^{l-1} \varphi\big(\|T\|^j\big) \Big[\phi\big(\|F(\cdot + (l-j)\tau; x) - TF(\cdot + (l-j-1)\tau; x)\|_Y\big)\Big]_{L^{p}(\Lambda_t)}. \end{aligned} \quad (11)$$

Basically, the required conclusion follows from this estimate and the fact that we can use the substitution $\cdot \mapsto \cdot + (l-j-1)\tau$ here since $p(\cdot)$ has a constant value; in actual fact, (11) implies for each $t > 0$ and $x \in X$ one has:

$$\begin{aligned} &\Big[\phi\big(\|F(\cdot + l\tau; x) - T^l F(\cdot; x)\|_Y\big)\Big]_{L^{p}(\Lambda_t)} \\ &\leq d_l \sum_{j=0}^{l-1} \varphi\big(\|T\|^j\big) \Big[\phi\big(\|F(\cdot + \tau; x) - TF(\cdot; x)\|_Y\big)\Big]_{L^{p}(\Lambda_{t+(l-j-1)|\tau|})}, \end{aligned}$$

and we only need to follow the corresponding definitions. □

Concerning the convolution invariance of multi-dimensional Doss ρ-almost periodicity, we will clarify the following extension of [18] (Proposition 6):

Proposition 6. *Suppose that $h \in L^1(\mathbb{R}^n)$, $supp(h) \subseteq K$ for some compact subset K of $\mathbb{R}^n$, and $F : \mathbb{R}^n \times X \to Y$ is a measurable function satisfying that for each $B \in \mathcal{B}$ we have $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B} \|F(\mathbf{t}; x)\|_Y < +\infty$. Suppose, further, that $\rho = A$ is a closed linear operator on Y satisfying that:*

(B) For each $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, the function $\mathbf{s} \mapsto AF(\mathbf{t} - \mathbf{s}; x)$, $\mathbf{s} \in \mathbb{R}^n$ is Bochner integrable.

Then the function

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X \quad (12)$$

is well defined and for each $B \in \mathcal{B}$ we have $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$.

Suppose, further, that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\varphi : [0, \infty) \to [0, \infty)$, $\phi : [0, \infty) \to [0, \infty)$ is a convex, monotonically increasing function satisfying $\phi(xy) \leq \varphi(x)\phi(y)$ for all $x, y \geq 0$, $p, q \in \mathcal{P}(\mathbb{R}^n)$ and $1/p(x) + 1/q(x) = 1$. Suppose, further, that the function $F(\cdot; \cdot)$ is Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', A)$-almost periodic, resp. Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', A)$-uniformly recurrent, as well as that $p_1 \in \mathcal{P}(\mathbb{R}^n)$, $\mathrm{F}_1 : (0, \infty) \to (0, \infty)$ and, for every $\epsilon > 0$, there exists a positive real number $t_1(\epsilon) > 0$ such that

$$\int_{-t}^{t} \varphi_{p_1(\mathbf{u})} \left(2F_1(t) \varphi(m(K)) [m(K)]^{-1} \frac{\big\|\varphi\big(|h(\mathbf{u} - \mathbf{v})|\big)\big\|_{L^{q(\mathbf{v})}(\mathbf{u}-K)}}{F(t + diam(K))} \right) d\mathbf{u} \leq 1, \quad (13)$$

for any $t \geq t_1(\epsilon)$. Then the function $(h * F)(\cdot; \cdot)$ is Doss-$(p_1, \phi, \mathrm{F}_1, \mathcal{B}, \Lambda', A)$-almost periodic, resp. Doss-$(p_1, \phi, \mathrm{F}_1, \mathcal{B}, \Lambda', A)$-uniformly recurrent.

Proof. We will consider only Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', A)$-almost periodic functions. It is clear that the function $(h * F)(\cdot; \cdot)$ is well defined and $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$ for all $B \in \mathcal{B}$. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. Then there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists a point $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$, the element $y_{\cdot;x} = A(F(\cdot; x))$ satisfies (1). Since A is a closed linear operator and condition (B) holds, for every $\mathbf{t} \in \mathbb{R}^n$ and $x \in B$, we have $z_{\mathbf{t},x} := A((h * F)(\mathbf{t}; x)) = \int_{\mathbb{R}^n} h(\mathbf{s}) A(F(\mathbf{t} - \mathbf{s}; x))\, d\mathbf{s}$. Let $t > 0$ and $x \in B$ be fixed. The prescribed assumptions together with the well-known Jensen integral inequality and the Hölder inequality (see e.g., [7] and Lemma 1(i)) imply:

$$
\begin{aligned}
&\left\| \phi\Big(\|(h * F)(\cdot + \tau; x) - z_{\cdot,x}\|_Y \Big) \right\| \\
&\leq \phi\left(m(K)[m(K)]^{-1} \int_K |h(\sigma)| \cdot \|F(\cdot + \tau - \sigma; x) - AF(\cdot - \sigma; x)\|_Y \, d\sigma \right) \\
&\leq \varphi(m(K))[m(K)]^{-1} \int_K \phi\Big(|h(\sigma)| \cdot \|F(\cdot + \tau - \sigma; x) - AF(\cdot - \sigma; x)\|_Y \Big)\, d\sigma \\
&\leq \varphi(m(K))[m(K)]^{-1} \int_K \varphi(|h(\sigma)|) \phi\Big(\|F(\cdot + \tau - \sigma; x) - AF(\cdot - \sigma; x)\|_Y \Big)\, d\sigma \\
&= \varphi(m(K))[m(K)]^{-1} \int_{\cdot - K} \varphi(|h(\cdot - \mathbf{v})|) \phi\Big(\|F(\mathbf{v} + \tau; x) - AF(\mathbf{v}; x)\|_Y \Big)\, d\mathbf{v} \\
&\leq 2\varphi(m(K))[m(K)]^{-1} \big\| \varphi(|h(\cdot - \mathbf{v})|) \big\|_{L^{q(\mathbf{v})}(\cdot - K)} \\
&\times \left\| \phi\Big(\|F(\mathbf{v} + \tau; x) - AF(\mathbf{v}; x)\|_Y \Big) \right\|_{L^{p(\mathbf{v})}(\cdot - K)},
\end{aligned}
$$

for any $\cdot \in (\mathbb{R}^n)_t$. Now the final conclusion simply follows as in the proof of [18] (Proposition 6) using the corresponding definition of Doss-$(p_1, \phi, \mathrm{F}_1, \mathcal{B}, \Lambda', A)$-almost periodicity and the definition of Luxemburg norm. $\square$

Unfortunately, the assumption $supp(h) \subseteq K$ for some compact subset K of $\mathbb{R}^n$ is almost inevitable here, so that we cannot so easily apply Theorem 6 in the qualitative analysis of solutions of the abstract inhomogeneous heat equation in $\mathbb{R}^n$; see [19] for more details. The statements of [14] (Proposition 2.5, Proposition 2.20) and the conclusions from [14] (Example 2.5), showing that the assumption $\Lambda' \subseteq \Lambda$ is redundant, can be simply formulated in our new context.

