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Basic Fundamental Formulas for Wiener Transforms Associated with a Pair of Operators on Hilbert Space

Hyun Soo Chung

Department of Mathematics, Dankook University, Cheonan 31116, Korea; hschung@dankook.ac.kr

Abstract: Segal introduce the Fourier–Wiener transform for the class of polynomial cylinder functions on Hilbert space, and Hida then develop this concept. Negrin define the extended Wiener transform with Hayker et al. In recent papers, Hayker et al. establish the existence, the composition formula, the inversion formula, and the Parseval relation for the Wiener transform. But, they do not establish homomorphism properties for the Wiener transform. In this paper, the author establishes some basic fundamental formulas for the Wiener transform via some concepts and motivations introduced by Segal and used by Hayker et al. We then state the usefulness of basic fundamental formulas as some applications.

Keywords: Hilbert space; convolution product; first variation; integration by parts formula; translation theorem

MSC: 60J65; 28C20

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1. Introduction

Let X be a normed space and let T be a operator on X. In functional analysis theory and algebraic structures, the homomorphism properties

$$T(f*g) = T(f)T(g) \tag{1}$$

and

$$(T(f)*T(g)) = T(fg) \tag{2}$$

are very important subjects to various fields of mathematics for $f, g \in X$, where $*$ denotes a corresponding convolution product of T.

In [1–3], Segal introduce the Fourier–Wiener transform for the class of polynomial cylinder functions on Hilbert space. Hida then develop this concept via the Fourier analysis on the dual space of nuclear spaces [4,5]. In addition, Negrin obtain an explicit integral representation of the second quantization by use of an integral operator and hence the Wiener transform [6] is extended. Later, Hayker et al. analyze and study some results and formulas of them via the matrix expressions [7].

In [8,9], the authors establish the existence, the composition formula, the inversion formula and the parseval relationship for the Wiener transform. But, they do not establish homomorphism properties (1) and (2) for the Wiener transform.

In this paper, we shall establish homomorphism properties for the Wiener transform. In addition, we obtain an integration by parts formula, and give some applications of it with respect to the Wiener transform. Our integration by parts formula takes a different form than in the Euclidean space. The reason is that the measure used in this paper is a probability measure, unlike the Lebesgue measure.

2. Definitions and Preliminaries

In this section, we first state some definitions and notations to understand the paper.

Let $\mathbf{H}'$ be a real Hilbert space and $\mathbf{H}$ be a complexification of $\mathbf{H}'$. The inner product on $\mathbf{H}$ is given by the formula

$$\langle x+iy, x'+iy'\rangle_{\mathbf{H}} = \langle x, x'\rangle_{\mathbf{H}'} + \langle y, y'\rangle_{\mathbf{H}'} + i\langle y, x'\rangle_{\mathbf{H}'} - i\langle x, y'\rangle_{\mathbf{H}'}.$$

Let A and B be operators defined on $\mathbf{H}$ such that there exists an orthonormal basis $\mathcal{B} = \{e_\alpha\}_{\alpha\in\mathcal{A}}$ of $\mathbf{H}$ ($\mathcal{A}$ being some index set) consisting of elements of $\mathbf{H}$ with

$$Ae_\alpha = \mu_\alpha e_\alpha, \qquad Be_\alpha = \lambda_\alpha e_\alpha \tag{3}$$

for some complex numbers μ_α and λ_α. Then we note that for each $x \in \mathbf{H}$,

$$x = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} e_\alpha$$

and so

$$Ax = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} \mu_\alpha e_\alpha$$

and

$$Bx = \sum_{\alpha\in\mathcal{A}} \langle x, e_\alpha\rangle_{\mathbf{H}} \lambda_\alpha e_\alpha.$$

We now state a class of functions used in this paper.

Definition 1. *Let f be a polynomial function on $\mathbf{H}'$ defined by the formula*

$$f(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} \langle x, e_{\alpha_2}\rangle_{\mathbf{H}}^{n_2} \cdots \langle x, e_{\alpha_r}\rangle_{\mathbf{H}}^{n_r} \tag{4}$$

where $n_1, \cdots, n_r \in \mathbb{N}\cup\{0\}$. Let $\mathcal{P}$ be the space of all complex-valued polynomial on $\mathbf{H}'$.

We are ready to state definitions of the Wiener transform, the convolution product and the first variation for functions in $\mathcal{P}$.

Definition 2. *For each pair of operators A and B on $\mathbf{H}$, we define the Wiener transform $\mathcal{F}_{c,A,B}(f)$ of f by the formula*

$$\mathcal{F}_{c,A,B}(f)(y) = \int_{\mathbf{H}'} f(Ax+By)\,dg_c(x) \tag{5}$$

where f is in $\mathcal{P}$ and the integration on $\mathbf{H}'$ is performed with respect to the normalized distribution g_c of the variance parameter $c>0$. In addition, we define the convolution product $(f_1 * f_2)_A$ of f_1 and f_2 by the formula

$$(f_1 * f_2)_A(y) = \int_{\mathbf{H}'} f_1\left(\frac{y+Ax}{\sqrt{2}}\right) f_2\left(\frac{y-Ax}{\sqrt{2}}\right) dg_c(x) \tag{6}$$

and the first variation $\delta_B f$ of f is defined by the formula

$$\delta_B f(x|u) = \frac{\partial}{\partial k} f(x+kBu)\bigg|_{k=0} \tag{7}$$

where $f, f_1, f_2 \in \mathcal{P}$ if they exist.

