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Perturbation Theory for Property (V_E) and Tensor Product

Elvis Aponte [1,*], José Sanabria [2] and Luis Vásquez [3]

[1] Escuela Superior Politécnica del Litoral (ESPOL), Facultad de Ciencias Naturales y Matemáticas, Departamento de Matemáticas, Campus Gustavo Galindo km. 30.5 Vía Perimetral, Guayaquil EC090112, Ecuador
[2] Departamento de Matemáticas, Facultad de Educación y Ciencias, Universidad de Sucre, Carrera 28 No. 5-267 Barrio Puerta Roja, Sincelejo 700001, Colombia; jesanabri@gmail.com
[3] Instituto Superior de Formación Docente Salomé Ureña-ISFODOSU, Recinto Emilio Prud'Homme, Calle R.C. Tolentino # 51, Esquina 16 de Agosto, Los Pepines, Santiago de los Caballeros 51000, Dominican Republic; luis.vasquez@isfodosu.edu.do
* Correspondence: ecaponte@espol.edu.ec

Abstract: Given a complex Banach space $\mathcal{X}$, we investigate the stable character of the property (V_E) for a bounded linear operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, under commuting perturbations that are Riesz, compact, algebraic and hereditarily polaroid. We also analyze sufficient conditions that allow the transfer of property (V_E) from the tensorial factors $\mathcal{T}$ and $\mathcal{S}$ to its tensor product.

Keywords: semi-Fredholm operator; property (V_E); commuting perturbations; tensor product

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1. Introduction

In 1900, E. Fredholm published his famous article *On a new method for the solution of Dirichlet's problem*, which changed the study of the solution of integral equations. This article served as inspiration for F. Riesz, in 1918, to establish Fredholm's abstract methods in the form of compact operators, thereby initiating what is now known as Fredholm theory for operators. In this theory, there are two classes of operators that play a fundamental role; these are the so-called Browder operators (also classically known as Riez–Schauder operators) and the Weyl operators, which have been the subject of a range of studies. In the last decades, numerous investigations have been developed on Fredholm theory, where some authors have introduced and studied several spectral properties similar to Weyl's theorem formulated by L. Coburn in [1]. The study of the spectra of the semi B-Fredholm and B-Weyl operators allowed M. Berkani and J. Koliha [2] to introduce two properties known as the generalized Weyl's and generalized a-Weyl's theorems, which are generalizations of the classical versions of the Weyl's and a-Weyl's theorems, respectively. Recently, other properties have been introduced and studied involving the different spectra of the Fredholm and B-Fredholm theories (started by M. Berkani), which together with the classical properties are known today as Weyl-type theorems. The stability of strong variations of Weyl-type theorems under direct sums and restrictions has been studied, as well as the transmission of spectral properties between a Drazin invertible operator and its Drazin inverse; for example, see [3,4]. In addition, the study of Weyl-type theorems under commuting perturbations has been considered by several authors, among which we can mention Oudghiri [5,6], Berkani et al. [7,8], Aiena and Triolo [9]. Elsewhere, the stability of Weyl's theorem under the tensor product has been studied by Kubrusly and Duggal in [10]. Subsequently, studies in this direction have been expanded by Duggal [11], Rashid [12] and Rashid and Prasad [13], involving new Weyl-type theorems. This article follows the same line of research as the works referenced above, but now we consider a strong variation of the Weyl-type theorems that was introduced by Sanabria et al. [3,14], namely property (V_E). According to [14], if an operator $\mathcal{T}$ satisfies property (V_E), then $\mathcal{T}$ satisfies equivalently another forty-four spectral properties, among which are Weyl-type theorems such as the properties (V_Π) and (gaz) recently studied in [15,16], respectively. This arouses the interest of studying property (V_E) from different points of view. In this paper, we focus our interest on obtaining conditions so that the property (V_E) remains stable under perturbations that are commutative and tensor products for some classes of operators.

2. Preliminaries

Let $\mathcal{L}(\mathcal{X}) = \{\mathcal{T} : \mathcal{X} \to \mathcal{X} | \mathcal{T}$ is a bounded linear operator$\}$, where $\mathcal{X}$ is a complex Banach space. For $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, let $\mathcal{T}^*$, $\mathcal{T}(\mathcal{X})$ and $N(\mathcal{T})$ be the dual operator, the range and the kernel of $\mathcal{T}$, respectively. We will use the following spectra of $\mathcal{T}$: $\sigma(\mathcal{T})$ (classical), $\sigma_a(\mathcal{T})$ (approximate point), $\sigma_s(\mathcal{T})$ (surjectivity), $\sigma_p(\mathcal{T})$ (point), $\sigma_e(\mathcal{T})$ (essential), $\sigma_{SF_+}(\mathcal{T})$ (upper semi-Fredholm), $\sigma_{SF_-}(\mathcal{T})$ (lower semi-Fredholm), $\sigma_b(\mathcal{T})$ (Browder), $\sigma_{ub}(\mathcal{T})$ (upper semi-Browder), $\sigma_{lb}(\mathcal{T})$ (lower semi-Browder), $\sigma_W(\mathcal{T})$ (Weyl), $\sigma_{SF_+^-}(\mathcal{T})$ (upper semi-Weyl), $\sigma_{SF_-^+}(\mathcal{T})$ (lower semi-Weyl), $\sigma_{BF}(\mathcal{T})$ (B-Fredholm), $\sigma_{SBF_+}(\mathcal{T})$ (upper semi B-Fredholm), $\sigma_{SBF_-}(\mathcal{T})$ (lower semi B-Fredholm), $\sigma_D(\mathcal{T})$ (Drazin invertible), $\sigma_{LD}(\mathcal{T})$ (left Drazin invertible), $\sigma_{RD}(\mathcal{T})$ (right Drazin invertible), $\sigma_{BW}(\mathcal{T})$ (B-Weyl) and $\sigma_{SBF_+^-}(\mathcal{T})$ (upper semi B-Weyl). See [17,18] for definitions and other details.

For $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, let $\beta(\mathcal{T})$ be the codimension of $\mathcal{T}(\mathcal{X})$, $\alpha(\mathcal{T})$ the dimension of $N(\mathcal{T})$, $p(\mathcal{T})$ the ascent of $\mathcal{T}$ and $q(\mathcal{T})$ the descent of $\mathcal{T}$. The resolvent set of $\mathcal{T}$ is denoted by $\rho(\mathcal{T})$ and the quasi-nilpotent part of $\mathcal{T}$ by $H_0(\mathcal{T}) := \{x \in \mathcal{X} : \lim_{n\to\infty} \|\mathcal{T}^n x\|^{1/n} = 0\}$. In addition, we put:

$$\Delta(\mathcal{T}) := \{n \in \mathbb{N} : \mathcal{T}^n(\mathcal{X}) \cap N(\mathcal{T}) \subseteq \mathcal{T}^m(X) \cap N(\mathcal{T}) \text{ if } m \geq n\}$$

and

$$\mathrm{dis}(\mathcal{T}) := \begin{cases} \inf \Delta(\mathcal{T}), & \text{if } \Delta(\mathcal{T}) \neq \emptyset \\ \infty, & \text{if } \Delta(\mathcal{T}) = \emptyset. \end{cases}$$

Definition 1. *An operator $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ is* quasi-Fredholm (of degree k) *if for some $k \in \mathbb{N}$:*

1. $\mathrm{dis}(\mathcal{T}) = k$,
2. *For any $n \geq k$, $\mathcal{T}^n(\mathcal{X})$ is a closed subspace of $\mathcal{X}$,*
3. *$\mathcal{T}(\mathcal{X}) + N(\mathcal{T}^k)$ is a closed subspace of $\mathcal{X}$.*

In [19], Finch introduced the following property. An operator $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ checks *the single valued extension property* at $\mu_0 \in \mathbb{C}$ (briefly, SVEP at μ_0), if for each open disc $\mathbb{D}_{\mu_0} \subseteq \mathbb{C}$ with center at μ_0 the unique analytic function $f : \mathbb{D}_{\mu_0} \to \mathcal{X}$, which satisfies the equation

$$(\mu I - \mathcal{T})f(\mu) = 0 \quad \text{for each } \mu \in \mathbb{D}_{\mu_0},$$

is $f \equiv 0$ on $\mathbb{D}_{\mu_0}$. We say that $\mathcal{T}$ satisfies SVEP if $\mathcal{T}$ satisfies SVEP at each point $\mu \in \mathbb{C}$. We put

$$\Sigma(\mathcal{X}, \mu) := \{\mathcal{T} \in \mathcal{L}(\mathcal{X}) : \mathcal{T} \text{ satisfies SVEP at } \mu\},$$

$$\Sigma(\mathcal{X}, \mathcal{A}) := \{\mathcal{T} \in \mathcal{L}(\mathcal{X}) : \mathcal{T} \text{ satisfies SVEP at each } \lambda \in \mathcal{A}\}$$

and

$$\Sigma(\mathcal{X}) := \{\mathcal{T} \in \mathcal{L}(\mathcal{X}) : \mathcal{T} \text{ satisfies SVEP}\}.$$