Concerning the extensions of Doss ρ-almost periodic type functions (see [7,13,14] for some results established for various classes of multi-dimensional ρ-almost periodic type functions), we first observe that any Doss-(p, c)-almost periodic function $F : [0, \infty) \to Y$, where $p \in [1, \infty)$ and $c \in \mathbb{C}$, can be extended to a Doss-(p, c)-almost periodic function $\tilde{F} : \mathbb{R} \to Y$ defined by $\tilde{F}(t) := 0$, $t < 0$ (and, obviously, $\tilde{F}(t) := F(t)$ for all $t \geq 0$). Without going into full details, we will only note that a similar type of extension can be achieved in a much more general situation; for example, a very simple argumentation shows that any Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodic function $F : \Lambda \times X \to Y$ can be extended to a Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho_1)$-almost periodic function $\tilde{F} : \mathbb{R}^n \times X \to Y$, defined by $\tilde{F}(\mathbf{t}) := 0$, $t \notin \Lambda$, $\tilde{F}(\mathbf{t}) := F(\mathbf{t})$, $t \in \Lambda$, with $\rho_1 := \rho \cup \{(0, 0)\}$, if the following conditions hold:

(i) $\phi(\cdot)$ is locally bounded;
(ii) The Lebesgue measure of $\partial\Lambda$ is equal to zero;
(iii) For each set $B \in \mathcal{B}$ we have $\sup_{\mathbf{t} \in \Lambda; x \in B} \|F(\mathbf{t}; x)\|_Y < +\infty$, or there exists $t_0 > 0$ such that, for every $t \geq 0$, $B \in \mathcal{B}$ and $\tau \in \Lambda'$, there exists a compact set $K_\tau \subseteq \mathbb{R}^n$ such that $(\mathbb{R}^n \setminus \Lambda) \cup (\Lambda - \tau) \subseteq K_\tau$ and

$$\sup_{x \in B} \big[\phi(\|F(\cdot + \tau; x)\|_Y) \big]_{L^{p(\cdot)}(K_\tau)} < +\infty;$$

(iv) $\lim_{t \to +\infty} \mathrm{F}(t) = 0$.

We will state only one composition principle for Doss ρ-almost periodic type functions. The following result for one-dimensional Doss (p,c)-almost periodic type functions can be deduced following the lines of the proof of [12] (Theorem 2.28):

Proposition 7. *Suppose that* $1 \leq p < \infty$, $c \in \mathbb{C}$ *and* $F : \Lambda \times X \to Y$ *satisfies that there exists a finite real number* $L > 0$ *such that*

$$\|F(t;x) - F(t;y)\|_Y \leq L\|x-y\|, \quad t \in \Lambda,\ x,\ y \in X. \tag{14}$$

(i) *Suppose that* $f : \Lambda \to X$ *is Doss* (p,Λ',c)*-uniformly recurrent, where* $\Lambda' := \{\alpha_k : k \in \mathbb{N}\}$ *for some strictly increasing sequence* (α_k) *of positive reals tending to plus infinity. If*

$$\lim_{k\to+\infty} \limsup_{t\to+\infty} \frac{1}{t}\int_{[-t,t]\cap\Lambda} \Bigl\| F\Bigl(s+\alpha_k; cf(s)\Bigr) - cF(s;f(s))\Bigr\|^p\, ds = 0, \tag{15}$$

then the mapping $\mathcal{F}(t) := F(t;f(t))$, $t \in \Lambda$ *is Doss* (p,Λ',c)*-uniformly recurrent.*

(ii) *Suppose that* $f : \Lambda \to X$ *is Doss* (p,Λ',c)*-almost periodic. If for each* $\epsilon > 0$ *the set of all positive real numbers* $\tau > 0$ *such that*

$$\limsup_{t\to+\infty} \frac{1}{t}\int_{[-t,t]\cap\Lambda} \bigl\|f(s+\tau) - cf(s)\bigr\|^p\, ds < \epsilon$$

and

$$\limsup_{t\to+\infty} \frac{1}{t}\int_{[-t,t]\cap\Lambda} \Bigl\| F\Bigl(s+\tau; cf(s)\Bigr) - cF(s;f(s))\Bigr\|^p\, ds < \epsilon,$$

is relatively dense in $[0,\infty)$*, then the mapping* $\mathcal{F}(t) := F(t;f(t))$, $t \in \Lambda$ *is Doss* (p,Λ',c)*-almost periodic.*

We can similarly analyze the composition principles for multi-dimensional Doss c-almost periodic functions (see also [14] for related results concerning the general class of multi-dimensional ρ-almost periodic functions). In combination with Proposition 6, this enables one to analyze the existence and uniqueness of bounded, continuous, Doss-(p,c)-almost periodic solutions of the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$:

$$y(\mathbf{t}) = \int_{\mathbb{R}^n} k(\mathbf{t}-\mathbf{s})F(\mathbf{s};y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

where the kernel $k(\cdot)$ has compact support; see also the issue [19] (4., Section 3).

2.1. Relationship between Weyl Almost Periodicity and Doss Almost Periodicity

It is worth noting that Proposition 3 can be formulated for multi-dimensional ρ-almost periodic functions and their Stepanov generalizations considered recently in [16]. This is very predictable and details can be left to the interested readers.