3. Existence

In this section, we establish the existence of the convolution product and the first variation for function f of the form (4). Before doing this, we give a theorem for some formulas with respect to the Wiener transform $\mathcal{F}_{c,A,B}$ which are established by Hayker et al. [9].

Theorem 1. *Let A, B, A', B', A'' and B'' be operators on $\mathbf{H}$ given by*

$$
\begin{aligned}
&Ae_\alpha = \mu_\alpha e_\alpha,\ Be_\alpha = \lambda_\alpha e_\alpha,\ A'e_\alpha = \mu'_\alpha e_\alpha,\ B'e_\alpha = \lambda'_\alpha e_\alpha \\
&A''e_\alpha = \mu''_\alpha e_\alpha,\ B''e_\alpha = \lambda''_\alpha e_\alpha
\end{aligned}
$$

where $\mu_\alpha, \mu'_\alpha, \mu''_\alpha, \lambda_\alpha, \lambda'_\alpha$ and λ''_α are complex numbers. Then we have the following assertions.

(a) *(Existence): for any $f \in \mathcal{P}$,*

$$
\mathcal{F}_{c,A,B}(f)(y) = \prod_{j=1}^{r}\left(\sum_{p=0}^{[n_j/2]} {}_{n_j}C_p \mu_{\alpha_j}^{2p} \lambda_{\alpha_j}^{n_j-2p} \langle y, e_{\alpha_j}\rangle_{\mathbf{H}}^{n_j-2p} \frac{(2p)!}{p!}\left(\frac{c}{2}\right)^p\right) \tag{8}
$$

and $\mathcal{F}_{c,A,B}(f) \in \mathcal{P}$.

(b) *(Composition formula [9], Theorem 1):*

$$
\mathcal{F}_{c,A',B'}(\mathcal{F}_{c,A,B}(f))(y) = \mathcal{F}_{c,A'',B''}(f)(y)
$$

if and only if

$$
\mu_\alpha^2 + (\mu'_\alpha \lambda_\alpha)^2 = (\mu''_\alpha)^2 \text{ and } \lambda_\alpha \lambda'_\alpha = \lambda''_\alpha
$$

for $\alpha \in \mathcal{A}$.

(c) *(Inversion formula [9], Corollary 2):*

$$
\mathcal{F}_{c,A',B'}(\mathcal{F}_{c,A,B}(f))(y) = f(y) \tag{9}
$$

if and only if

$$
\mu_\alpha^2 + (\mu'_\alpha \lambda_\alpha)^2 = 0 \text{ and } \lambda_\alpha \lambda'_\alpha = 1
$$

for $\alpha \in \mathcal{A}$.

(d) *(Parseval relation [9], Theorem 2):*

$$
\int_{\mathbf{H}'} \mathcal{F}_{c,A,B}(f_1)(y) f_2(y) dg_c(y) = \int_{\mathbf{H}'} \mathcal{F}_{c,A,B}(f_2)(y) f_1(y) dg_c(y)
$$

if and only if

$$
\mu_\alpha^2 + \lambda_\alpha^2 = 1
$$

for $\alpha \in \mathcal{A}$. Furthermore, they show that it can be extended to the Unitary extension.

We shall obtain the existence of the convolution product and the first variation. To do this, we need an observation as below.

Remark 1. *For any f_1 and f_2 in $\mathcal{P}$, we can always express f_1 by Equation (4) and f_2 by*

$$
f_2(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{m_1} \langle x, e_{\alpha_2}\rangle_{\mathbf{H}}^{m_2} \cdots \langle x, e_{\alpha_r}\rangle_{\mathbf{H}}^{m_r} \tag{10}
$$

using the same nonnegative integer r and α_j's. Because, if $f_1(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} \langle x, e_{\alpha_3}\rangle_{\mathbf{H}}^{n_3}$ and $f_2(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} \langle x, e_{\alpha_2}\rangle_{\mathbf{H}}^{n_2}$, then we can set

$$
f_1(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} \langle x, e_{\alpha_2}\rangle_{\mathbf{H}}^{0} \langle x, e_{\alpha_3}\rangle_{\mathbf{H}}^{n_3}
$$

and

$$
f_2(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{m_1} \langle x, e_{\alpha_2}\rangle_{\mathbf{H}}^{m_2} \langle x, e_{\alpha_3}\rangle_{\mathbf{H}}^{0}.
$$

In addition, if $f_1(x) = \langle x, e_\alpha\rangle_{\mathbf{H}}^{n}$ and $f_2(x) = \langle x, e_\beta\rangle_{\mathbf{H}}^{m}$ for $n \neq m$, then we can set

$$
f_1(x) = \langle x, e_{\gamma_1}\rangle_{\mathbf{H}}^{n_1} \langle x, e_{\gamma_2}\rangle_{\mathbf{H}}^{0}
$$

and

$$f_2(x) = \langle x, e_{\gamma_1}\rangle_{\mathbf{H}}^{m_1} \langle x, e_{\gamma_2}\rangle_{\mathbf{H}}^{0}$$

where $\gamma_1 = \alpha, \gamma_2 = \beta, n_1 = n, n_2 = 0, m_1 = 0$ *and* $m_2 = m$.