Obviously, $\mathcal{T} \in \Sigma(\mathcal{X}, \rho(\mathcal{T}))$. In addition, $\mathcal{T} \in \Sigma(\mathcal{X}, \mathrm{Fr}\,\sigma(\mathcal{T}))$, where $\mathrm{Fr}\,\sigma(\mathcal{T})$ is the frontier of $\sigma(\mathcal{T})$. Note that, $\mathcal{T} \in \Sigma(\mathcal{X}, \mu)$ and $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$ for all μ being an isolated point of the spectrum of $\mathcal{T}$. We also have

$$p(\mu I - \mathcal{T}) \text{ is finite } \Rightarrow \mathcal{T} \in \Sigma(\mathcal{X}, \mu) \tag{1}$$

and dually,

$$q(\mu I - \mathcal{T}) \text{ is finite } \Rightarrow \mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu), \tag{2}$$

see [17] (Theorem 3.8). Furthermore,

$$\mu \text{ is not a limit point of } \sigma_a(\mathcal{T}) \Rightarrow \mathcal{T} \in \Sigma(\mathcal{X}, \mu), \tag{3}$$

and dually,

$$\mu \text{ is not a limit point of } \sigma_s(\mathcal{T}) \Rightarrow \mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu). \tag{4}$$

Observe that, in general, $H_0(\mathcal{T})$ is not necessarily closed, and by [17] (Theorem 2.31),

$$H_0(\mu I - \mathcal{T}) \text{ is closed} \Rightarrow \mathcal{T} \in \Sigma(\mathcal{X}, \mu). \tag{5}$$

Remark 1. It is well known that if $\mu I - \mathcal{T}$ is quasi-Fredholm, the implications (1)–(5) become equivalences; in particular, this happens when $\mu I - \mathcal{T}$ is a semi B-Fredholm operator [20].

Let $\operatorname{Iso} \mathcal{D} := \{\mu \in \mathbb{C} | \mu \text{ is an isolated point of } \mathcal{D}\}$. For $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, we consider the following sets:

$$E^0(\mathcal{T}) := \{\mu \in \operatorname{Iso} \sigma(\mathcal{T}) : 0 < \alpha(\mu I - \mathcal{T}) < \infty\},$$
$$E(\mathcal{T}) := \{\mu \in \operatorname{Iso} \sigma(\mathcal{T}) : 0 < \alpha(\mu I - \mathcal{T})\}.$$

We also define

$$\Pi_a^0(\mathcal{T}) := \sigma_a(\mathcal{T}) \setminus \sigma_{ub}(\mathcal{T}), \quad \Pi_+^0(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_{ub}(\mathcal{T}),$$
$$\Pi^0(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_b(\mathcal{T}), \quad \Pi(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_D(\mathcal{T}).$$

According to [21], $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ has the *a-Browder's theorem* if $\sigma_a(\mathcal{T}) \setminus \sigma_{SF_+^-}(\mathcal{T}) = \Pi_a^0(\mathcal{T})$. Following [2], $\mathcal{T}$ has the *generalized Weyl's theorem* if $\sigma(\mathcal{T}) \setminus \sigma_{BW}(\mathcal{T}) = E(\mathcal{T})$. Following [22] (resp. [23]), $\mathcal{T}$ is said to satisfy *property* (w) (resp. *property* (gw)) if $\sigma_a(\mathcal{T}) \setminus \sigma_{SF_+^-}(\mathcal{T}) = E^0(\mathcal{T})$ (resp. $\sigma_a(\mathcal{T}) \setminus \sigma_{SBF_+^-}(\mathcal{T}) = E(\mathcal{T})$).

3. Perturbation Theory for Property (V_E)

For $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, put $\Delta_+(\mathcal{T}) := \sigma(\mathcal{T}) \setminus \sigma_{SF_+^-}(\mathcal{T})$. Following [14], $\mathcal{T}$ satisfies *property* (V_E) if $\Delta_+(\mathcal{T}) = E(\mathcal{T})$. Next, we establish several results related to property (V_E) for an operator $\mathcal{T}$ (resp. $\mathcal{T}^*$) satisfying SVEP at each point that does not belong to the lower (resp. upper) semi-Weyl spectrum of $\mathcal{T}$ and such that $\operatorname{Iso} \sigma_a(\mathcal{T}) = \emptyset$. Later, these results will be useful to analyze the stability of property (V_E) for certain perturbations. Let $V_E(\mathcal{X}) := \{\mathcal{T} \in \mathcal{L}(\mathcal{X}) : \mathcal{T} \text{ satisfies property } (V_E)\}$.

Theorem 1. *Let $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ with $\operatorname{Iso} \sigma_a(\mathcal{T}) = \emptyset$. If $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$ for each $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$, then $\mathcal{T} \in V_E(\mathcal{X})$.*

Proof. As $E(\mathcal{T}) = \emptyset$ whenever $\operatorname{Iso} \sigma_a(\mathcal{T}) = \emptyset$, it remains to show that $\sigma(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T})$. Now, if $\mu \in \sigma(\mathcal{T})$ and $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$, then $q(\mu I - \mathcal{T}) < \infty$ (since $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$) and as $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$, also $p(\mu I - \mathcal{T}) < \infty$. Hence, $\mu \in \operatorname{Iso} \sigma(\mathcal{T})$, so then $\mu \in \operatorname{Iso} \sigma_a(\mathcal{T})$, which is not possible. Thus, $\sigma(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T})$ and hence $\mathcal{T} \in V_E(\mathcal{X})$. □

Corollary 1. $\mathcal{T} \in V_E(\mathcal{X})$ *whenever* $\operatorname{Int}(\Delta^+(\mathcal{T})) = \operatorname{Iso} \sigma_a(\mathcal{T}) = \emptyset$.

Proof. Let $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$. If $\mu \notin \sigma(\mathcal{T})$, obviously $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$. If $\mu \in \sigma(\mathcal{T})$, then $\mu \in \Delta^+(\mathcal{T})$ and since the set of all upper semi-Weyl operators is open in $\mathcal{L}(\mathcal{X})$, from hypothesis $\operatorname{Int}(\Delta^+(\mathcal{T})) = \emptyset$ it follows that $\mu \in \operatorname{Fr} \sigma(\mathcal{T})$. Hence, $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$ again. Now, by Theorem 1, $\mathcal{T} \in V_E(\mathcal{X})$. □

The following example points out that the converse of the previous theorem is not true.

Example 1. Let $\mathcal{T}$ be the Volterra operator on $\mathbf{C}[0,1]$ given by $\mathcal{T}(g)(z) := \int_0^z g(w)dw$ for each $g \in \mathbf{C}[0,1]$. Observe that $\mathcal{T}$ is quasinilpotent and injective. So, $\sigma(\mathcal{T}) = \{0\}$, $\alpha(\mathcal{T}) = 0$ and hence $E(\mathcal{T}) = \emptyset$. As $R(\mathcal{T})$ is not closed, we have $\sigma_a(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T}) = \{0\}$ and $\sigma(\mathcal{T}) \setminus \sigma_{SF_+^-}(\mathcal{T}) = E(\mathcal{T})$, i.e., $\mathcal{T} \in V_E(\mathcal{X})$. However, $\operatorname{Iso}\sigma_a(\mathcal{T}) \neq \emptyset$. Note that $\mathcal{T}^* \in \Sigma(\mathcal{X}^*)$, because it is quasinilpotent.

Theorem 2. *If $\mathcal{T} \in \Sigma(\mathcal{X},\mu)$ for each $\mu \notin \sigma_{SF_-^+}(\mathcal{T})$ and $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, then $\mathcal{T}^* \in V_E(\mathcal{X}^*)$.*

Proof. Clearly, since $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, we have $E(\mathcal{T}^*) = \emptyset$. Assume that $\mu \in \sigma(\mathcal{T}^*)$ and $\mu \notin \sigma_{SF_+^-}(\mathcal{T}^*)$. According to this, $\mu \in \sigma(\mathcal{T})$ and $\mu \notin \sigma_{SF_-^+}(\mathcal{T})$. As $\mathcal{T} \in \Sigma(\mathcal{X},\mu)$, $p(\mu I - \mathcal{T}) < \infty$ and since $\mu \notin \sigma_{SF_-^+}(\mathcal{T})$, $q(\mu I - \mathcal{T}) < \infty$. Thus, $\mu \in \operatorname{Iso}\sigma(\mathcal{T})$ and therefore $\mu \in \operatorname{Iso}\sigma_a(\mathcal{T})$, contradicting that $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$. Therefore, $\sigma(\mathcal{T}^*) = \sigma_{SF_+^-}(\mathcal{T}^*)$ and we conclude that $\mathcal{T}^* \in V_E(\mathcal{X}^*)$. □

