

Article

Some Inequalities of Extended Hypergeometric Functions

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Abstract: Hypergeometric functions and their inequalities have found frequent applications in various fields of mathematical sciences. Motivated by the above, we set up certain inequalities including extended type Gauss hypergeometric function and confluent hypergeometric function, respectively, by virtue of Hölder integral inequality and Chebyshev's integral inequality. We also studied the monotonicity, log-concavity, and log-convexity of extended hypergeometric functions, which are derived by using the inequalities on an extended beta function.



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1. Introduction and Preliminaries

In mathematics, the theory of special functions is an area that plays a vital role due to its applications in real analysis, functional analysis, geometry, physics, and many more subjects of science. Special functions can be defined by power series, generating functions, infinite products, and other series in orthogonal functions. In the past few years, many researchers and authors have been engaged in working on theory and applications of the special functions [1–8]. Inequalities and extensions are both important topics in the theory of special functions, but from a theoretical point of view, very few inequalities involving hypergeometric functions and extended hypergeometric functions seem to have appeared in the literature until now. Here, we aim to introduce some inequalities of extended hypergeometric functions.

Very recently, Goyal and Jain et al. [9,10] have extended the beta function, Gauss hypergeometric function, confluent hypergeometric function and studied various properties of these extended functions. They also studied the increasing or decreasing nature (monotonicity), log-concavity, and log-convexity of extended beta function in [10].

The extended beta function is defined as [9]:

$$B_{(s_1, s_2)}^{(s)}(w_1, w_2) = \int_0^1 t^{w_1-1} (1-t)^{w_2-1} E_{s_1, s_2} \left(-r(t(1-t))^{-1} \right) dt, \quad (1)$$

where $\min\{\Re(w_1), \Re(w_2)\} > 0$, $r \geq 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $E_{s_1, s_2}(z)$ is the 2-parameter Mittag–Leffler function.

The extended Gauss hypergeometric function is defined as [10]:

$$F_{(s_1, s_2)}^{(s)}(q_0, q_1, q_2; z) = \sum_{k=0}^{\infty} \frac{B_{(s_1, s_2)}^{(s)}(q_1 + k, q_2 - p_1)}{B(q_1, q_2 - q_1)} (q_0)_k \frac{z^k}{k!}, \quad (2)$$

where $\Re(q_2) > \Re(q_1) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$, $s \geq 0$, $|z| < 1$ and $B_{(s_1, s_2)}^{(s)}(w_1, w_2)$ is the extended beta function.

Integral representation of the extended Gauss hypergeometric function is defined as [10]:

$$\begin{aligned} F_{(s_1, s_2)}^{(s)}(q_0, q_1, q_2; z) = \\ \frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} (1-zt)^{-q_0} E_{s_1, s_2} \left(-s(t(1-t))^{-1} \right) dt, \end{aligned} \quad (3)$$

where ($\Re(q_2) > \Re(q_1) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$, $s \geq 0$ and $|z| < 1$).

$$\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; z) = \sum_{k=0}^{\infty} \frac{B_{(s_1, s_2)}^{(s)}(q_1 + k, q_2 - q_1)}{B(q_1, q_2 - q_1)} \frac{z^k}{k!}, \quad (4)$$

where ($\Re(q_2) > \Re(q_1) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $s \geq 0$).

Integral representation of the extended confluent hypergeometric function is defined as [10]:

$$\begin{aligned} \Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; z) = \\ \frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} e^{zt} E_{s_1, s_2} \left(-s(t(1-t))^{-1} \right) dt, \end{aligned} \quad (5)$$

where ($\Re(q_2) > \Re(q_1) > 0$, $\Re(s_1) > 0$, $\Re(s_2) > 0$ and $s \geq 0$).

We also require some important results that were published earlier in [10–17] to obtain our main results.

Theorem 1 ([10]). Assume that:

- x'', y'', x''_1, y''_1 are non-zero and non-negative numbers such that $(x'' - x''_1)(y'' - y''_1) \geq 0$,
- $u_1 \in [0, 1]$ and $u_2 \in [0, 1]$.