In Theorem 1, we obtain the existence of the convolution product and the first variation for functions in $\mathcal{P}$.

Theorem 2. *Let f_1 and f_2 be elements of $\mathcal{P}$ and A as in Theorem 1. Then the convolution product $(f_1 * f_2)_A$ of f_1 and f_2 exists, belongs to $\mathcal{P}$ and is given by the formula*

$$\begin{aligned}&(f_1 * f_2)_A(y) \\ &= \left(\frac{1}{2\pi c}\right)^{\frac{r}{2}} \prod_{j=1}^{r} \left[\int_{\mathbb{R}} \left(\frac{1}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} + \frac{\lambda_{\alpha_j}}{\sqrt{2}} u_j\right)^{n_j} \right. \\ &\qquad \left. \times \left(\frac{1}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} - \frac{\lambda_{\alpha_j}}{\sqrt{2}} u_j\right)^{m_j} \exp\left\{-\frac{u_j^2}{2c}\right\} du_j\right].\end{aligned} \tag{11}$$

Furthermore, the first variation $\delta_A f$ of f exists, belongs to $\mathcal{P}$ and is given by the formula

$$\delta_A f(x|u) = \sum_{j=1}^{r} n_j \lambda_{\alpha_j} \langle u, e_{\alpha_j}\rangle_{\mathbf{H}} f_j(x) \tag{12}$$

where

$$f_j(x) = \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} \times \cdots \times \langle x, e_{\alpha_j}\rangle_{\mathbf{H}}^{n_j - 1} \times \cdots \times \langle x, e_{\alpha_r}\rangle_{\mathbf{H}}^{n_r}. \tag{13}$$

Proof. Using Equations (5) and (6), we have

$$\begin{aligned}&(f_1 * f_2)_A(y) \\ &= \int_{\mathbf{H}'} \prod_{j=1}^{r} \left(\frac{1}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} + \frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}}\right)^{n_j} \left(\frac{1}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} - \frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}}\right)^{m_j} dg_c(x) \\ &= \left(\frac{1}{2\pi c}\right)^{\frac{r}{2}} \prod_{j=1}^{r} \left[\int_{\mathbb{R}} \left(\frac{1}{\sqrt{2}} v_j + \frac{\lambda_{\alpha_j}}{\sqrt{2}} u_j\right)^{n_j} \left(\frac{1}{\sqrt{2}} v_j - \frac{\lambda_{\alpha_j}}{\sqrt{2}} u_j\right)^{m_j} \exp\left\{-\frac{u_j^2}{2c}\right\} du_j\right]\end{aligned}$$

where $v_j = \langle y, e_{\alpha_j}\rangle_{\mathbf{H}}$ for $j = 1, 2, \cdots, r$. The last integral always exists because

$$\int_{\mathbb{R}} p(u) \exp\left\{-\frac{u_j^2}{2c}\right\} du < \infty$$

for any polynomial function p. In addition, it is a polynomial in the variables

$$\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}, \cdots, \langle y, e_{\alpha_r}\rangle_{\mathbf{H}}.$$

We next establish Equation (12). From Equation (7), we have

$$\begin{aligned}\delta_A f(x|u) &= \frac{\partial}{\partial k} \prod_{j=1}^{r} (\langle x, e_{\alpha_j}\rangle_{\mathbf{H}} + k\lambda_{\alpha_j}\langle u, e_{\alpha_j}\rangle_{\mathbf{H}})^{n_j}\Big|_{k=0} \\ &= \sum_{j=1}^{r} n_j \lambda_{\alpha_j} \langle u, e_{\alpha_j}\rangle_{\mathbf{H}} f_j(x).\end{aligned}$$

Finally, $\delta_A f$ is in $\mathcal{P}$ since $f_j \in \mathcal{P}$ for all $j = 1, 2 \cdots, r$. □

4. Homomorphism Properties and Basic Relationships

In this section, we establish some basic relationships among the Wiener transform, the convolution product and the first variation.

Theorem 3 tells us that the Wiener transform of the convolution product is the product of their Wiener transforms.

Theorem 3. *Let f_1, f_2, A, B and A' be as in Theorem 1. Then*

$$\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y) = \mathcal{F}_{c,A,B}(f_1)\Big(\frac{y}{\sqrt{2}}\Big)\mathcal{F}_{c,A,B}(f_2)\Big(\frac{y}{\sqrt{2}}\Big). \tag{14}$$

Furthermore, under the hypothesis of Theorem 1, we have

$$(\mathcal{F}_{c,A,B}(f_1) * \mathcal{F}_{c,A,B}(f_2))_{A'}(y) = \mathcal{F}_{c,A,B}\Big(f_1(\frac{\cdot}{\sqrt{2}})f_2(\frac{\cdot}{\sqrt{2}})\Big)(y). \tag{15}$$