Corollary 2. *For $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, we have:*

1. *If $\mathcal{T} \in \Sigma(\mathcal{X})$, then $\mathcal{T}^* \in V_E(\mathcal{X}^*)$.*
2. *If $\mathcal{T}^* \in \Sigma(\mathcal{X}^*)$, then $\mathcal{T} \in V_E(\mathcal{X})$.*

Corollary 3. *If $\mathcal{T} \in \Sigma(\mathcal{X},\mu)$ for each $\mu \notin \sigma_{SF_+}(\mathcal{T})$ and $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, then we have the following equalities: $\sigma_{SF_+}(\mathcal{T}) = \sigma_e(\mathcal{T}) = \sigma_W(\mathcal{T}) = \sigma_{SF_-^+}(\mathcal{T}) = \sigma_{ub}(\mathcal{T}) = \sigma_b(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma_a(\mathcal{T}) = \sigma_D(\mathcal{T}) = \sigma_{SBF_+}(\mathcal{T}) = \sigma_{BF}(\mathcal{T}) = \sigma_{SBF_+^-}(\mathcal{T}) = \sigma_{BW}(\mathcal{T}) = \sigma_{LD}(\mathcal{T}) \overset{*}{=} \sigma_{RD}(\mathcal{T}) = \sigma_{lb}(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T}) = \sigma_{SF_-}(\mathcal{T})$.*

Proof. The equalities before $\overset{*}{=}$ are followed by [24] (Corollary 2.18). By hypothesis and Theorem 2, $\mathcal{T}^* \in V_E(\mathcal{X}^*)$, so from [24] (Theorem 2.10), $\sigma_{SF_-}(\mathcal{T}) = \sigma_e(\mathcal{T})$. Hence, we deduce that equalities after $\overset{*}{=}$ are valid (see [14] (Theorem 2.27)). □

The exploration of the perturbations is very important in the spectral theory of the linear operators, because through them is studied the behavior of the spectral properties when the operators undergo a small change. This topic has occupied a place in applied mathematics, and over time has evolved into a self-interested mathematical discipline. An outstanding aspect of conducting studies of operators under commuting perturbations is that these could be used in harmonic analysis; for example, concerning the Wiener–Pitt phenomenon. In what follows, we mainly analyze the stable character of property (V_E) through a perturbation that commutes with the operator and is of finite range (resp. Riesz, compact, algebraic). We say that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ is *isoloid* if each $\mu \in \operatorname{Iso}\sigma(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$; while $\mathcal{T}$ is called *finitely isoloid* if each $\mu \in \operatorname{Iso}\sigma(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ with finite multiplicity.

Theorem 3. *If $\mathcal{T} \in V_E(\mathcal{X})$ is isoloid and $\mathcal{F}$ is a finite rank operator such that $\mathcal{T}\mathcal{F} = \mathcal{F}\mathcal{T}$, then $\mathcal{T} + \mathcal{F} \in V_E(\mathcal{X})$.*

Proof. By hypothesis and [14] (Theorem 2.8), $\mathcal{T}$ has the generalized Weyl's theorem and $\sigma_{SF_+^-}(\mathcal{T}) = \sigma_{BW}(\mathcal{T})$. Since $\mathcal{T}$ is isoloid, by [25] (Theorem 3.4), $\mathcal{T} + \mathcal{F}$ has the generalized Weyl's theorem. Moreover, as $\mathcal{F}$ is of finite rank, by [17] (Theorem 3.39), $\sigma_{SF_+^-}(\mathcal{T} + \mathcal{F}) = \sigma_{SF_+^-}(\mathcal{T})$, and by [26] (Theorem 3.2), we get that $\sigma_{BW}(\mathcal{T}) = \sigma_{BW}(\mathcal{T} + \mathcal{F})$. Thus, $\sigma_{SF_+^-}(\mathcal{T} + \mathcal{F}) = \sigma_{BW}(\mathcal{T} + \mathcal{F})$, and again, by [14] (Theorem 2.8), we deduce that $\mathcal{T} + \mathcal{F} \in V_E(\mathcal{X})$. □

Corollary 4. *If $\mathcal{T} \in V_E(\mathcal{X})$ is quasi-nilpotent, which has 0 as an eigenvalue, and $\mathcal{F}$ is of finite rank such that $\mathcal{T}\mathcal{F} = \mathcal{F}\mathcal{T}$, then $\mathcal{T} + \mathcal{F} \in V_E(\mathcal{X})$.*

Proof. From hypothesis $\mathcal{T}$ is isoloid, so the proof is completed using Theorem 3. □

According to [27] (Theorem 7), $\mathcal{R} \in \mathcal{L}(\mathcal{X})$ satisfies $\sigma_{ub}(\mathcal{T}+\mathcal{R}) = \sigma_{ub}(\mathcal{T})$ for each $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ such that $\mathcal{R}\mathcal{T} = \mathcal{T}\mathcal{R}$ if and only if $\mathcal{R}$ is a Riesz operator. In addition, $\sigma_b(\mathcal{T}+\mathcal{R}) = \sigma_b(\mathcal{T})$ by [27] (Corollary 7). In the case that $\mathcal{T} \in V_E(\mathcal{X})$, $\sigma_{ub}(\mathcal{T}) = \sigma_b(\mathcal{T})$ and $\sigma_{ub}(\mathcal{T}+\mathcal{R}) = \sigma_b(\mathcal{T}+\mathcal{R})$. In particular, these results hold for finite rank operators.

Theorem 4. *Let $\mathcal{T} \in V_E(\mathcal{X})$ and $\mathcal{F}$ be of finite rank such that $\mathcal{T}\mathcal{F} = \mathcal{F}\mathcal{T}$. The following are equivalent:*

1. $\mathcal{T}+\mathcal{F} \in V_E(\mathcal{X})$.
2. $E(\mathcal{T}+\mathcal{F}) = \Pi^0_+(\mathcal{T}+\mathcal{F})$.
3. $E(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T}) \subseteq \Pi^0_+(\mathcal{T})$.

Proof. (1)⇔(2) Since $\mathcal{T} \in V_E(\mathcal{X})$ if and only if $E(\mathcal{T}) = \Pi^0_+(\mathcal{T})$ and $\sigma_{SF_+^-}(\mathcal{T}) = \sigma_{ub}(\mathcal{T})$ (see [14] (Theorem 2.23)), the proof is completed using the fact that for finite rank operators, $\sigma_{SF_+^-}(\mathcal{T}+\mathcal{F}) = \sigma_{SF_+^-}(\mathcal{T})$, see [17] (Theorem 3.39).

(2)⇔(3) Assume that $E(\mathcal{T}+\mathcal{F}) = \Pi^0_+(\mathcal{T}+\mathcal{F})$. If $\mu \in E(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T})$, then $\mu \in \Pi^0_+(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T})$, whereby $\mu \notin \sigma_{ub}(\mathcal{T}+\mathcal{F})$. However, we have $\sigma_{ub}(\mathcal{T}+\mathcal{F}) = \sigma_{ub}(\mathcal{T})$, so $\mu \in \Pi^0_+(\mathcal{T})$. Therefore, $E(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T}) \subseteq \Pi^0_+(\mathcal{T})$. Reciprocally, since $\mathcal{T} \in V_E(\mathcal{X})$, $\sigma_{ub}(\mathcal{T}+\mathcal{F}) = \sigma_b(\mathcal{T}+\mathcal{F})$ and hence, $\Pi^0_+(\mathcal{T}+\mathcal{F}) = \Pi^0(\mathcal{T}+\mathcal{F}) \subseteq E(\mathcal{T}+\mathcal{F})$. Thus, it remains to show that $E(\mathcal{T}+\mathcal{F}) \subseteq \Pi^0_+(\mathcal{T}+\mathcal{F})$. If $\mu \in E(\mathcal{T}+\mathcal{F})$, then $\mu \in \sigma(\mathcal{T}+\mathcal{F})$. First, we note that $\mathcal{F}$ is Riesz, and this way $\sigma_{ub}(\mathcal{T}) = \sigma_{ub}(\mathcal{T}+\mathcal{F})$. Now, we consider two cases.

Case 1: $\mu \notin \sigma(\mathcal{T})$.

Case 2: $\mu \in \sigma(\mathcal{T})$.

For Case 1, obviously $\mu \notin \sigma_{ub}(\mathcal{T}) = \sigma_{ub}(\mathcal{T}+\mathcal{F})$, whereby $\mu \in \Pi^0_+(\mathcal{T}+\mathcal{F})$. For Case 2, we have $\mu \in E(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T}) \subseteq \Pi^0_+(\mathcal{T})$ and so, $\mu \notin \sigma_{ub}(\mathcal{T}) = \sigma_{ub}(\mathcal{T}+\mathcal{F})$, which implies that $\mu \in \Pi^0_+(\mathcal{T}+\mathcal{F})$ again. Thus, by both cases, if $\mu \in E(\mathcal{T}+\mathcal{F})$ then $\mu \in \Pi^0_+(\mathcal{T}+\mathcal{F})$ and hence, we deduce that $E(\mathcal{T}+\mathcal{F}) = \Pi^0_+(\mathcal{T}+\mathcal{F})$. □

Remark 2. The equivalence (1)⇔(2) of Theorem 4 holds if we replace $\mathcal{F}$ by $\mathcal{K} \in \mathcal{L}(\mathcal{X})$ being compact and commuting with $\mathcal{T}$.