Then

$$B_{(u_1, u_2)}^{(u)}(x'', y''_1) \cdot B_{(u_1, u_2)}^{(u)}(x''_1, y'') \leq B_{(u_1, u_2)}^{(u)}(x'', y'') B_{(u_1, u_2)}^{(u)}(x''_1, y''_1). \quad (6)$$

Theorem 2 ([10]). The map $(x', y') \mapsto B_{(u_1, u_2)}^{(u)}(x', y')$ is logarithmically convex on $\mathbb{R}^+ \times \mathbb{R}^+$ $\forall u \geq 0$, with $u_1 \in [0, 1]$ and $u_2 \in [0, 1]$. Moreover:

$$\left[B_{(u_1, u_2)}^{(u)} \left(\frac{x'_1 + x'_2}{2}, \frac{y'_1 + y'_2}{2} \right) \right]^2 \leq B_{(u_1, u_2)}^{(u)}(x'_1, y'_1) \cdot B_{(u_1, u_2)}^{(u)}(x'_2, y'_2). \quad (7)$$

Lemma 1 ([11]). consider $f'_1(y) = \sum_{k=0}^{\infty} a'_k y^k$ and $g'_1(y) = \sum_{k=0}^{\infty} b'_k y^k$ with $a'_k \in R$ and $b'_k > 0$ $\forall k$, converges on $(-\beta, \beta)$. Then, if the sequence $(\frac{a'_k}{b'_k})_{k \geq 0}$ is decreasing (or increasing, respectively), subsequently the mapping $y \mapsto \frac{f'_1(y)}{g'_1(y)}$ is also decreasing (or increasing, respectively) on $(0, \beta)$.

Lemma 2 ([10,12,13] Hölder Inequality). Let θ'_1 and θ'_2 be positive numbers such that

$$\frac{1}{\theta'_1} + \frac{1}{\theta'_2} = 1.$$

Let $f'_1, f'_2 : [c', d'] \rightarrow (-\infty, +\infty)$ be integrable functions. Then,

$$\int_{c'}^{d'} f'_1(x) f'_2(x) dx \leq \left(\int_{c'}^{d'} f'_1(x)^{\theta'_1} dx \right)^{\frac{1}{\theta'_1}} \cdot \left(\int_{c'}^{d'} f'_2(x)^{\theta'_2} dx \right)^{\frac{1}{\theta'_2}}. \quad (8)$$

Lemma 3 ([10,14–16] Chebyshev's integral inequality). Let $f'_1, f'_2 : [c', d'] \subseteq (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ be integrable functions. Assume that:

$$[f'_1(x) - f'_1(y)][f'_2(x) - f'_2(y)] \geqslant 0, \quad \forall x', y' \in [c', d'].$$

Let $h'(x) : [c', d'] \subseteq (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ be a positive integrable function. Then:

$$\int_{c'}^{d'} h'(x) f'_1(x) dx \cdot \int_{c'}^{d'} h'(x) f'_2(x) dx \geqslant \int_{c'}^{d'} h'(x) dx \cdot \int_{c'}^{d'} h'(x) f'_1(x) f'_2(x) dx. \quad (9)$$

Theorem 3 ([17]). Let

$$f(b, y) = \sum_{k \geq 0} f'_k \frac{(b)_k}{k!} y^k, \quad (10)$$

where, f'_k is not depend on b , and we consider that $c > b > 0$ and $\delta > 0$, then the mapping $f(b + \delta, y)f(c, y) - f(c + \delta, y)f(b, y) = \sum_{m' \geq 2} \phi'_{m'} y^m$ has negative power series coefficient $\phi'_{m'} < 0$, so that $b \mapsto f(b, y)$ is strictly log-convex for $y > 0$ if the sequence $(\frac{f'_k}{f'_{k-1}})$ is increasing.

2. Main Results

In this section, we are introducing some inequalities, monotonicity, log-convexity and log-concavity of the extended confluent hypergeometric function $\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)$ and extended Gauss hypergeometric function $F_{(s_1, s_2)}^{(s)}(q_0, q_1, q_2; x)$, respectively.

2.1. Inequalities of Confluent Hypergeometric Function $\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)$

Theorem 4. Consider $q_1 \geq 0$ and $q_2, q_3 > 0$ then for $q_2 - q_3 \geq 0$ the mapping

$$x \mapsto \frac{\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)}{\Phi_{(s_1, s_2)}^{(s)}(q_1, q_3; x)}, \quad (11)$$

is increasing on $(0, \infty)$.

Proof of Theorem 4. By the definition of the extended confluent hypergeometric function (4), we have:

$$\frac{\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)}{\Phi_{(s_1, s_2)}^{(s)}(q_1, q_3; x)} = \frac{\sum_{k=0}^{\infty} \frac{B_{(s_1, s_2)}^{(s)}(q_1+k, q_2-q_1)}{B(q_1, q_2-q_1)} x^k}{\sum_{k=0}^{\infty} \frac{B_{(s_1, s_2)}^{(s)}(q_1+k, q_3-q_1)}{B(q_1, q_3-q_1)} x^k}. \quad (12)$$

Now, consider $a'_k(z) = \frac{B_{(s_1,s_2)}^{(s)}(q_1+k,z-q_1)}{B(q_1,z-q_1)}$ then, we have:

$$\frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)} = \frac{\sum_{k=0}^{\infty} a'_k(q_2)x^k}{\sum_{k=0}^{\infty} a'_k(q_3)x^k}. \quad (13)$$