Proof. Using Equations (2), (6) and (11), we have

$$\begin{aligned}
&\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y)\\
&= \int_{\mathbf{H}'}\int_{\mathbf{H}'} f_1\Big(\frac{Ax+By+Az}{\sqrt{2}}\Big) f_2\Big(\frac{Ax+By-Az}{\sqrt{2}}\Big) dg_c(x)dg_c(z)\\
&= \int_{\mathbf{H}'}\int_{\mathbf{H}'} \prod_{j=1}^{r}\Big(\frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}} + \frac{\mu_{\alpha_j}}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} + \frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle z, e_{\alpha_j}\rangle_{\mathbf{H}}\Big)^{n_j}\\
&\qquad \times \prod_{j=1}^{r}\Big(\frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}} + \frac{\mu_{\alpha_j}}{\sqrt{2}}\langle y, e_{\alpha_j}\rangle_{\mathbf{H}} - \frac{\lambda_{\alpha_j}}{\sqrt{2}}\langle z, e_{\alpha_j}\rangle_{\mathbf{H}}\Big)^{m_j} dg_c(x)dg_c(z)\\
&= \Big(\frac{1}{2\pi c}\Big)^{r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\prod_{j=1}^{r}\Big(\frac{\lambda_{\alpha_j}}{\sqrt{2}}u_j + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j + \frac{\lambda_{\alpha_j}}{\sqrt{2}}w_j\Big)^{n_j}\\
&\qquad \times \prod_{j=1}^{r}\Big(\frac{\lambda_{\alpha_j}}{\sqrt{2}}u_j + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j - \frac{\lambda_{\alpha_j}}{\sqrt{2}}w_j\Big)^{m_j} \exp\Big\{-\sum_{j=1}^{r}\frac{u_j^2+w_j^2}{2c}\Big\} d\vec{u}d\vec{w}
\end{aligned}$$

where $v_j = \langle y, e_{\alpha_j}\rangle_{\mathbf{H}}$ for $j = 1, 2, \cdots, r$. Now let $u_j' = \frac{u_j+w_j}{\sqrt{2}}$ and $w_j' = \frac{u_j-w_j}{\sqrt{2}}$ for $j = 1, 2, \cdots, r$. Then we have

$$\begin{aligned}
&\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y)\\
&= \Big(\frac{1}{2\pi c}\Big)^{r}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}\prod_{j=1}^{r}\Big(\lambda_{\alpha_j}u_j' + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j\Big)^{n_j}\\
&\qquad \times \prod_{j=1}^{r}\Big(\lambda_{\alpha_j}w_j' + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j\Big)^{m_j} \exp\Big\{-\sum_{j=1}^{r}\frac{(u_j')^2+(w_j')^2}{2c}\Big\} d\vec{u'}d\vec{w'}\\
&= \Big(\frac{1}{2\pi c}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\prod_{j=1}^{r}\Big(\lambda_{\alpha_j}u_j' + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j\Big)^{n_j} \exp\Big\{-\sum_{j=1}^{r}\frac{(u_j')^2}{2c}\Big\} d\vec{u'}\\
&\times \Big(\frac{1}{2\pi c}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\prod_{j=1}^{r}\Big(\lambda_{\alpha_j}w_j' + \frac{\mu_{\alpha_j}}{\sqrt{2}}v_j\Big)^{m_j} \exp\Big\{-\sum_{j=1}^{r}\frac{(w_j')^2}{2c}\Big\} d\vec{w'}
\end{aligned}$$

where $v_j = \langle y, e_{\alpha_j}\rangle_{\mathbf{H}}$ for $j = 1, 2, \cdots, r$. Hence, using Equation (8), we can conclude that

$$\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y) = \mathcal{F}_{c,A,B}(f_1)\Big(\frac{y}{\sqrt{2}}\Big)\mathcal{F}_{c,A,B}(f_2)\Big(\frac{y}{\sqrt{2}}\Big).$$

In addition, using Equation (9), we have

$$\begin{aligned}
&\mathcal{F}_{c,A',B'}(\mathcal{F}_{c,A,B}(f_1)*\mathcal{F}_{c,A,B}(f_2))_{A'}(y)\\
&=\mathcal{F}_{c,A',B'}(\mathcal{F}_{c,A,B}(f_1))\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{c,A',B'}(\mathcal{F}_{c,A,B}(f_2))\left(\frac{y}{\sqrt{2}}\right)\\
&=f_1\left(\frac{y}{\sqrt{2}}\right)f_2\left(\frac{y}{\sqrt{2}}\right),
\end{aligned}$$

which yields Equation (15) as desired, where $\mathcal{F}_{c,A',B'}$ is as in Theorem 1. □

In our next theorem, we show that the Wiener transform and the first variation are commutable.

Theorem 4. *Let f be as in Theorem 1 and let A and B be as in Theorem 1. Let S be an operator on $\mathbf{H}$ with $Se_\alpha=\gamma_\alpha e_\alpha$ for $\alpha\in\mathcal{A}$. Then*

$$\delta_S\mathcal{F}_{c,A,B}(f)(y|u)=\mathcal{F}_{c,A,B}(\delta_{BS}f(\cdot|u))(y). \tag{16}$$