Corollary 5. *Let $\mathcal{T} \in \Sigma(\mathcal{X}, \mu)$ for each $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$ and $\mathcal{F}$ be of finite rank commuting with $\mathcal{T}$. Then, $\mathcal{T} \in V_E(\mathcal{X})$ is equivalent to $\mathcal{T}+\mathcal{F} \in V_E(\mathcal{X})$.*

Proof. Since $\mathcal{T} \in V_E(\mathcal{X})$, $E(\mathcal{T}) = \Pi^0_+(\mathcal{T})$, thus $E(\mathcal{T}) \cap \sigma(\mathcal{T}) \subseteq \Pi^+_0(\mathcal{T})$. By [28] (Lemma 2.1), $\mu \in \operatorname{Iso}\sigma(\mathcal{T}) \Leftrightarrow \mu \in \operatorname{Iso}\sigma(\mathcal{T}+\mathcal{F})$, which implies that $E(\mathcal{T}) = E(\mathcal{T}+\mathcal{F})$. Therefore, $E(\mathcal{T}+\mathcal{F}) \cap \sigma(\mathcal{T}) \subseteq \Pi^0_+(\mathcal{T})$, and by Theorem 4, $\mathcal{T}+\mathcal{F} \in V_E(\mathcal{X})$. Reciprocally, assume that $\mathcal{T}+\mathcal{F} \in V_E(\mathcal{X})$. So, using symmetry, we have $\mathcal{T} = (\mathcal{T}+\mathcal{F}) - \mathcal{F} \in V_E(\mathcal{X})$. □

It is well known that if $\mathcal{N}$ is a nilpotent operator that commutes with $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, we have $\sigma(\mathcal{T}+\mathcal{N}) = \sigma(\mathcal{T})$ and $E(\mathcal{T}+\mathcal{N}) = E(\mathcal{T})$, see [29]. According to this, we establish the next result.

Theorem 5. *Assume that $\mathcal{N} \in \mathcal{L}(\mathcal{X})$ is nilpotent and commutes with $\mathcal{T}$. Then, $\mathcal{T} \in V_E(\mathcal{X})$ is equivalent to $\mathcal{T}+\mathcal{N} \in V_E(\mathcal{X})$.*

Proof. Assume that $\mathcal{T} \in V_E(\mathcal{X})$. By [30] (Theorem 2.13), $\sigma_{SF_+^-}(\mathcal{T}+\mathcal{N}) = \sigma_{SF_+^-}(\mathcal{T})$. Since $\sigma(\mathcal{T}+\mathcal{N}) = \sigma(\mathcal{T})$ and $E(\mathcal{T}+\mathcal{N}) = E(\mathcal{T})$, then $\sigma(\mathcal{T}+\mathcal{N}) \setminus \sigma_{SF_+^-}(\mathcal{T}+\mathcal{N}) = E(\mathcal{T}+\mathcal{N})$ and hence, $\mathcal{T}+\mathcal{N} \in V_E(\mathcal{X})$. The converse is obtained by symmetry. □

In the following example we show that the hypothesis of commutativity cannot be omitted from Theorem 5.

Example 2. Let $\mathcal{T},\mathcal{N} \in \mathcal{L}(\ell^2(\mathbb{N}))$ be defined as

$$\mathcal{T}(a_1,a_2,\dots) = \left(0,\frac{a_1}{2},\frac{a_2}{3},\dots\right) \text{ and } \mathcal{N}(a_1,a_2,\dots) = \left(0,\frac{-a_1}{2},0,0,\dots\right).$$

Obviously $\mathcal{N}$ is nilpotent and $\mathcal{N}\mathcal{T} \neq \mathcal{T}\mathcal{N}$. As $\sigma(\mathcal{T}) = \{0\} = \sigma_{SF_+^-}(\mathcal{T})$ and $E(\mathcal{T}) = \emptyset$, we have $\mathcal{T} \in V_E(\mathcal{X})$. However, $\sigma(\mathcal{T}+\mathcal{N}) = \sigma_{SF_+^-}(\mathcal{T}+\mathcal{N}) = \{0\}$ and $E(\mathcal{T}+\mathcal{N}) = \{0\}$, whereby $\mathcal{T}+\mathcal{N} \notin V_E(\mathcal{X})$.

Corollary 6. *Let $\mathcal{F} \in \mathcal{L}(\mathcal{X})$ be of finite rank and commuting with a quasi-nilpotent operator $\mathcal{T}$ such that $0 \notin \sigma_p(\mathcal{T})$. Then, $\mathcal{T} \in V_E(\mathcal{X})$ is equivalent to $\mathcal{T}+\mathcal{F} \in V_E(\mathcal{X})$.*

Proof. The hypothesis about $\mathcal{T}$ and $\mathcal{F}$ implies that $\mathcal{F}$ is nilpotent. Indeed, $\mathcal{T}$ is injective because $0 \notin \sigma_p(\mathcal{T})$. As $\mathcal{F}\mathcal{T} = \mathcal{T}\mathcal{F}$ and $\mathcal{T}$ is quasi-nilpotent, $\mathcal{T}\mathcal{F}$ is quasi-nilpotent and of finite rank. Thus, $\mathcal{T}\mathcal{F}$ is nilpotent, and as $\mathcal{T}$ is injective, we have that $\mathcal{F}$ is nilpotent. Therefore, the proof is completed using Theorem 5. □

The stable character of property (V_E) seen in Theorem 5 does not hold for compact or quasi-nilpotent operators.

Example 3. Let us consider the operators $\mathcal{T}$ and $\mathcal{K}$ on $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ given by

$$\mathcal{T} = 0 \oplus S \quad \text{and} \quad \mathcal{K} = \mathcal{S} \oplus 0,$$

with $\mathcal{S}$ defined on $\ell^2(\mathbb{N})$ as $S(a_1,a_2,\dots) = \left(\frac{a_2}{2},\frac{a_3}{3},\dots\right)$. Note that $\mathcal{K}$ is a compact quasi-nilpotent operator and $\mathcal{T}\mathcal{K} = \mathcal{K}\mathcal{T} = 0$. On the other hand, $\mathcal{T} \in V_E(\mathcal{X})$, because $\sigma(\mathcal{T}) = \{0\} = \sigma_{SF_+^-}(\mathcal{T})$ and $E(\mathcal{T}) = \emptyset$. However, $\mathcal{T}+\mathcal{K} = S \oplus \mathcal{S} \notin V_E(\mathcal{X})$, because $\sigma(\mathcal{T}+\mathcal{K}) = \sigma_{SF_+^-}(\mathcal{T}+\mathcal{K}) = E(\mathcal{T}+\mathcal{K}) = \{0\}$.

Theorem 6. *If $\mathcal{Q} \in \mathcal{L}(\mathcal{X})$ is quasi-nilpotent and commutes with the operator $\mathcal{T}$ such that $\operatorname{Int}(\Delta_+(\mathcal{T})) = \operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, then $\mathcal{T}+\mathcal{Q} \in V_E(\mathcal{X})$.*

Proof. By [31] (Corollary 3.24), $\sigma_a(\mathcal{T}+\mathcal{Q}) = \sigma_a(\mathcal{T})$ and $\sigma(\mathcal{T}+\mathcal{Q}) = \sigma(\mathcal{T})$. Since $\mathcal{Q}$ is quasi-nilpotent, it follows that $\mathcal{Q}$ is of Riesz, and from [31] (Corollary 3.18), we get that $\sigma_{SF_+^-}(\mathcal{T}+\mathcal{Q}) = \sigma_{SF_+^-}(\mathcal{T})$. Thus, $\operatorname{Int}(\Delta_+(\mathcal{T}+\mathcal{Q})) = \operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{Q}) = \emptyset$ and by Corollary 1, $\mathcal{T}+\mathcal{Q} \in V_E(\mathcal{X})$. □

Corollary 7. *Let $\mathcal{S} \in \mathcal{L}(\mathcal{X})$ commute with $\mathcal{T}$ and suppose that there exists $k \in \mathbb{N}$ such that $\mathcal{S}^k$ is an operator of finite rank. If $\operatorname{Int}(\Delta_+(\mathcal{T})) = \operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$, then $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$.*

Proof. We have $\sigma_a(\mathcal{T}+\mathcal{S}) = \sigma_a(\mathcal{T})$ by [31] (Lemma 5.106), and $\sigma(\mathcal{T}+\mathcal{S}) = \sigma(\mathcal{T})$ by [31] (Theorem 3.27). Since $\mathcal{S}$ is a Riesz operator, the remainder of the proof follows as the proof of Theorem 6. □

The proof of the following theorem is obtained using the stability of $\sigma_{SF_+^-}(\mathcal{T})$ under Riesz commuting perturbations, see [31] (Corollary 3.18).