In this subsection, we would like to point out the following much more important fact with regards to Proposition 3: It is well known that, in the one-dimensional setting, the class of Doss-p-almost periodic functions provides a proper extension of the class of Besicovitch-p-almost periodic functions; see [6] for more details. On the other hand, the class of Weyl-p-almost periodic functions taken in the generalized approach of A. S. Kovanko [24] is not contained in the class of Besicovitch-p-almost periodic functions, as clearly marked in [7]. A very simple observation shows that the class of Doss-p-almost periodic functions extends the class of Weyl-p-almost periodic functions, as well, which is defined as follows ($1 \leq p < \infty$): Let $\Lambda = \mathbb{R}$ or $\Lambda = [0,\infty)$, and let $f \in L^p_{loc}(\Lambda : Y)$. Then we say that the function $f(\cdot)$ is Weyl-p-almost periodic if and only if for each $\epsilon > 0$ we can find a real number $L > 0$ such that any interval $\Lambda_0 \subseteq \Lambda$ of length L contains a point $\tau \in \Lambda_0$ such that

$$\limsup_{l \to \infty} \sup_{x \in \Lambda} \left[\frac{1}{l} \int_x^{x+l} \| f(t+\tau) - f(t) \|^p \, dt \right]^{1/p} \leq \epsilon. \tag{16}$$

So, let $\Lambda = \mathbb{R}$, let $f(\cdot)$ be Weyl-p-almost periodic, and let a number $\epsilon > 0$ be given. Then there exists a finite real number $L > 0$ such that such that any interval $\Lambda_0 \subseteq \Lambda$ of length L contains a point $\tau \in \Lambda_0$ such that (16) holds; hence, there exists a finite real number $l_0(\epsilon, \tau) > 0$ such that, for every $l \geq l_0(\epsilon, \tau)$ and $x \in \mathbb{R}$, we have

$$\int_x^{x+l} \| f(t+\tau) - f(t) \|^p \, dt \leq l\epsilon^p.$$

Plugging $x = 0$ and $x = -l$ here, we easily get that, for each real number $l \geq l_0(\epsilon, \tau)$, we have:

$$\int_{-l}^{l} \| f(s+\tau) - f(s) \|^p \, ds \leq 2l\epsilon^p,$$

which simply implies that $f(\cdot)$ is Doss-p-almost periodic.

In [7] (Theorem 8.3.8), we have particularly proved the following: Suppose that $\sigma \in (0,1)$, $p \in [1, \infty)$, $(1-\sigma)p < 1$ and $a > 1 - (1-\sigma)p$. Define $f(x) := |x|^\sigma$, $x \in \mathbb{R}$. Then the function $f(\cdot)$ is Weyl-p-almost periodic, Besicovitch p-unbounded and has no mean value (see [7] for the notion). As a consequence, we have that a Weyl-p-almost periodic function (Doss-p-almost periodic function) is not necessarily Besicovitch-p-almost periodic; also, a Doss-p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ has no mean value and can be Besicovitch p-unbounded in general ($1 \leq p < \infty$).

The above consideration can be simply extended to the multi-dimensional setting. In order to do that, we will first recall the following definition from [16]:

Definition 2. *Assume that the following condition holds:*

(WM): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$ and $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$ and $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0, \infty) \to [0, \infty)$ and $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$.

By $W^{(p(\mathbf{u}), \phi, \mathbb{F}), \rho}_{\Omega, \Lambda', \mathcal{B}}(\Lambda \times X : Y)$, we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\limsup_{l \to +\infty} \sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Big(\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \|_Y \Big)_{L^{p(\mathbf{u})}(\mathbf{t} + l\Omega)} < \epsilon.$$

The usual concept of multi-dimensional Weyl-p-almost periodicity is obtained by plugging $p(\cdot) \equiv p \in [1, \infty)$, $\phi(x) \equiv x$, $x \geq 0$, $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$, $l > 0$, $\mathbf{t} \in \Lambda$, $\Omega = [0,1]^n$, $\Lambda' = \Lambda = [0, \infty)^n$ or $\mathbb{R}^n$ and $\rho = \mathrm{I}$. The proof of following proposition is quite simple and therefore omitted (we employ almost all of the above-mentioned conditions but we allow the situation in which $\Lambda' \neq \Lambda$ and $\phi(x)$ is not identically equal to x for all $x \geq 0$):

Proposition 8. *Suppose that* (WM) *holds with $\Lambda = [0, \infty)^n$ or $\mathbb{R}^n$, $p(\cdot) \equiv p \in [1, \infty)$, $\mathbb{F}(l, \mathbf{t}) \equiv \mathbb{F}(l)$, $l > 0$, $\mathbf{t} \in \Lambda$, $\Omega = [0,1]^n$, and ρ is single-valued on $R(F)$. Suppose that for each $l_0 > 0$ there exists a finite real number $t_0 \geq l_0$ such that*

$$(t/l_0)^{n/p} \leq \mathbb{F}(l_0)/\mathrm{F}(t), \quad t \geq t_0. \tag{17}$$

If $F \in W^{(p,\phi,\mathbb{F}),\rho}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$, then $F(\cdot;\cdot)$ is Doss-$(p,\phi,\mathrm{F},\mathcal{B},\Lambda',\rho)$-almost periodic.

It is worth noting that condition (17) holds in the classical situation $\mathbb{F}(l,\mathbf{t}) \equiv l^{-n/p}$ and $\mathrm{F}(t) \equiv t^{-n/p}$ $(l,\ t > 0;\ \mathbf{t} \in \Lambda)$.