Proof. Using Equations (5) and (7), we have

$$\begin{aligned}
&\delta_S\mathcal{F}_{c,A,B}(f)(y|u)\\
&=\frac{\partial}{\partial k}\mathcal{F}_{c,A,B}(f)(u+kSu)\bigg|_{k=0}\\
&=\frac{\partial}{\partial k}\int_{\mathbf{H}'}f(Ax+By+kBSu)dg_c(x)\bigg|_{k=0}\\
&=\frac{\partial}{\partial k}\int_{\mathbf{H}'}\prod_{j=1}^{r}(\lambda_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}+\mu_{\alpha_j}\langle y,e_{\alpha_j}\rangle_{\mathbf{H}}+k\mu_{\alpha_j}\gamma_{\alpha_j}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}})^{n_j}dg_c(x)\bigg|_{k=0}\\
&=\sum_{j=1}^{r}n_j\mu_{\alpha_j}\gamma_{\alpha_j}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}\mathcal{F}_{c,A,B}(f_j)(y)
\end{aligned}$$

where f_j is as in Equation (13). We next use Equations (5) and (7) again to get

$$\begin{aligned}
&\mathcal{F}_{c,A,B}(\delta_S f(\cdot|u))(y)\\
&=\int_{\mathbf{H}'}\frac{\partial}{\partial k}f(Ax+By+kSu)\bigg|_{k=0}dg_c(x)\\
&=\frac{\partial}{\partial k}\int_{\mathbf{H}'}\prod_{j=1}^{r}(\lambda_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}+\mu_{\alpha_j}\langle y,e_{\alpha_j}\rangle_{\mathbf{H}}+k\gamma_{\alpha_j}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}})^{n_j}dg_c(x)\bigg|_{k=0}\\
&=\sum_{j=1}^{r}n_j\gamma_{\alpha_j}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}\mathcal{F}_{c,A,B}(f_j)(y)
\end{aligned}$$

where f_j is as in Equation (13). Comparing two expressions, we obtain Equation (16) as desired. □

From Equations (14) and (16) in Theorems 3 and 4, we have the following basic relationships.

Theorem 5. *Let f_1 and f_2 be as in Theorem 3. Let A and B as in Theorem 1 and let S as in Theorem 4. Then we have*

$$\delta(f_1*f_2)_S(y|u)=(\delta f_1(\cdot|u/\sqrt{2})*f_2)_S(y)+(f_1*\delta f_2(\cdot|u/\sqrt{2}))_S(y), \tag{17}$$

$$\begin{aligned}&\mathcal{F}_{c,A,B}(\delta_{BS}f_1(\cdot|u)*\delta_{BS}f_2(\cdot|u))_A(y)\\&=\delta_S\mathcal{F}_{c,A,B}f_1(y/\sqrt{2}|u)\delta_S\mathcal{F}_{c,A,B}f_2(y/\sqrt{2}|u),\end{aligned}\tag{18}$$

$$\begin{aligned}&\mathcal{F}_{c,A,B}(\delta_{BS}(f_1*f_2)_A(\cdot|u))(y)\\&=\delta_S(\mathcal{F}_{c,A,B}f_1(\cdot/\sqrt{2})\mathcal{F}_{c,A,B}f_2(\cdot/\sqrt{2}))(y|u)\\&=\delta_S\mathcal{F}_{c,A,B}(f_1*f_2)_A(y|u)\end{aligned}\tag{19}$$

and

$$(\mathcal{F}_{c,A,B}\delta_{BS}f_1(\cdot|u)*\mathcal{F}_{c,A,B}\delta_{BS}f_2(\cdot|u))_A(z)=(\delta_S\mathcal{F}_{c,A,B}f_1(\cdot|u)*\delta_S\mathcal{F}_{c,A,B}f_2(\cdot|u))_A(y).\tag{20}$$

Proof. We first note that Equation (17) follows directly from the definition of the first variation given by (7). Next we note that Equations (18) and (19) follow from Equations (14)–(16). Finally we note that Equation (20) follows immediately from Equations (14) and (16). □

5. Integration by Parts Formula with an Application

In this section, we obtain an integration by part formula, and give an application with respect to the Wiener transform.

Since the Lebesgue measure m_L on $\mathbb{R}^r$ is an uniform measure and so we see that

$$\int_{\mathbb{R}^r}h(\vec{u}+\vec{v})dm_L(\vec{u})=\int_{\mathbb{R}^r}h(\vec{w})dm_L(\vec{w})$$

by substitution for $w_j=u_j+v_j$ for $j=1,2,\cdots,r$ if the integrals exist. It is called the translation theorem for the Lebesgue integrals. However, the distribution measure g_c used in this paper is the Gaussian measure and hence, in generally,

$$\int_{\mathbf{H}'}h(x+y)dg_c(x)\neq\int_{\mathbf{H}'}h(z)dg_c(z)$$

even if the integrals exist, see [10–14]. For this reason, a different form of formula is obtained in this paper.

Lemma 1. *Let s be a non-negative integer and let p be a function on $\mathbf{H}$ defined by the formula*

$$p(x)=\langle x,e_\alpha\rangle_{\mathbf{H}}^s\tag{21}$$

for some $e_\alpha\in\mathcal{B}$. Then for all $x_0\in\mathbf{H}'$,

$$\begin{aligned}&\int_{\mathbf{H}'}p(x+x_0)dg_c(x)\\&=\exp\left\{-\frac{1}{2c}\langle x_0,e_\alpha\rangle_{\mathbf{H}}^2\right\}\int_{\mathbf{H}}p(x)\exp\left\{\frac{1}{c}\langle x,e_\alpha\rangle_{\mathbf{H}}\langle x_0,e_\alpha\rangle_{\mathbf{H}'}\right\}dg_c(x).\end{aligned}\tag{22}$$