Theorem 7. *Let $\mathcal{T} \in V_E(\mathcal{X})$ be finitely isoloid and let $\mathcal{S}$ be a Riesz operator such that $\mathcal{T}\mathcal{S} = \mathcal{S}\mathcal{T}$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}+\mathcal{S})$. Then, $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$ is equivalent to $E(\mathcal{T}) = E(\mathcal{T}+\mathcal{S})$.*

Theorem 8. *Let $\mathcal{T} \in V_E(\mathcal{X})$ and let $\mathcal{S}$ be a Riesz operator such that $\mathcal{T}\mathcal{S} = \mathcal{S}\mathcal{T}$. Then, $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$ is equivalent to $E(\mathcal{T}+\mathcal{S}) = \Pi_+^0(\mathcal{T}+\mathcal{S})$.*

Proof. If $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$, then $E(\mathcal{T}+\mathcal{S}) = \Pi_+^0(\mathcal{T}+\mathcal{S})$. For the converse, suppose that $E(\mathcal{T}+\mathcal{S}) = \Pi_+^0(\mathcal{T}+\mathcal{S})$. As $\mathcal{T} \in V_E(\mathcal{X})$, it has the a-Browder's theorem, so from [5] (Corollary 2.3), $\mathcal{T}+\mathcal{S}$ has the a-Browder's theorem. Consequently, $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$. □

Theorem 9. *If $\mathcal{T} \in V_E(\mathcal{X})$ is isoloid and $\mathcal{S}$ is a Riesz operator such that $\mathcal{TS} = \mathcal{ST}$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}+\mathcal{S})$, then $\mathcal{T}+\mathcal{S} \in V_E(\mathcal{X})$.*

Proof. If $\mathcal{S}$ is Riesz, then from [31] (Corollary 3.18), we have $\sigma_{SF_+^-}(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S})$. Let $\mu \in E(\mathcal{T})$. As $\mathcal{T} \in V_E(\mathcal{X})$, $\mu \notin \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S})$ and so, $(\mu I - \mathcal{T} - \mathcal{S})(\mathcal{X})$ is closed. We also have $\mu \in \operatorname{Iso}\sigma(\mathcal{T}) = \operatorname{Iso}\sigma(\mathcal{T}+\mathcal{S}) \subseteq \operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{S})$, whereby $\alpha(\mu I - \mathcal{T} - \mathcal{S}) > 0$, so $E(\mathcal{T}) \subseteq E(\mathcal{T}+\mathcal{S})$ and hence $\sigma(\mathcal{T}+\mathcal{S}) \setminus \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S}) \subseteq E(\mathcal{T}+\mathcal{S})$, because $\sigma(\mathcal{T}) = \sigma(\mathcal{T}+\mathcal{S})$. For the other inclusion, let $\mu \in E(\mathcal{T}+\mathcal{S})$. Then $\mu \in \operatorname{Iso}\sigma(\mathcal{T}+\mathcal{S}) = \operatorname{Iso}\sigma(\mathcal{T})$ and as $\mathcal{T}$ is isoloid, $\alpha(\mu I - \mathcal{T}) > 0$ and $\mu \in E(\mathcal{T})$. Consequently, $\mu \notin \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S})$ and hence, $\mu \in \sigma(\mathcal{T}+\mathcal{S}) \setminus \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S})$. Thus, $E(\mathcal{T}+\mathcal{S}) \subseteq \sigma(\mathcal{T}+\mathcal{S}) \setminus \sigma_{SF_+^-}(\mathcal{T}+\mathcal{S})$. □

Recall that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ is *algebraic* [31] (Section 3.5) if $p(\mathcal{T}) = 0$ for some complex nontrivial polynomial p. Obviously, each nilpotent operator is algebraic. According to [31] (Theorem 3.72), if $\mathcal{T}$ is an algebraic operator and $\alpha(p(\mathcal{T})) < \infty$ for each polynomial p, then there exists $k \in \mathbb{N}$ such that $\mathcal{T}^k$ has finite rank and hence, $\mathcal{T}$ is Riesz. In addition, $\mathcal{T}$ being algebraic is equivalent to $\mathcal{T}^*$ being algebraic. Given $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ and an open subset $\mathcal{O}$ of $\mathbb{C}$, we put

$$\mathcal{H}(\sigma(\mathcal{T})) := \{f : \mathcal{O} \to \mathbb{C} \mid f \text{ is a analytic function and } \sigma(\mathcal{T}) \subset \mathcal{O}\}.$$

Theorem 10. *Suppose that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, $\mathcal{S}$ is algebraic such that $\mathcal{ST} = \mathcal{TS}$ and $f \in \mathcal{H}(\sigma(\mathcal{T}+\mathcal{S}))$. Then:*

1. *If $\mathcal{T}^* \in \Sigma(\mathcal{X}^*)$ and $\operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{S}) = \emptyset$, then $f(\mathcal{T}+\mathcal{S}) \in V_E(\mathcal{X})$.*
2. *If $\mathcal{T} \in \Sigma(\mathcal{X})$ and $\operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{S}) = \emptyset$, then $f(\mathcal{T}^*+\mathcal{S}^*) \in V_E(\mathcal{X}^*)$.*

Proof. (1) Suppose that $\mathcal{S}$ is an algebraic operator. Then, $\mathcal{S}^*$ is algebraic and since $\mathcal{T}^* \in \Sigma(\mathcal{X}^*)$, by [32] (Theorem 2.3) it follows that $\mathcal{T}^*+\mathcal{S}^* = (\mathcal{T}+\mathcal{S})^* \in \Sigma(\mathcal{X}^*)$. Thus, by [17] (Theorem 2.40), we have $f(\mathcal{T}^*+\mathcal{S}^*) \in \Sigma(\mathcal{X}^*)$, and as $\operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{S}) = \emptyset$, by Corollary 2, we get that $f(\mathcal{T}+\mathcal{S}) \in V_E(\mathcal{X})$.

(2) Can be proved similarly to (1). □

Theorem 11. *Let $\mathcal{T} \in \Sigma(\mathcal{X})$ and $f \in \mathcal{H}(\sigma(\mathcal{T}))$. Then:*

1. *If $\operatorname{Iso}\sigma_a(\mathcal{T}) = \emptyset$ and $\mathcal{Q}$ is quasi-nilpotent such that $\mathcal{QT} = \mathcal{TQ}$, then both $f(\mathcal{T})^* + \mathcal{Q}^*$ and $f(\mathcal{T}^*+\mathcal{Q}^*)$ belong to $V_E(\mathcal{X}^*)$.*
2. *If $\operatorname{Iso}\sigma_a(f(\mathcal{T})+\mathcal{S}) = \emptyset$ and $\mathcal{S}$ is algebraic (or Riesz) such that $\mathcal{ST} = \mathcal{TS}$, then $f(\mathcal{T})^* + \mathcal{S}^*$ belongs to $V_E(\mathcal{X}^*)$.*

Proof. (1) If $\mathcal{T} \in \Sigma(\mathcal{X})$, then $f(\mathcal{T}) \in \Sigma(\mathcal{X})$, by [17] (Theorem 2.40). Since $\mathcal{Q}$ is quasi-nilpotent and commutes with $\mathcal{T}$, from [17] (Corollary 2.12), we have that both $\mathcal{T}+\mathcal{Q}$ and $f(\mathcal{T})+\mathcal{Q}$ belong to $\Sigma(\mathcal{X})$. By [31] (Corollary 3.24), $\sigma(\mathcal{T}+\mathcal{Q}) = \sigma(\mathcal{T})$ and so $f(\mathcal{T}+\mathcal{Q}) \in \Sigma(\mathcal{X})$. Observe that $\operatorname{Iso}\sigma_a(f(\mathcal{T})) = \emptyset$. Again, by using [31] (Corollary 3.24), we have $\sigma_a(\mathcal{T}+\mathcal{Q}) = \sigma_a(\mathcal{T})$ and $\sigma_a(f(\mathcal{T})+\mathcal{Q}) = \sigma_a(f(\mathcal{T}))$, which implies that $\operatorname{Iso}\sigma_a(\mathcal{T}+\mathcal{Q}) = \emptyset$ and hence, $\operatorname{Iso}\sigma_a(f(\mathcal{T}+\mathcal{Q})) = \operatorname{Iso}\sigma_a(f(\mathcal{T})+\mathcal{Q}) = \emptyset$. By Corollary 2, we conclude that both $f(\mathcal{T})^*+\mathcal{Q}^*$ and $f(\mathcal{T}^*+\mathcal{Q}^*)$ belong to $V_E(\mathcal{X}^*)$.