We continue with the following instructive example, which has not been published in any research article by now and which will be published soon as [7] (Example 3.2.14):

Example 7. *Let $\zeta \geq 1$ and $0^{\zeta} := 0$. Define the complex-valued function:*

$$f_{\zeta}(t) := \sum_{l=1}^{\infty} \frac{1}{l} \sin^{\zeta}\Big(\frac{t}{2^l}\Big), \quad t \in \mathbb{R}.$$

Then the function $f_{\zeta}(\cdot)$ is Lipschitz continuous and uniformly recurrent. To prove the Lipschitz continuity of function $f_{\zeta}(\cdot)$, it suffices to observe that the function $t \mapsto \sin^{\zeta}(t)$, $t \in \mathbb{R}$ is continuous and

$$\Big|\sin^{\zeta} x - \sin^{\zeta} y\Big| \leq \zeta|x-y|, \quad x,\ y \in \mathbb{R}. \tag{18}$$

To see that the function $f_{\zeta}(\cdot)$ is uniformly recurrent (cf. [7] for the notion), it suffices to see that for each integer $k \in \mathbb{N} \setminus \{1\}$ we have

$$\begin{aligned}
\Big|f_{\zeta}(t+2^k\pi) - f_{\zeta}(t)\Big| &= \Bigg|\sum_{l=1}^{\infty} \frac{1}{l}\Bigg[\sin^{\zeta}\Big(\frac{t+2^k\pi}{2^l}\Big) - \sin^{\zeta}\Big(\frac{t}{2^l}\Big)\Bigg]\Bigg| \\
&= \Bigg|\sum_{l=1}^{k-1} \frac{1}{l}\Bigg[\sin^{\zeta}\Big(\frac{t+2^k\pi}{2^l}\Big) - \sin^{\zeta}\Big(\frac{t}{2^l}\Big)\Bigg]\Bigg| + \Bigg|\sum_{l=k}^{\infty} \frac{1}{l}\Bigg[\sin^{\zeta}\Big(\frac{t+2^k\pi}{2^l}\Big) - \sin^{\zeta}\Big(\frac{t}{2^l}\Big)\Bigg]\Bigg| \\
&= \Bigg|\sum_{l=k}^{\infty} \frac{1}{l}\Bigg[\sin^{\zeta}\Big(\frac{t+2^k\pi}{2^l}\Big) - \sin^{\zeta}\Big(\frac{t}{2^l}\Big)\Bigg]\Bigg| \leq \sum_{l=k}^{\infty} \frac{1}{l}\Bigg|\sin^{\zeta}\Big(\frac{t+2^k\pi}{2^l}\Big) - \sin^{\zeta}\Big(\frac{t}{2^l}\Big)\Bigg| \\
&\leq \sum_{l=k}^{\infty} \frac{\zeta}{l} 2^{k-l}\pi = \frac{2\pi\zeta}{k}, \quad t \in \mathbb{R},
\end{aligned}$$

where we have applied (18) in the last line of computation. In the case that $\zeta = 2v$ for some integer $v \in \mathbb{N}$, we have that the function $f_{\zeta}(\cdot)$ is Besicovitch unbounded. This can be inspected as in the proof of [30] (Theorem 1.1), with the additional observation that:

$$\int_0^{2^{k-l}\pi} \sin^{2v} t\, dt = \frac{2}{3}\frac{(2v-1)!!}{(2v)!!}\int_0^{2^{k-l}\pi} \sin^2 t\, dt \ \ (k \in \mathbb{N} \setminus \{1\},\ 1 \leq l \leq k);$$

here, we have used the well known recurrent formula:

$$\int_0^{2^{k-l}\pi} \sin^{2v} t\, dt = \frac{2v-1}{2v}\int_0^{2^{k-l}\pi} \sin^{2v-2} t\, dt,$$

which can be deduced with the help of the partial integration. We would like to ask whether the function $f_{\zeta}(\cdot)$ is Besicovitch unbounded in general case and for which functions $p \in D_+(\mathbb{R})$ we have that $f(\cdot)$ is Doss-$p(\cdot)$-almost periodic, that is, Doss-$(p(\cdot),1)$-almost periodic?

We continue by observing that the functions of the form:

$$f(t) := \sum_{l=1}^{\infty} a_l \sin^{\zeta_l}\Big(\frac{t}{b_l}\Big), \quad t \in \mathbb{R},$$

where $\zeta_l \geq 0$ for all $l \in \mathbb{N}$, and (a_l) and (b_l) are real sequences such that the above series is absolutely convergent, are still very unexplored within the theory of almost periodic functions.

For example, we know that the sequence of partial sums of the series (therefore, the sequence of trigonometric polynomials):

$$x \mapsto f(x) := \sum_{l=1}^{\infty} \frac{1}{l} \sin \frac{x}{l}, \quad x \in \mathbb{R}$$

is a Cauchy sequence with respect to the Weyl metric W^2 but its sum, which is clearly an essentially bounded function, is not equi-Weyl-2-almost periodic; see for example [8] (p. 247) and [6] for the notion. On the other hand, using the identity

$$\sin \frac{x+\tau}{l} - \sin \frac{x}{l} = 2\Big[\cos \frac{x}{l} \cos \frac{\tau}{2l} - \sin \frac{x}{l} \sin \frac{\tau}{2l}\Big] \sin \frac{\tau}{2l}, \quad x \in \mathbb{R},\ l \in \mathbb{N}, \tau \in \mathbb{R},$$

we can simply prove that for each $\tau \in \mathbb{R}$ and $p \geq 1$ we have

$$\lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{t} \big|f(x+\tau) - f(x)\big|^p \, dx = 0.$$

In particular, the function $f(\cdot)$ is Doss-p-almost periodic for any finite exponent $p \geq 1$. We would like to ask whether the function $f(\cdot)$ is equi-Weyl-p-almost periodic for some exponent $p \in [1, 2)$ or Weyl-p-almost periodic for some finite exponent $p \geq 1$?

We close this subsection with the observation that, for every finite exponent $p \geq 1$, there exists a (Besicovitch-)Doss-p-almost periodic function $f : \mathbb{R} \to \mathbb{R}$ which is not Weyl-p-almost periodic; see for example [11] (Example 6.24).

2.2. Invariance of Doss ρ-Almost Periodicity under the Actions of Convolution Products

This subsection investigates the invariance of Doss ρ-almost periodicity under the actions of infinite convolution products (for simplicity, we will not consider here the finite convolution products). From the application point of view, the one-dimensional framework is the most important and here we will only note that the established results admit straightforward extensions for the infinite convolution product

$$\mathbf{t} \mapsto \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s}) \, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

and the finite convolution product

$$\mathbf{t} \mapsto \int_{\alpha_1}^{t_1} \int_{\alpha_2}^{t_2} \cdots \int_{\alpha_n}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s}) \, ds, \quad \mathbf{t} \in [0, \infty)^n;$$

see [7,19] for more details.