Proof. We set $v = \langle x_0, e_\alpha \rangle_{\mathbf{H}}$. Using Equations (8) and (21), we have

$$\begin{aligned}
&\int_{\mathbf{H}'} p(x+x_0)dg_c(x)\\
&= \left(\frac{1}{2\pi c}\right)^{\frac{1}{2}} \int_{\mathbb{R}} (u+v)^s \exp\left\{-\frac{u^2}{2c}\right\}du\\
&= \left(\frac{1}{2\pi c}\right)^{\frac{1}{2}} \int_{\mathbb{R}} w^s \exp\left\{-\frac{(w-v)^2}{2c}\right\}dw\\
&= \exp\left\{-\frac{1}{2c}v^2\right\}\left(\frac{1}{2\pi c}\right)^{\frac{1}{2}} \int_{\mathbb{R}} w^s \exp\left\{-\frac{w^2}{2c}+\frac{1}{c}vw\right\}dw\\
&= \exp\left\{-\frac{1}{2c}\langle x_0, e_\alpha\rangle_{\mathbf{H}}^2\right\} \int_{\mathbf{H}'} p(x) \exp\left\{\frac{1}{c}\langle x, e_\alpha\rangle_{\mathbf{H}} \langle x_0, e_\alpha\rangle_{\mathbf{H}}\right\}dg_c(x).
\end{aligned}$$

Hence, we have the desired result. □

In Theorem 6, we obtain a translation theorem for **H**-integrals.

Theorem 6 (Translation theorem for **H**-integrals)**.** *Let f be as in Equation (4) and let $x_0 \in \mathbf{H}'$. Then*

$$\begin{aligned}
&\int_{\mathbf{H}'} f(x+x_0)dg_c(x)\\
&= \exp\left\{-\frac{1}{2c}\sum_{j=1}^{r}\langle x_0, e_\alpha\rangle_{\mathbf{H}}^2\right\} \int_{\mathbf{H}'} f(x) \exp\left\{\frac{1}{c}\sum_{j=1}^{r}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}} \langle x_0, e_{\alpha_j}\rangle_{\mathbf{H}}\right\}dg_c(x).
\end{aligned} \tag{23}$$

Proof. First, by using fact that

$$\int_{\mathbf{H}'} f(x)dg_c(x) = \int_{\mathbf{H}'} \langle x, e_{\alpha_1}\rangle_{\mathbf{H}}^{n_1} dg_c(x) \cdots \int_{\mathbf{H}'} \langle x, e_{\alpha_r}\rangle_{\mathbf{H}}^{n_r} dg_c(x),$$

and Equation (22) in Lemma 1 we can establish Equation (23) as desired. □

The following theorem is one of main results in this paper.

Theorem 7 (Integration by parts formula)**.** *Let f be as in Theorem 6 and let S be as in Theorem 4. Then*

$$\begin{aligned}
&c\int_{\mathbf{H}'} \delta_S f(x|u)dg_c(x)\\
&= c\int_{\mathbf{H}'} f(x)dg_c(x) + \int_{\mathbf{H}'} f(x)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}}\langle u, e_{\alpha_j}\rangle_{\mathbf{H}} dg_c(x).
\end{aligned} \tag{24}$$

Proof. Using Equations (1) and (7), we have

$$\begin{aligned}
&\int_{\mathbf{H}'} \delta_S f(x|u)dg_c(x)\\
&= \frac{\partial}{\partial k}\int_{\mathbf{H}'} f(x+kSu)dg_c(x)\Big|_{k=0}\\
&= \frac{\partial}{\partial k}\left[\exp\left\{-\frac{k^2}{2c}\sum_{j=1}^{r}\gamma_{\alpha_j}^2\langle u, e_\alpha\rangle_{\mathbf{H}}^2\right\}\right.\\
&\qquad\left.\times\int_{\mathbf{H}'} f(x)\exp\left\{\frac{k}{c}\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}}\langle u, e_{\alpha_j}\rangle_{\mathbf{H}}\right\}dg_c(x)\right]\Big|_{k=0}\\
&= \int_{\mathbf{H}'} f(x)dg_c(x) + \frac{1}{c}\int_{\mathbf{H}'} f(x)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x, e_{\alpha_j}\rangle_{\mathbf{H}}\langle u, e_{\alpha_j}\rangle_{\mathbf{H}} dg_c(x),
\end{aligned}$$

which yields Equation (24) as desired. □

Finally, we give an application of Theorem 7.

Theorem 8 (Application of Theorem 7)**.** *Let f and S be as in Theorem 7. Let A and B as in Theorem 5. Then*

$$\begin{aligned}&c\mathcal{F}_{c,A,B}(\delta_A f(\cdot|u))(y)\\&= c\mathcal{F}_{c,A,B}(f)(y) + \int_{\mathbf{H}'} f(Ax+By)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}dg_c(x).\end{aligned} \tag{25}$$

Proof. Using Equations (5) and (7), we have

$$\mathcal{F}_{c,A,B}(\delta_A f(\cdot|u))(y) = \frac{\partial}{\partial k}\int_{\mathbf{H}'} f(Ax+By+kAu)dg_c(x)\Big|_{k=0}.$$