(2) Since $\mathcal{S}$ is algebraic (resp. Riesz) commuting with $\mathcal{T}$ and $f(\mathcal{T}) \in \Sigma(\mathcal{X})$, by [33] (Theorem 2.14) (resp. [31] (Theorem 2.129)) we get that $f(\mathcal{T})+\mathcal{S}$ belongs to $\Sigma(\mathcal{X})$. Thus, by Corollary 2, $f(\mathcal{T})^*+\mathcal{S}^*$ belongs to $V_E(\mathcal{X}^*)$. □

We say that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ is called *polaroid* if $\operatorname{Iso}\sigma(\mathcal{T}) = \Pi(\mathcal{T})$; while $\mathcal{T}$ is called *hereditary polaroid* if each part of $\mathcal{T}$ is polaroid, where a part of $\mathcal{T}$ means the restriction of $\mathcal{T}$ to a closed $\mathcal{T}$-invariant subspace. Let $\mathcal{H}_{nc}(\sigma(\mathcal{T})) := \{f \in \mathcal{H}(\sigma(\mathcal{T})) :\ f$ is non-constant on each component of its domain.

Theorem 12. *Suppose that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, $\mathcal{S}$ is algebraic commuting with $\mathcal{T}$ and $f \in \mathcal{H}_{nc}(\sigma(\mathcal{T} + \mathcal{K}))$. If $\mathcal{T} + \mathcal{K}$ is finitely isoloid, then we have:*

1. *If $\mathcal{T}^*$ is hereditarily polaroid, then $f(\mathcal{T} + \mathcal{K}) \in V_E(\mathcal{X})$.*
2. *If $\mathcal{T}$ is hereditarily polaroid, then $f(\mathcal{T}^* + \mathcal{K}^*) \in V_E(\mathcal{X}^*)$.*

Proof. (1) Since $\mathcal{T}^*$ is hereditarily polaroid, $\mathcal{T}^* \in \Sigma(\mathcal{X}^*)$ by [31] (Theorem 4.31), and as $\mathcal{K}^*$ is algebraic, by [32] (Theorem 2.3), we get that $\mathcal{T}^* + \mathcal{K}^* = (\mathcal{T} + \mathcal{K})^* \in \Sigma(\mathcal{X}^*)$. In addition, $\mathcal{T}^*$ is polaroid, which is equivalent to saying that $\mathcal{T}$ is polaroid, which implies, by [31] (Theorem 4.24), that $\mathcal{T} + \mathcal{K}$ is polaroid, or equivalently, $f(\mathcal{T} + \mathcal{K})$ is polaroid, by [31] (Theorem 4.19). Now, $\mathcal{T} + \mathcal{K}$ polaroid and $(\mathcal{T} + \mathcal{K})^* \in \Sigma(\mathcal{X}^*)$ entails that $f(\mathcal{T} + \mathcal{K})$ satisties properties (w) and (gw), by [34] (Theorem 3.12). Since $\mathcal{T} + \mathcal{K}$ is finitely isoloid and polaroid, $\sigma_{LD}(\mathcal{T} + \mathcal{K}) = \sigma_b(\mathcal{T} + \mathcal{K})$ and hence, $\sigma_{LD}(f(\mathcal{T} + \mathcal{K})) = f(\sigma_{LD}(\mathcal{T} + \mathcal{K})) = f(\sigma_b(\mathcal{T} + \mathcal{K})) = \sigma_b(f(\mathcal{T} + \mathcal{K}))$. However, $f(\mathcal{T} + \mathcal{K})$ polaroid implies, by [15] (Theorem 4.12), that $f(\mathcal{T} + \mathcal{K})$ satisfies property (V_{Π}), or equivalently, $f(\mathcal{T} + \mathcal{K}) \in V_E(\mathcal{X})$, by [15] (Theorem 4.5).

(2) This is proved similar to (1). □

Following the proof of [32] (Theorem 2.3) we can get:

Lemma 1. *Let $\mathcal{S}, \mathcal{T} \in \mathcal{L}(\mathcal{X})$ be such that $\mathcal{T}\mathcal{S} = \mathcal{S}\mathcal{T}$. If $\mathcal{S}$ is algebraic, we have:*

1. *If $\mu \in \sigma(\mathcal{S})$ and $\mathcal{T}^* \in \Sigma(\mathcal{X}^*, \mu)$, then $\mathcal{T}^* + \mathcal{S}^* \in \Sigma(\mathcal{X}^*, \mu)$.*
2. *If $\mu \in \sigma(\mathcal{S})$ and $\mathcal{T} \in \Sigma(\mathcal{X}, \mu)$, then $\mathcal{T} + \mathcal{S} \in \Sigma(\mathcal{X}, \mu)$.*

Theorem 13. *Suppose that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, $\mathcal{S}$ is algebraic such that $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S}$ and $\sigma_{SF_+^-}(\mathcal{T}) \bigcap \sigma(\mathcal{S}) = \emptyset$. If $\mathcal{T} \in V_E(\mathcal{X})$ with $\mathrm{Iso}\,\sigma_a(\mathcal{T} + \mathcal{S}) = \emptyset$ and $\sigma_{SF_+^-}(\mathcal{T}) = \sigma_{SF_+^-}(\mathcal{T} + \mathcal{S})$, then $\mathcal{T} + \mathcal{S} \in V_E(\mathcal{X})$.*

Proof. If $\mathcal{T} \in V_E(\mathcal{X})$, then $\mathcal{T}^*$ satisfies SVEP at $\mu \notin \sigma_{SF_+^-}(\mathcal{T})$. Since $\mathcal{S}$ is algebraic and $\sigma_{SF_+^-}(\mathcal{T}) \bigcap \sigma(\mathcal{S}) = \emptyset$, from Lemma 1 it follows that $\mathcal{T}^* + \mathcal{S}^* = (\mathcal{T} + \mathcal{S})^*$ satisfies SVEP at $\mu \notin \sigma_{SF_+^-}(\mathcal{T} + \mathcal{S})$. Thus, by Theorem 1, we conclude that $\mathcal{T} + \mathcal{S}$ satisfies property (V_E). □

Theorem 14. *Suppose that $\mathcal{T} \in \mathcal{L}(\mathcal{X})$, $\mathcal{S}$ is algebraic such that $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S}$ and $\sigma_{SF_-^+}(\mathcal{T}) \bigcap \sigma(\mathcal{S}) = \emptyset$. If $\mathcal{T}^* \in V_E(\mathcal{X}^*)$ with $\mathrm{Iso}\,\sigma_a(\mathcal{T} + \mathcal{S}) = \emptyset$ and $\sigma_{SF_-^+}(\mathcal{T}) = \sigma_{SF_-^+}(\mathcal{T} + \mathcal{S})$, then $\mathcal{T} + \mathcal{S} \in V_E(\mathcal{X})$.*

4. Property (V_E) under Tensor Products

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces and $\mathcal{X} \otimes \mathcal{Y}$ be the algebraic completion (in some reasonable uniform cross norm) of the tensor product of $\mathcal{X}$ and $\mathcal{Y}$. The tensor product of $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ and $\mathcal{S} \in \mathcal{L}(\mathcal{Y})$ on $\mathcal{X} \otimes \mathcal{Y}$ is the operator defined as $(\mathcal{T} \otimes \mathcal{S})(\sum_i x_i \otimes y_i) = \sum_i \mathcal{T}x_i \otimes \mathcal{S}y_i$ for each $\sum_i x_i \otimes y_i \in \mathcal{X} \otimes \mathcal{Y}$. In this section, we analyze some conditions that allow property (V_E) to be transmitted from the tensor factors $\mathcal{T}$ and $\mathcal{S}$ to the tensor product $\mathcal{T} \otimes \mathcal{S}$ and vice versa. For this, we consider the following three lemmas.