We start by stating the following extension of [18] (Theorem 3):

Theorem 1. *Suppose that $\psi : (0, \infty) \to (0, \infty)$, $\varphi : [0, \infty) \to [0, \infty)$, $\phi : [0, \infty) \to [0, \infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leq \varphi(x)\phi(y)$ for all $x,\ y \geq 0$ and $p \in \mathcal{P}(\mathbb{R})$. Suppose, further, $\emptyset \neq \Lambda' \subseteq \mathbb{R}$, A is a closed linear operator commuting with $R(\cdot)$, $\check{f} : \mathbb{R} \to X$ is Doss-$(p, \phi, F, \Lambda', A)$-almost periodic, resp. Doss-$(p, \phi, F, \Lambda', A)$-uniformly recurrent, and measurable, $F_1 : (0, \infty) \to (0, \infty)$, $q \in \mathcal{P}(\mathbb{R})$, $1/p(x) + 1/q(x) = 1$, $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family and for every real number $x \in \mathbb{R}$ we have*

$$\int_{-x}^{\infty} \|R(v+x)\| \|\check{f}(v)\| \, dv < \infty \tag{19}$$

and

$$\int_{-x}^{\infty} \|R(v+x)\| \|A\check{f}(v)\| \, dv < \infty. \tag{20}$$

Suppose, further, that for each $\epsilon > 0$ there exists an increasing sequence (a_m) of positive real numbers tending to plus infinity and a number $t_0(\epsilon) > 0$ satisfying that, for every $t \geq t_0(\epsilon)$, we have:

$$\int_{-t}^{t} \varphi_{p(x)}\left(2\varphi(a_m)a_m^{-1}F_1(t)\limsup_{m\to+\infty}\left[\left[\varphi(\|R(\cdot+x)\|)\right]_{L^{q(\cdot)}[-x,-x+a_m]}F(t+a_m)^{-1}\right]\right)dx \leq 1.$$

Then the function $F : \mathbb{R} \to X$, given by:

$$F(t) := \int_{-\infty}^{t} R(t-s)f(s)\,ds, \quad t \in \mathbb{R}, \tag{21}$$

is well-defined and Doss-$(p, \phi, F_1, \Lambda', A)$-almost periodic, resp. Doss-$(p, \phi, F_1, \Lambda', A)$-uniformly recurrent.

Proof. It is clear that $F(x) = \int_{-x}^{\infty} R(v+x)\check{f}(v)\,dv$, $x \in \mathbb{R}$; hence, (19) implies that the function $F(\cdot)$ is well-defined as well as that the integrals in the definitions of $F(x)$ and $F(x+\tau) - F(x)$ converge absolutely ($x \in \mathbb{R}$). Furthermore, since A is a closed linear operator commuting with $R(\cdot)$, and since we have assumed (20), we have $AF(x) = \int_{-x}^{\infty} R(v+x)A\check{f}(v)\,dv$, $x \in \mathbb{R}$. The remainder of proof is almost the same as the proof of the corresponding part of [18] (Theorem 3), with the distance $\check{f}(v+\tau) - \check{f}(v)$ replaced therein with the distance $\check{f}(v+\tau) - A\check{f}(v)$. $\square$

Using a similar argumentation and inspecting carefully the proof of [6] (Theorem 2.13.10), we may conclude that the following result holds true:

Theorem 2. *Let $\emptyset \neq \Lambda' \subseteq \mathbb{R}$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ satisfy:*

$$\|R(t)\| \leq \frac{Mt^{\beta-1}}{1+t^{\gamma}},\ t > 0 \text{ for some finite constants } \gamma > 1,\ \beta \in (0,1],\ M > 0.$$

Let A be a closed linear operator commuting with $R(\cdot)$, let a function $g : \mathbb{R} \to X$ be Doss-(p, Λ', A)-almost periodic, resp. Doss-(p, Λ', A)-uniformly recurrent, and Stepanov p-bounded, and let $q(\beta-1) > -1$ provided that $p > 1$, resp. $\beta = 1$, provided that $p = 1$. Assume that the function $t \mapsto Ag(t)$, $t \in \mathbb{R}$ is Stepanov p-bounded. Then the function $G : \mathbb{R} \to X$, defined through (21) with $f = g$ therein, is bounded, continuous and Doss-(p, Λ', A)-almost periodic, resp. Doss-(p, Λ', A)-uniformly recurrent. Furthermore, if $g(\cdot)$ is B^p-continuous, then $G(\cdot)$ is B^p-continuous, as well.

Remark 2. *If $A = c\mathrm{I}$ for some $c \in \mathbb{C}$, then we can consider two different pivot spaces X and Y in Theorems 1 and 2. See also [6] (Theorem 2.13.7), where we have used the estimate*

$$\int_0^{+\infty} (1+t)\|R(t)\|\,dt < +\infty,$$

which cannot be satisfied for fractional solution operator families.

3. Applications to Abstract Volterra Integro-Differential Equations and Partial Differential Equations

In this section, we aim to present some applications of our abstract results to the abstract Volterra integro-differential equations and the partial differential Equations.

1. We start by observing that our results about the invariance of Doss ρ-almost periodicity under the actions of convolution products, established in Section 2.2, can be applied in the analysis of the existence and uniqueness of Doss-(p, A)-almost periodic solutions in the time variable for various kinds of the abstract (degenerate) Volterra integro-differential equations (see e.g., [6] for more details). For example, we can apply Theorem 2

in the analysis of the existence and uniqueness of Doss (p,c)-almost periodic solutions of the following fractional Poisson heat equation with Weyl-Liouville fractional derivatives:

$$D^{\gamma}_{t,+}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), \quad t \in \mathbb{R},\ x \in \Omega,$$

where $\gamma \in (0,1)$, $1 \leq p < \infty$ and $c \in \mathbb{C}$; possible applications can be also given to the higher-order differential operators in Hölder spaces. All this has been seen many times and details can be omitted.