Now, let $f_y(x) = f(x+y)$ and $f^A(x) = f(Ax)$. Then

$$f(Ax+By+kAu) = (f_{By})^A(x+ku)$$

and, hence, using Equation (24) by replacing f with $(f_{By})^A$, we have

$$\begin{aligned}&\mathcal{F}_{c,A,B}(\delta_A f(\cdot|u))(y)\\&= \frac{\partial}{\partial k}\int_{\mathbf{H}'}(f_{By})^A(x+ku)dg_c(x)\Big|_{k=0}\\&= \int_{\mathbf{H}'}(f_{By})^A(x)dg_c(x) + \frac{1}{c}\int_{\mathbf{H}'}(f_{By})^A(x)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}dg_c(x)\\&= \mathcal{F}_{c,A,B}(f)(y) + \frac{1}{c}\int_{\mathbf{H}'} f(Ax+By)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}dg_c(x).\end{aligned}$$

Hence, we have the desired results. □

6. Applications

In this section, we give some applications to apply our fundamental formulas obtained in previous sections.

6.1. *Application of Theorem 3*

We first give an application to illustrate the usefulness of Equations (14) and (15) in Theorem 3.

Example 1. *Let $r = 2$. Let $f_1(x) = \langle x,e_{\alpha_2}\rangle^2$ and let $f_2(x) = \langle x,e_{\alpha_1}\rangle^2\langle x,e_{\alpha_2}\rangle$. Let A and B be as in Theorem 3. From Equation (8) we have*

$$\mathcal{F}_{c,A,B}(f_1)(y) = \lambda_{\alpha_2}^2\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2$$

and

$$\mathcal{F}_{c,A,B}(f_2)(y) = [\lambda_{\alpha_1}^2\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2][\mu_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}].$$

Hence, using Equation (14), we have

$$\begin{aligned}&\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y)\\&= \left[\frac{\lambda_{\alpha_2}^2}{2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}\right].\end{aligned} \tag{26}$$

Furthermore, we note that

$$f_1(x)f_2(x) = \langle x, e_{\alpha_1}\rangle^2 \langle x, e_{\alpha_2}\rangle^3$$

and so

$$\begin{aligned}&\mathcal{F}_{c,A,B}\left(f_1(\frac{\cdot}{\sqrt{2}})f_2(\frac{\cdot}{\sqrt{2}})\right)(y)\\&= \left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\lambda_{\alpha_2}^3}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^3 + \frac{3c(\mu_{\alpha_2}^2+\lambda_{\alpha_2})}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right].\end{aligned}$$

Hence, using Equation (15), we have

$$\begin{aligned}&(\mathcal{F}_{c,A,B}(f_1) * \mathcal{F}_{c,A,B}(f_2))_{A'}(y)\\&= \left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\lambda_{\alpha_2}^3}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^3 + \frac{3c(\mu_{\alpha_2}^2+\lambda_{\alpha_2})}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right].\end{aligned}$$

These tell us that the Wiener transform of convolution product and the convolution product of Wiener transforms can be calculated without concept of convolution product very easily.

6.2. Application of Theorem 5

We next give an application of Equation (19) in Theorem 5.

Example 2. *Let f_1, f_2, A and B be as in Example 1. Using Equation (26), we have*

$$\begin{aligned}&\delta_S\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y|u)\\&= \frac{\partial}{\partial k}\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y + kSu)\Big|_{k=0}\\&= \frac{\partial}{\partial k}\Bigg(\left[\frac{\lambda_{\alpha_2}^2}{2}(\langle y, e_{\alpha_2}\rangle_{\mathbf{H}} + k\gamma_{\alpha_2}\langle u, e_{\alpha_2}\rangle_{\mathbf{H}})^2 + 2c\mu_{\alpha_2}^2\right]\\&\qquad\times\left[\frac{\lambda_{\alpha_1}^2}{2}(\langle y, e_{\alpha_1}\rangle_{\mathbf{H}} + k\gamma_{\alpha_1}\langle u, e_{\alpha_1}\rangle_{\mathbf{H}})^2 + 2c\mu_{\alpha_1}^2\right]\\&\qquad\qquad\times\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}(\langle y, e_{\alpha_2}\rangle_{\mathbf{H}} + k\gamma_{\alpha_2}\langle u, e_{\alpha_2}\rangle_{\mathbf{H}})\right]\Bigg)\Bigg|_{k=0}\\&= \lambda_{\alpha_2}^2\langle u, e_{\alpha_2}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right]\\&\quad + \lambda_{\alpha_1}^2\langle u, e_{\alpha_1}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right]\\&\quad + \frac{\mu_{\alpha_2}}{\sqrt{2}}\langle u, e_{\alpha_1}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right].\end{aligned}$$