Lemma 2 ([35], Theorem 3)**.** *If $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ and $\mathcal{S} \in \mathcal{L}(\mathcal{Y})$ have the Browder's theorem, then the following statements are equivalent:*

1. *$\mathcal{T} \otimes \mathcal{S}$ has the Browder's theorem.*
2. *$\sigma_W(\mathcal{T} \otimes \mathcal{S}) = \sigma(\mathcal{T})\sigma_W(\mathcal{S}) \cup \sigma_W(\mathcal{T})\sigma(\mathcal{S})$.*

Lemma 3. *If $\mathcal{T} \in \mathcal{L}(\mathcal{X})$ and $\mathcal{S} \in \mathcal{L}(\mathcal{Y})$, then*

$$\begin{aligned} \sigma_{SF_+^-}(\mathcal{T} \otimes \mathcal{S}) &\subseteq \sigma_{SF_+^-}(\mathcal{T})\sigma(\mathcal{S}) \cup \sigma_{SF_+^-}(\mathcal{S})\sigma(\mathcal{T}) \\ &\subseteq \sigma_b(\mathcal{T})\sigma(\mathcal{S}) \cup \sigma_b(\mathcal{S})\sigma(\mathcal{T}) = \sigma_b(\mathcal{T} \otimes \mathcal{S}). \end{aligned}$$

Proof. By virtue of [35] (Lemma 5), $\sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S})\subseteq\sigma_{SF_+^-}(\mathcal{T})\sigma_a(\mathcal{S})\cup\sigma_{SF_+^-}(\mathcal{S})\sigma_a(\mathcal{T})$. Thus, the proof follows from the facts that $\sigma_a(\mathcal{R})\subseteq\sigma(\mathcal{R})$ and $\sigma_{SF_+^-}(\mathcal{R})\subseteq\sigma_b(\mathcal{R})$ for every operator $\mathcal{R}$. $\square$

Lemma 4. *If $\mathcal{T}\in\mathcal{L}(\mathcal{X})$ and $\mathcal{S}\in\mathcal{L}(\mathcal{Y})$ are isoloid and $0\notin\sigma_p(\mathcal{T}\otimes\mathcal{S})$, then*

$$E(\mathcal{T}\otimes\mathcal{S})\subseteq E(\mathcal{T})E(\mathcal{S}).$$

Proof. Since $\mathcal{T}\in\mathcal{L}(\mathcal{X})$ and $\mathcal{S}\in\mathcal{L}(\mathcal{Y})$ are isoloid, then $\mathcal{T}\otimes\mathcal{S}$ is an isoloid operator. According to this, we have $E(\mathcal{T})=\mathrm{Iso}\sigma(\mathcal{T})$, $E(\mathcal{S})=\mathrm{Iso}\sigma(\mathcal{S})$ and $E(\mathcal{T}\otimes\mathcal{S})=\mathrm{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})$.

Suppose that $\mathrm{Iso}\sigma(\mathcal{T})\subseteq\{0\}$ or $\mathrm{Iso}\sigma(\mathcal{S})\subseteq\{0\}$. By [36] (Proposition 3), $\mathrm{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})\subseteq\{0\}$, and as $0\notin\sigma_p(\mathcal{T}\otimes\mathcal{S})$, whereby $E(\mathcal{T}\otimes\mathcal{S})=\emptyset$, and so $E(\mathcal{T}\otimes\mathcal{S})\subseteq E(\mathcal{T})E(\mathcal{S})$ holds. Now, suppose that $\mathrm{Iso}\sigma(\mathcal{T})\nsubseteq\{0\}$ and $\mathrm{Iso}\sigma(\mathcal{T})\nsubseteq\{0\}$. Then, by [36] (Proposition 3(a)), $E(\mathcal{T}\otimes\mathcal{S})\subseteq E(\mathcal{T})E(\mathcal{S})$. $\square$

The following Theorem was proved in [13], but here we give a simpler proof.

Theorem 15. *Let $\mathcal{T}\in\mathcal{L}(\mathcal{X})$ and $\mathcal{S}\in\mathcal{L}(\mathcal{Y})$ satisfy property (Sb). Then, $\mathcal{T}\otimes\mathcal{S}$ satisfies property (Sb), which is equivalent to*

$$\sigma_{SBF_+^-}(\mathcal{T}\otimes\mathcal{S})=\sigma(\mathcal{T})\sigma_{SBF_+^-}(\mathcal{S})\cup\sigma_{SBF_+^-}(\mathcal{T})\sigma(\mathcal{S}).$$

Proof. It is well known that properties (Sb) and (V_Π) are equivalent (see [3] (Corollary 2.5)). In addition, property (V_Π) implies the equality of the Browder spectrum and the upper semi B-Weyl spectrum (see [3] (Theorem 2.27)). Thus, the proof follows from the identity $\sigma_b(\mathcal{T}\otimes\mathcal{S})=\sigma(\mathcal{T})\sigma_b(\mathcal{S})\cup\sigma_b(\mathcal{T})\sigma(\mathcal{S})$ (see [37] (Theorem 4.2(a))). $\square$

Theorem 16. *Let $\mathcal{T}\in V_E(\mathcal{X})$ and $\mathcal{S}\in V_E(\mathcal{Y})$ be two isoloid operators and $0\notin\sigma_p(\mathcal{T}\otimes\mathcal{S})$. Then, $\mathcal{T}\otimes\mathcal{S}\in V_E(\mathcal{X}\otimes\mathcal{Y})$ is equivalent to*

$$\sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S})=\sigma_{SF_+^-}(\mathcal{T})\sigma(\mathcal{S})\cup\sigma_{SF_+^-}(\mathcal{S})\sigma(\mathcal{T}).$$

Proof. Since property (V_E) implies the equality between upper semi-Weyl and Browder spectra (see [3], Theorem 2.27), the direct sense is immediate from [37] (Theorem 3.5).

Conversely, suppose that the identity

$$(1)\qquad \sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S})=\sigma_{SF_+^-}(\mathcal{T})\sigma(\mathcal{S})\cup\sigma_{SF_+^-}(\mathcal{S})\sigma(\mathcal{T})$$

holds. Then, again by [3] (Theorem 2.27), we get that

$$\sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S})=\sigma_b(\mathcal{T})\sigma(\mathcal{S})\cup\sigma_b(\mathcal{S})\sigma(\mathcal{T})=\sigma_b(\mathcal{T}\otimes\mathcal{S}).$$

Thus, we obtain that $\Delta_+(\mathcal{T}\otimes\mathcal{S})=\Pi^0(\mathcal{T}\otimes\mathcal{S})\subseteq E(\mathcal{T}\otimes\mathcal{S})$. However, we will show that $E(\mathcal{T}\otimes\mathcal{S})\subseteq\Delta_+(\mathcal{T}\otimes\mathcal{S})$. If $\mu\in E(\mathcal{T}\otimes\mathcal{S})$, then $\mu\in E(\mathcal{T})E(\mathcal{S})$ by Lemma 4. Hence, if $\mu=\xi\nu$ with $\xi\in\sigma(\mathcal{T})$ and $\nu\in\sigma(\mathcal{S})$, then $\xi\in\sigma(\mathcal{T})\setminus\sigma_{SF_+^-}(\mathcal{T})$ and $\nu\in\sigma(\mathcal{S})\setminus\sigma_{SF_+^-}(\mathcal{S})$, and since the identity (1) holds, we get that $\mu\in\Delta_+(\mathcal{T}\otimes\mathcal{S})$. Hence, $\mathcal{T}\otimes\mathcal{S}\in V_E(\mathcal{X}\otimes\mathcal{Y})$. $\square$

Recall that, if $\mathcal{A}_1\in\mathcal{L}(\mathcal{X})$ and $\mathcal{A}_2\in\mathcal{L}(\mathcal{Y})$ are quasinilpotent commuting with $\mathcal{T}$ and $\mathcal{S}$, respectively, then

$$(\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2)=(\mathcal{T}\otimes\mathcal{S})+\mathcal{Q},$$

where $\mathcal{Q}=\mathcal{A}_1\otimes\mathcal{S}+\mathcal{T}\otimes\mathcal{A}_2+\mathcal{A}_1\otimes\mathcal{A}_2\in\mathcal{L}(\mathcal{X}\otimes\mathcal{Y})$ is quasinilpotent.

Theorem 17. *Let $\mathcal{A}_1\in\mathcal{L}(\mathcal{X})$ and $\mathcal{A}_2\in\mathcal{L}(\mathcal{Y})$ be quasinilpotent commuting with $\mathcal{T}$ and $\mathcal{S}$, respectively. If $\mathcal{T}\otimes\mathcal{S}\in V_E(\mathcal{X}\otimes\mathcal{Y})$ is isoloid, then $(\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2)\in V_E(\mathcal{X}\otimes\mathcal{Y})$.*

Proof. We know that

$$\sigma((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))=\sigma(\mathcal{T}\otimes\mathcal{S}),$$
$$\sigma_{SF_+^-}((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))=\sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S}).$$

We also have that an operator satisfies SVEP if and only if any perturbation of it by a commuting quasinilpotent operator satisfies SVEP. Assume that $\mathcal{T}\otimes\mathcal{S}\in V_E(\mathcal{X}\otimes\mathcal{Y})$. Then

$$\begin{aligned} E(\mathcal{T}\otimes\mathcal{S}) &= \sigma(\mathcal{T}\otimes\mathcal{S})\setminus\sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S}) \\ &= \sigma((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))\setminus\sigma_{SF_+^-}((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2)). \end{aligned}$$