2. It is worth noting that Proposition 4, Proposition 7 and Theorem 2 can be implemented in the analysis of the existence and uniqueness of Doss-(p,Λ',c)-uniformly recurrent solutions for various classes of abstract fractional semilinear Cauchy inclusions and equations. Suppose, for instance, that $\gamma \in (0,1)$, a closed multivalued linear operator $\mathcal{A}$ on X satisfies all requirements from [6] (Subsection 2.9.2) and the solution family $P_{\gamma}(\cdot)$ is defined as therein. Define $R_{\gamma}(t) := t^{\gamma-1}P_{\gamma}(t)$, $t > 0$. Then we know that $\|R_{\gamma}(t)\| = O(t^{\gamma-1}/(1+t^{2\gamma}))$, $t > 0$. Let $p \in (1,\infty)$, let $1/p + 1/q = 1$, and let $q(\gamma-1) > -1$. Fix now a strictly increasing sequence (α_k) of positive reals tending to plus infinity, and define:

$$\begin{aligned} BCD_{(\alpha_k);c}(\mathbb{R}:X) :=\Big\{ & f : \mathbb{R} \rightarrow X;\ f(\cdot) \text{ is bounded and} \\ & (p,\Lambda',c) - \text{uniformly recurrent,} \Big\}, \end{aligned}$$

where $\Lambda' := \{\alpha_k : k \in \mathbb{N}\}$. By Proposition 4(iv), the set $BCD_{(\alpha_k);c}(\mathbb{R}:X)$ equipped with the metric $d(\cdot,\cdot) := \|\cdot - \cdot\|_{\infty}$ is a complete metric space. Suppose now that a mapping $F : \Lambda \times X \rightarrow Y$ satisfies the estimate (15). We say that a continuous function $u : \mathbb{R} \rightarrow X$ is a mild solution of the semilinear Cauchy inclusion

$$D^{\gamma}_{t,+}u(t) \in \mathcal{A}u(t) + F(t;u(t)),\ t \in \mathbb{R}, \tag{22}$$

if and only if

$$u(t) = \int_{-\infty}^{t} R_{\gamma}(t-s)F\big(s;u(s)\big)\,ds, \quad t \in \mathbb{R}.$$

Keeping in mind Proposition 7 and Theorem 2, we can simply prove the following analogue of [12] (Theorem 3.1):

Theorem 3. *Suppose that the above requirements hold as well as that the function $F : \mathbb{R} \times X \rightarrow X$ satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{x\in B} \|F(t;x)\| \leq M_B$. If there exists a finite real number $L > 0$ such that: (14) holds, and there exists an integer $m \in \mathbb{N}$ such that: $M_m < 1$, where*

$$\begin{aligned} M_m :=& L^m \sup_{t\geq 0} \int_{-\infty}^{t}\int_{-\infty}^{x_m}\cdots\int_{-\infty}^{x_2}\Big\|R_{\gamma}(t-x_m)\Big\| \\ & \times \prod_{i=2}^{m}\Big\|R_{\gamma}(x_i - x_{i-1})\Big\|\,dx_1\,dx_2\cdots dx_m, \end{aligned}$$

then the abstract semilinear fractional Cauchy inclusion (22) has a unique bounded Doss-(p,Λ',c)-uniformly recurrent solution which belongs to the space $BCD_{(\alpha_k);c}(\mathbb{R}:X)$.

3. In this issue, we continue our analysis of the famous d'Alembert formula. Let $a > 0$; then we know that the regular solution of the wave equation $u_{tt} = a^2u_{xx}$ in domain $\{(x,t) : x \in \mathbb{R},\ t > 0\}$, equipped with the initial conditions $u(x,0) = f(x) \in C^2(\mathbb{R})$ and $u_t(x,0) = g(x) \in C^1(\mathbb{R})$, is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}\big[f(x-at)+f(x+at)\big] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds, \quad x \in \mathbb{R},\ t>0.$$

Suppose now that the function $x \mapsto (f(x), g^{[1]}(x))$, $x \in \mathbb{R}$ is Doss-(p,c)-almost periodic for some $p \in [1,\infty)$ and $c \in \mathbb{C}$, where: $g^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\,ds$. Clearly, the solution $u(x,t)$ can be extended to the whole real line in the time variable; we will prove that the solution $u(x,t)$ is Doss-(p,c)-almost periodic in $(x,t) \in \mathbb{R}^2$. In actual fact, we have $(x,\ t,\ \tau_1,\ \tau_2 \in \mathbb{R})$:

$$\begin{aligned}
&\Big|u\big(x+\tau_1, t+\tau_2\big) - cu(x,t)\Big| \\
&\quad \leq \frac{1}{2}\Big|f\big((x-at)+(\tau_1-a\tau_2)\big) - cf(x-at)\Big| \\
&\quad + \frac{1}{2}\Big|f\big((x+at)+(\tau_1+a\tau_2)\big) - cf([x+at+(\tau_1+a\tau_2)] - (\tau_1+a\tau_2))\Big| \\
&\quad + \frac{1}{2a}\Big|g^{[1]}\big((x-at)+(\tau_1-a\tau_2)\big) - cg^{[1]}(x-at)\Big| \\
&\quad + \frac{1}{2a}\Big|g^{[1]}\big((x+at)-(\tau_1-a\tau_2)\big) - cg^{[1]}(x+at)\Big|.
\end{aligned} \tag{23}$$

If $\tau_1 - a\tau_2$ satisfies that $\limsup_{l\to+\infty}(1/l)\int_{-l}^{l}|f(v+\tau_1-a\tau_2)-f(v)|^p\,dv \leq \epsilon^p$, then there exists a finite real number $l_0(\epsilon,\tau_1,\tau_2)>0$ such that $\int_{-l}^{l}|f(v+\tau_1-a\tau_2)-f(v)|^p\,dv \leq \epsilon^p l$, $l \geq l_0(\epsilon,\tau_1,\tau_2)$ and therefore:

$$\begin{aligned}
&\int_{|(x,t)|\leq l}\Big|f\big((x-at)+(\tau_1-a\tau_2)\big)-cf(x-at)\Big|^p\,dx\,dt \\
&\quad \leq \int_{-l}^{l}\int_{-l}^{l}\Big|f\big((x-at)+(\tau_1-a\tau_2)\big)-cf(x-at)\Big|^p\,dx\,dt \\
&\quad = \int_{-l}^{l}\left[\int_{-l}^{l}\Big|f\big((x-at)+(\tau_1-a\tau_2)\big)-cf(x-at)\Big|^p\,dt\right]dx \\
&\quad = \frac{1}{a}\int_{-l}^{l}\left[\int_{x-al}^{x+al}\Big|f\big(v+(\tau_1-a\tau_2)\big)-cf(v)\Big|^p\,dv\right]dx \\
&\quad \leq \frac{1}{a}\int_{-l}^{l}\left[\int_{-l(1+a)}^{l(1+a)}\Big|f\big(v+(\tau_1-a\tau_2)\big)-cf(v)\Big|^p\,dv\right]dx \\
&\quad \leq \frac{1}{a}\epsilon^p l(1+a)\int_{-l}^{l}dx = \frac{1}{a}\epsilon^p l^2(1+a), \quad l \geq (1+a)^{-1}l_0(\epsilon,\tau_1,\tau_2),
\end{aligned}$$

where we have applied the Fubini theorem in the third line of computation. The remaining three addends in (23) can be estimated similarly, so that the final conclusion simply follows as in the final part of [12] (Example 1.2).