Using this, we obtain that

$$\begin{aligned}&\mathcal{F}_{c,A,B}(\delta_{BS}(f_1 * f_2)_A(\cdot|u))(y)\\&= \lambda_{\alpha_2}^2\langle u, e_{\alpha_2}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right]\\&\quad + \lambda_{\alpha_1}^2\langle u, e_{\alpha_1}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right]\\&\quad + \frac{\mu_{\alpha_2}}{\sqrt{2}}\langle u, e_{\alpha_1}\rangle_{\mathbf{H}}\left[\frac{\lambda_{\alpha_2}^2}{2}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_2}^2\right]\left[\frac{\lambda_{\alpha_1}^2}{2}\langle y, e_{\alpha_1}\rangle_{\mathbf{H}}^2 + 2c\mu_{\alpha_1}^2\right]\left[\frac{\mu_{\alpha_2}}{\sqrt{2}}\langle y, e_{\alpha_2}\rangle_{\mathbf{H}}\right]\\&= \left(\lambda_{\alpha_2}^2\langle u, e_{\alpha_2}\rangle_{\mathbf{H}} + \lambda_{\alpha_1}^2\langle u, e_{\alpha_1}\rangle_{\mathbf{H}} + \frac{\mu_{\alpha_2}}{\sqrt{2}}\langle u, e_{\alpha_1}\rangle_{\mathbf{H}}\right)\mathcal{F}_{c,A,B}(f_1 * f_2)_A(y).\end{aligned}$$

6.3. Application of Theorem 7

We finish this paper by giving an application of Equation (25) in Theorem 7. Equation (25) tells us that

$$\begin{aligned}&\int_{\mathbf{H}'} f(Ax+By)\sum_{j=1}^{r}\gamma_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}dg_c(x)\\&=c\mathcal{F}_{c,A,B}(\delta_A f(\cdot|u))(y)-c\mathcal{F}_{c,A,B}(f)(y).\end{aligned} \tag{27}$$

The left-hand side of Equation (27) contains some polynomial-weight and so it is not easy to calculate. However, by using Equation (27), we can calculate it very easy via the Wiener transform and the first variation. We shall explain this as example.

Example 3. *Let f_1, f_2, A and B be as in Example 1. Then we have*

$$\mathcal{F}_{c,A,B}(\delta_A f_1(\cdot|u))(y)=2\mu_{\alpha_2}\lambda_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}$$

and

$$\mathcal{F}_{c,A,B}(f_1)(y)=\lambda_{\alpha_2}^2\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}^2+2c\mu_{\alpha_2}^2.$$

Hence, using Equation (27), we obtain that

$$\begin{aligned}&\int_{\mathbf{H}'}[\mu_{\alpha_2}\langle x,e_{\alpha_2}\rangle_{\mathbf{H}}+\lambda_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}]^2\mu_{\alpha_2}\langle x,e_{\alpha_2}\rangle_{\mathbf{H}}\langle u,e_{\alpha_2}\rangle_{\mathbf{H}}dg_c(x)\\&=2c\mu_{\alpha_2}\lambda_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}-c\lambda_{\alpha_2}^2\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}^2+2c^2\mu_{\alpha_2}^2.\end{aligned}$$

In addition, we have

$$\begin{aligned}\mathcal{F}_{c,A,B}(\delta_A f_2(\cdot|u))(y)&=2c\mu_{\alpha_1}^3\lambda_{\alpha_2}\langle u,e_{\alpha_1}\rangle_{\mathbf{H}}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}+2\mu_{\alpha_1}^3\lambda_{\alpha_1}^2\langle u,e_{\alpha_1}\rangle_{\mathbf{H}}\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}\\&\quad+\mu_{\alpha_2}\langle u,e_{\alpha_2}\rangle_{\mathbf{H}}(c\mu_{\alpha_1}^2+\lambda_{\alpha_1}^2\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2).\end{aligned}$$

and

$$\mathcal{F}_{c,A,B}(f_2)(y)=[\lambda_{\alpha_1}^2\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2+2c\mu_{\alpha_1}^2][\mu_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}].$$

Thus, from Equation (27) we conclude that

$$\begin{aligned}&\int_{\mathbf{H}'}[\mu_{\alpha_1}\langle x,e_{\alpha_1}\rangle_{\mathbf{H}}+\lambda_{\alpha_1}\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}]^2\\&\qquad\times[\mu_{\alpha_2}\langle x,e_{\alpha_2}\rangle_{\mathbf{H}}+\lambda_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}]\sum_{j=1}^{2}\gamma_{\alpha_j}\langle x,e_{\alpha_j}\rangle_{\mathbf{H}}\langle u,e_{\alpha_j}\rangle_{\mathbf{H}}dg_c(x)\\&=2c^2\mu_{\alpha_1}^3\lambda_{\alpha_2}\langle u,e_{\alpha_1}\rangle_{\mathbf{H}}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}+2c\mu_{\alpha_1}^3\lambda_{\alpha_1}^2\langle u,e_{\alpha_1}\rangle_{\mathbf{H}}\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}\\&\quad+c\mu_{\alpha_2}\langle u,e_{\alpha_2}\rangle_{\mathbf{H}}(c\mu_{\alpha_1}^2+\lambda_{\alpha_1}^2\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2)\\&\qquad-c[\lambda_{\alpha_1}^2\langle y,e_{\alpha_1}\rangle_{\mathbf{H}}^2+2c\mu_{\alpha_1}^2][\mu_{\alpha_2}\langle y,e_{\alpha_2}\rangle_{\mathbf{H}}].\end{aligned}$$

7. Conclusions

According to some results and formula in previous papers [1–3,7–9,15] and our results and formulas in previous Sections 3–5, we note that all results can be explained by the eigenvalue of operators on Hilbert space. As you can see from the results of the previous Sections 3–5, we are able to obtain various relationships that are not found in the previous research results. We also see in Section 6 that our results can be applied to various functions in the application of various fields. Therefore, it can be seen that the results in this paper are structured in a generalized form.

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