We will show that $E(\mathcal{T}\otimes\mathcal{S})=E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$. Indeed, if $\mu\in E(\mathcal{T}\otimes\mathcal{S})$, then $\mu\in\sigma((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))\setminus\sigma_{SF_+^-}((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$, and also $\mu\in\operatorname{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})$. Since $\mu\in\operatorname{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})$ implies that $(\mathcal{T}^*+\mathcal{A}_1^*)\otimes(\mathcal{S}^*+\mathcal{A}_2^*)$ satisfies SVEP at μ, if follows that $\mu\notin\sigma_W((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$ and $\mu\in\operatorname{Iso}\sigma((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$. Hence, $\mu\in E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$ and so, $E(\mathcal{T}\otimes\mathcal{S})\subseteq E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$. To show the inclusion $E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))\subseteq E(\mathcal{T}\otimes\mathcal{S})$, let $\mu\in E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))$. Then $\mu\in\operatorname{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})$, and as $\mathcal{T}\otimes\mathcal{S}$ is isoloid, $\mu\in E(\mathcal{T}\otimes\mathcal{S})$. Therefore, $E((\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2))\subseteq E(\mathcal{T}\otimes\mathcal{S})$ and consequently $(\mathcal{T}+\mathcal{A}_1)\otimes(\mathcal{S}+\mathcal{A}_2)\in V_E(\mathcal{X}\otimes\mathcal{Y})$. □

Theorem 18. *Let $\mathcal{T}\in V_E(\mathcal{X})$ and $\mathcal{S}\in V_E(\mathcal{Y})$ be two isoloid operators and let $\mathcal{B}_1\in\mathcal{L}(\mathcal{X})$ and $\mathcal{B}_2\in\mathcal{L}(\mathcal{Y})$ be two Riesz operators commuting with $\mathcal{T}$ and $\mathcal{S}$, respectively. Suppose that $\sigma(\mathcal{T}+\mathcal{B}_1)=\sigma(\mathcal{T})$, $\sigma(\mathcal{S}+\mathcal{B}_2)=\sigma(\mathcal{S})$ and $\mathcal{T}\otimes\mathcal{S}\in V_E(\mathcal{X}\otimes\mathcal{Y})$. The following are equivalent:*

1. *a-Browder's theorem transfers from $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ to their tensor product.*
2. *$(\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)\in V_E(\mathcal{X}\otimes\mathcal{Y})$.*

Proof. First of all, let us observe that according to the hypothesis and Theorem 9, we have that both $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ satisfy property (V_E), which implies that $\sigma(\mathcal{T}+\mathcal{B}_1)=\sigma_a(\mathcal{T}+\mathcal{B}_1)$, $\sigma(\mathcal{S}+\mathcal{B}_2)=\sigma_a(\mathcal{S}+\mathcal{B}_2)$, $\sigma_{SF_+^-}(\mathcal{T}+\mathcal{B}_1)=\sigma_b(\mathcal{T}+\mathcal{B}_1)$ and $\sigma_{SF_+^-}(\mathcal{S}+\mathcal{B}_2)=\sigma_b(\mathcal{S}+\mathcal{B}_2)$. In addition, as $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{T}\otimes\mathcal{S}$ satisfy property (V_E), we get that

$$\begin{aligned} \sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S}) &= \sigma_b(\mathcal{T}\otimes\mathcal{S})=\sigma(\mathcal{T})\sigma_b(\mathcal{S})\cup\sigma_b(\mathcal{T})\sigma(\mathcal{S}) \\ &= \sigma(\mathcal{T})\sigma_{SF_+^-}(\mathcal{S})\cup\sigma_{SF_+^-}(\mathcal{T})\sigma(\mathcal{S}) \\ &= \sigma(\mathcal{T}+\mathcal{B}_1)\sigma_{SF_+^-}(S+B_2)\cup\sigma_{SF_+^-}(\mathcal{T}+\mathcal{B}_1)\sigma(\mathcal{S}+\mathcal{B}_2). \end{aligned}$$

Now, we will prove the required equivalences in the theorem.

(1)⇒ (2) Assume that a-Browder's theorem transfers from $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ to $(\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)$. Then, from the above and by [11] (Lemma 1), we have

$$\begin{aligned} \sigma_{SF_+^-}(\mathcal{T}\otimes\mathcal{S}) &= \sigma(\mathcal{T}+\mathcal{B}_1)\sigma_{SF_+^-}(\mathcal{S}+\mathcal{B}_2)\cup\sigma_{SF_+^-}(\mathcal{T}+\mathcal{B}_1)\sigma(\mathcal{S}+\mathcal{B}_2) \\ &= \sigma_a(\mathcal{T}+\mathcal{B}_1)\sigma_{SF_+^-}(\mathcal{S}+\mathcal{B}_2)\cup\sigma_{SF_+^-}(\mathcal{T}+\mathcal{B}_1)\sigma_a(\mathcal{S}+\mathcal{B}_2) \\ &= \sigma_{SF_+^-}((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)) \end{aligned}$$

and

$$E(\mathcal{T}\otimes\mathcal{S})=\sigma((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))\setminus\sigma_{SF_+^-}((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)).$$

Thus, to conclude this part of the proof, we will show that $E(\mathcal{T}\otimes\mathcal{S})=E((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$ holds. Let $\mu\in E(\mathcal{T}\otimes\mathcal{S})$. Then, there exist $\xi\in\sigma(\mathcal{T}+\mathcal{B}_1)\setminus\sigma_{SF_+^-}(\mathcal{T}+\mathcal{B}_1)$ and $\nu\in\sigma(\mathcal{S}+\mathcal{B}_2)\setminus\sigma_{SF_+^-}(\mathcal{S}+\mathcal{B}_2)$ with $\mu=\xi\nu$. As both $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ satisfy property (V_E), it follows that $\xi\in E(\mathcal{T}+\mathcal{B}_1)$ and $\nu\in E(\mathcal{S}+\mathcal{B}_2)$. Thus,

$\mu = \xi\nu \in \sigma_p(\mathcal{T}+\mathcal{B}_1)\sigma_p(\mathcal{S}+\mathcal{B}_2) \subseteq \sigma_p((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$, and using the fact that $\mu \in \sigma(\mathcal{T}\otimes\mathcal{S}) = \sigma((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$, we get that $\mu \in E((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$. Conversely, if $\mu \in E((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$ then $\mu \in \operatorname{Iso}\sigma((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2))$. Since $\sigma((\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)) = \sigma(\mathcal{T}\otimes\mathcal{S})$, we have $\mu \in \operatorname{Iso}\sigma(\mathcal{T}\otimes\mathcal{S})$, and as $\mathcal{T}\otimes\mathcal{S}$ is isoloid (because both $\mathcal{T}$ and $\mathcal{S}$ are isoloid), it follows that $\mu \in E(\mathcal{T}\otimes\mathcal{S})$.

(2)$\Rightarrow$ (1) As property (V_E) implies a-Browder's theorem, $(\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)$ has the Browder's theorem. As both $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ satisfy property (V_E), this tells us that a-Browder's theorem is transmitted from $\mathcal{T}+\mathcal{B}_1$ and $\mathcal{S}+\mathcal{B}_2$ to $(\mathcal{T}+\mathcal{B}_1)\otimes(\mathcal{S}+\mathcal{B}_2)$. $\square$

Remark 3. Let $\mathcal{X}$ be a Banach space and $\mathcal{M}$ be a proper closed subspace of $\mathcal{X}$. We consider the set $P(\mathcal{X},\mathcal{M}) = \{\mathcal{T} \in \mathcal{L}(\mathcal{X}) : \mathcal{T}(\mathcal{M}) \subseteq \mathcal{M}$ and there exists an integer $k \geq 1$ for which $\mathcal{T}^k(\mathcal{X}) \subseteq \mathcal{M}\}$. For every $\mathcal{T} \in P(\mathcal{X},\mathcal{M})$, let $\mathcal{T}_{\mathcal{M}}$ be the restriction of $\mathcal{T}$ on $\mathcal{M}$. According to the results established in [38], if $\mathcal{T} \in P(\mathcal{X},\mathcal{M})$ and $0 \notin \operatorname{Iso}\sigma(\mathcal{T})$, then $\mathcal{T} \in V_E(\mathcal{X})$ is equivalent to $\mathcal{T}_{\mathcal{M}} \in V_E(\mathcal{M})$. Hence, if $\mathcal{T} \in P(\mathcal{X},\mathcal{M})$ and $0 \notin \operatorname{Iso}\sigma(\mathcal{T})$, then the results given in this work can be preserved from $\mathcal{T}$ to $\mathcal{T}_{\mathcal{M}}$ and reciprocally.

5. Conclusions

The spectral property (V_E) implies a range of spectral properties, including the classical Weyl's theorems, so this property is somewhat strong. Some necessary conditions were obtained that guarantee the stable character of property (V_E) under the classic perturbations. Among other things, it was concluded that property (V_E) is stable under commuting perturbations: nilpotent, of finite range "but the operator is isoloid", of Riesz "but the operator is isoloid and the spectrum of the operator coincides with the spectrum of the sum of the operator with the Riesz perturbation", and algebraic when the operator satisfies SVEP at all points of the spectrum of the algebraic perturbation. Finally, the tensor product between two operators that satisfy the property (V_E) was analyzed and we concluded that under certain conditions it is stable for quasinilpotent (or Riesz) perturbations in the factors.

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