4. In [7], we have recently the existence and uniqueness of c-almost periodic type solutions of the wave equations in $\mathbb{R}^3$:

$$u_{tt}(t,x) = d^2\Delta_x u(t,x), \quad x \in \mathbb{R}^3,\ t>0;\ \ u(0,x)=g(x),\ \ u_t(0,x)=h(x), \tag{24}$$

where $d>0$, $g \in C^3(\mathbb{R}^3:\mathbb{R})$ and $h \in C^2(\mathbb{R}^3:\mathbb{R})$. Let us recall that the famous Kirchhoff formula (see e.g., [31] (Theorem 5.4, pp. 277–278); we will use the same notion and notation) says that the function:

$$
\begin{aligned}
u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} g(\sigma)\,d\sigma\right] + \frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} g(\sigma)\,d\sigma \\
&= \frac{1}{4\pi}\int_{\partial B_1(0)} g(x+dt\omega)\,d\omega + \frac{dt}{4\pi}\int_{\partial B_1(0)} \nabla g(x+dt\omega)\cdot\omega\,d\omega \\
&+ \frac{t}{4\pi}\int_{\partial B_1(0)} h(x+dt\omega)\,d\omega \\
&:= u_1(t,x)+u_2(t,x)+u_3(t,x), \quad t\geq 0,\ x\in\mathbb{R}^3,
\end{aligned}
$$

is a unique solution of problem (24) which belongs to the class $C^2([0,\infty)\times\mathbb{R}^3)$. Let us fix now a number $t_0>0$. Then the function $x\mapsto u(t_0,x)$, $x\in\mathbb{R}^3$ is Doss-$(1,x,\mathrm{F},\Lambda',c)$-almost periodic (Doss-$(1,x,\mathrm{F},\Lambda',c)$–uniformly recurrent) provided that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are of the same type ($\emptyset\neq\Lambda'\subseteq\mathbb{R}^3$; $c\in\mathbb{C}$). This is a simple consequence of the following computation, given here only for the function $u_3(t,\cdot)$:

$$
\begin{aligned}
&\int_{|x|\leq l}\big|u_3(t,x+\tau)-cu_3(x,t)\big|\,dx \\
&\quad\leq \frac{t}{4\pi}\int_{|x|\leq l}\int_{\partial B_1(0)}\big|h(x+\tau+dt\omega)-ch(x+dt\omega)\big|\,d\omega\,dx \\
&\quad= \frac{t}{4\pi}\int_{\partial B_1(0)}\int_{|x|\leq l}\big|h(x+\tau+dt\omega)-ch(x+dt\omega)\big|\,dx\,d\omega \\
&\quad\leq \frac{t\epsilon}{4\pi\mathrm{F}(l)}\int_{\partial B_1(0)} d\omega,
\end{aligned}
$$

provided that $l-dt\geq l_0(\epsilon,\tau)$, the last being determined from the Doss-$(1,x,\mathrm{F},\Lambda',c)$-almost periodicity of function $h(\cdot)$ with a number $\epsilon>0$ given in advance.

We can similarly analyze the existence and uniqueness of Doss-$(1,x,\mathrm{F},\Lambda',c)$-almost periodic (Doss-$(1,x,\mathrm{F},\Lambda',c)$–uniformly recurrent) solutions of the wave equations in $\mathbb{R}^2$:

$$
u_{tt}(t,x)=d^2\Delta_x u(t,x),\quad x\in\mathbb{R}^2,\ t>0;\ \ u(0,x)=g(x),\ \ u_t(0,x)=h(x), \tag{25}
$$

where $d>0$, $g\in C^3(\mathbb{R}^2:\mathbb{R})$ and $h\in C^2(\mathbb{R}^2:\mathbb{R})$. Let us only recall that the famous Poisson formula (see e.g., [31] (Theorem 5.5, pp. 280–281)) says that the function:

$$
\begin{aligned}
u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{2\pi d}\int_{\partial B_{dt}(x)}\frac{g(\sigma)}{\sqrt{d^2t^2-|x-y|^2}}\,d\sigma\right] + \frac{1}{2\pi d}\int_{\partial B_{dt}(x)}\frac{h(\sigma)}{\sqrt{d^2t^2-|x-y|^2}}\,d\sigma \\
&= d\int_{B_1(0)}\frac{g(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,d\sigma + d^2t\int_{B_1(0)}\frac{\nabla g(x+dt\sigma)\cdot\sigma}{\sqrt{1-|\sigma|^2}}\,d\sigma \\
&+ dt\int_{B_1(0)}\frac{h(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\,d\sigma, \quad t\geq 0,\ x\in\mathbb{R}^2,
\end{aligned}
$$

is a unique solution of problem (25) which belongs to the class $C^2([0,\infty)\times\mathbb{R}^3)$.

4. Conclusions and Final Remarks

In this paper, we have analyzed the multi-dimensional Doss ρ-almost periodic type functions of the form $F:\Lambda\times X\to Y$, where $n\in\mathbb{N}$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, X and Y are complex Banach spaces, and ρ is a binary relation on Y. The main structural properties of introduced classes of functions are presented, including some applications to the abstract Volterra integro-differential equations and the partial differential equations.

Concerning some drawbacks and research limitations of the class of Doss ρ-almost periodic type functions, we would like to emphasize that the usually considered Doss almost periodic type functions (ρ is equal to the identity operator) do not have a linear vector structure, which can be very unpleasant for providing certain applications. It is also clear that a Doss almost periodic function $F : \mathbb{R}^n \to X$ need not have a mean value, which is also a very unpleasant property of Doss almost periodic functions.

Concerning some practical implications of our work, we would like to emphasize that the various types of Doss almost periodicity are invariant under the actions of the convolution products. This enables us to consider the existence and uniqueness of Doss almost periodic solutions for various classes of abstract Voleterra integro-differential equations and inclusions; the abstract semilinear Cauchy problems and inclusions can be also analyzed since we can formulate composition principles in our framework. It is also worth noting that the class of Doss p-almost periodic functions provides, in the theoretical sense, a unification concept for the class of Besicovitch p-almost periodic functions and the class of Weyl p-almost periodic functions ($1 \leq p < \infty$). In our further investigations, we will analyze the multi-dimensional analogues of conditions (A)–(B) and results established by R. Doss [26,27] as well as the class of multi-dimensional semi-ρ-periodic functions and certain classes of (equi-)Weyl-(p, ρ)-uniformly recurrent functions. It could be also of importance to analyze the multi-dimensional Hartman almost periodic functions, as well.

We close the paper with the observation that we can further extend the notion introduced in Definition 1 by allowing that the function $\mathrm{F}(t)$ depends not only on $t > 0$ but also on $\tau \in \Lambda'$. For example, we can consider the following notion (with the exception of assumption $\mathrm{F} : (0, \infty) \to (0, \infty)$, which is replaced by the assumption $\mathrm{F} : (0, \infty) \times \Lambda' \to (0, \infty)$ here, we retain all remaining standing assumptions of ours):

Definition 3. *We say that the function $F(\cdot; \cdot)$ is Doss-$(p, \phi, \mathrm{F}, \mathcal{B}, \Lambda', \rho)$-almost periodic if and only if, for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $l > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists a point $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that, for every $t > 0$, $x \in B$ and $\cdot \in \Lambda_t$ we have the existence of an element $y_{\cdot;x} \in \rho(F(\cdot; x))$ such that:*

$$\limsup_{t \to +\infty} \mathrm{F}(t, \lambda) \sup_{x \in B} \Big[\phi\big(\| F(\cdot + \tau; x) - y_{\cdot;x} \|_Y \big) \Big]_{L^{p(\cdot)}(\Lambda_t)} < \epsilon. \tag{26}$$

In actual fact, sometimes it is very important to assume that the function F depends also on $\tau \in \Lambda'$. We will illustrate this fact by considering the second-order partial differential equation: $\Delta u = -f$, where $f \in C^2(\mathbb{R}^3)$ has a compact support. Let us recall that the Newtonian potential of $f(\cdot)$, defined by:

$$u(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(x - y)}{|y|} \, dy, \quad x \in \mathbb{R}^3,$$

is a unique function belonging to the class $C^2(\mathbb{R}^3)$, vanishing at infinity and satisfying $\Delta u = -f$; see, for example, [31] (Theorem 3.9, pp. 126–127).

In our final application, we will assume that $\rho = c\mathrm{I}$ for some $c \in \mathbb{C}$, $p(\cdot) \equiv 1$ and $\phi(x) \equiv x$, as well as that $\emptyset \neq \Lambda' \subseteq \Lambda = \mathbb{R}^3$ and there exists a finite real number $d > 0$ such that:

$$\mathrm{F}_1(t, \tau) \geq d \left[F(t, \tau) + \sup_{\cdot \in \Lambda_t} \int_{(\cdot + \tau - K) \setminus (\cdot - K)} \frac{dy}{|y|} \right], \quad t > 0, \ \lambda \in \Lambda'. \tag{27}$$

Then, we have the following:

Theorem 4. *Suppose that f is Doss-$(1, \phi, \mathrm{F}, \Lambda', c)$-almost periodic and $supp(f) \subseteq K$. Then u is Doss-$(1, \phi, \mathrm{F}_1, \Lambda', c)$-almost periodic.*

Proof. Let $\epsilon > 0$ be given, and let $\tau \in \Lambda'$ be as in (26). Using the Fubini theorem, we have the existence of a finite real number $t_1(\epsilon, \tau) > 0$ such that:

$$
\begin{aligned}
&\left\|u(x+\tau)-cu(x)\right\|_{L^1(\Lambda_t)}\\
&\quad\leq \frac{1}{4\pi}\int_{\Lambda_t}\int_{\mathbb{R}^3}\frac{|f(x+\tau-y)-cf(x-y)|}{|y|}\,dy\,dx\\
&\quad=\frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\int_{\Lambda_t}|f(x+\tau-y)-cf(x-y)|\,dx\right]\frac{dy}{|y|}\\
&=\frac{1}{4\pi}\int_{x-K}\left[\int_{\Lambda_t}|f(x+\tau-y)-cf(x-y)|\,dx\right]\frac{dy}{|y|}\\
&\quad+\frac{1}{4\pi}\int_{(x+\tau-K)\setminus(x-K)}\left[\int_{\Lambda_t}|f(x+\tau-y)-cf(x-y)|\,dx\right]\frac{dy}{|y|}\\
&\quad\leq \frac{1}{4\pi}m(K)\frac{\epsilon}{F(t,\tau)}+\frac{1}{4\pi}\frac{\epsilon}{F(t,\tau)}\int_{(x+\tau-K)\setminus(x-K)}\int_{\Lambda_t}\frac{dy}{|y|},\quad t\geq t_1(\epsilon,\tau).
\end{aligned}
$$

Keeping in mind the assumption (27) and the notion introduced in Definition 3, this simply implies the required statement. □

We can similarly analyze the two-dimensional analogue of this example by considering the logarithmic potential of $f(\cdot)$, given by:

$$u(x) := \frac{(-1)}{2\pi}\int_{\mathbb{R}^2}\ln(|y|)\cdot f(x-y)\,dy,\quad x\in\mathbb{R}^2;$$

see also [31] (Remark 3.7, p. 128) and [7].

Author Contributions: Writing original draft, M.K., W.-S.D. and V.E.F. All authors have read and agreed to the published version of the manuscript.

Funding: The first author is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. The second author is partially supported by Grant No. MOST 110-2115-M-017-001 of the Ministry of Science and Technology of the Republic of China. The third author is partially supported by the Russian Foundation for Basic Research, grant number 21-51-54003.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to express their hearty thanks to the anonymous referees for their valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

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