Now assume, $f_k = \frac{a'_k(q_2)}{a'_k(q_3)}$, then

$$f_k - f_{k+1} = \frac{a'_k(q_2)}{a'_k(q_3)} - \frac{a'_{k+1}(q_2)}{a'_{k+1}(q_3)}. \quad (14)$$

Then, by the using above equation, we have:

$$f_k - f_{k+1} = \frac{B(q_1,q_3-q_1)}{B(q_1,q_2-q_1)} \left[\frac{B_{(s_1,s_2)}^{(s)}(q_1+k,q_2-q_1)}{B_{(s_1,s_2)}^{(s)}(q_1+k,q_3-q_1)} - \frac{B_{(s_1,s_2)}^{(s)}(q_1+k+1,q_2-q_1)}{B_{(s_1,s_2)}^{(s)}(q_1+k+1,q_3-q_1)} \right]. \quad (15)$$

On taking $x' = q_1 + n$, $x'_1 = q_1 + k + 1$, $y' = q_3 - q_1$ and $y'_1 = q_2 - q_1$, we get: $(x' - x'_1).(y' - y'_1) \geq 0$ as $q_2 - q_3 \geq 0$

So, the above equation reduces to

$$f_k - f_{k+1} = \frac{B(x' - k, y')}{B(x' - k, y'_1)} \left[\frac{B_{(s_1,s_2)}^{(s)}(x', y'_1)}{B_{(s_1,s_2)}^{(s)}(x', y')} - \frac{B_{(s_1,s_2)}^{(s)}(x'_1, y'_1)}{B_{(s_1,s_2)}^{(s)}(x'_1, y')} \right]. \quad (16)$$

After using the Theorem (1), we have:

$$\frac{B_{(s_1,s_2)}^{(s)}(x', y'_1)B_{(s_1,s_2)}^{(s)}(x'_1, y')}{B_{(s_1,s_2)}^{(s)}(x'_1, y'_1)B_{(s_1,s_2)}^{(s)}(x', y')} \leq 1. \quad (17)$$

After re-arranging the terms, we have:

$$\frac{B_{(s_1,s_2)}^{(s)}(x', y'_1)}{B_{(s_1,s_2)}^{(s)}(x', y')} - \frac{B_{(s_1,s_2)}^{(s)}(x'_1, y'_1)}{B_{(s_1,s_2)}^{(s)}(x'_1, y')} \leq 0. \quad (18)$$

Then, using the above result in Equation (16), we have:

$$f_k - f_{k+1} \leq 0. \quad (19)$$

Which implies $f_k \leq f_{k+1}$ that means $\{f_k\}_{k \geq 0}$ is non-decreasing, non-constant sequence, i.e., increasing sequence. After the use of Lemma (1), we conclude that $x \mapsto \frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)}$ is increasing on $(0, \infty)$. Hence the proof of Theorem (4), is completed. \square

Theorem 5. For $q_2 - q_3 \geq 0$ the extended confluent hypergeometric function $\Phi_{(s_1,s_2)}^{(s)}(q_1, q_2; x)$ satisfied the inequality

$$q_3 \Phi_{(s_1,s_2)}^{(s)}(q_1 + 1, q_2 + 1; x) \Phi_{(s_1,s_2)}^{(s)}(q_1, q_3; x) \geq q_2 \Phi_{(s_1,s_2)}^{(s)}(q_1 + 1, q_3 + 1; x) \Phi_{(s_1,s_2)}^{(s)}(q_1, q_2; x). \quad (20)$$

Proof of Theorem 5. From the previous Theorem (4), we know that $x \mapsto \frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)}$ is increasing on $(0,\infty)$ for $q_2 - q_3 \geq 0$, then by the increasing property of the function, we have:

$$\frac{d}{dx} \left[\frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)} \right] \geq 0, \quad (21)$$

on $(0,\infty)$.

After differentiating the function $\frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)}$ w.r.t x , we get:

$$\begin{aligned} & \frac{\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x) \left(\frac{q_1}{q_2} \right) \Phi_{(s_1,s_2)}^{(s)}(q_1+1,q_2+1;x)}{[\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)]^2} - \\ & \frac{\left(\frac{q_1}{q_3} \right) \Phi_{(s_1,s_2)}^{(s)}(q_1+1,q_3+1;x) \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x)}{[\Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x)]^2} \geq 0. \end{aligned} \quad (22)$$

On some computation, we get our desired result.

$$q_3 \Phi_{(s_1,s_2)}^{(s)}(q_1+1,q_2+1;x) \Phi_{(s_1,s_2)}^{(s)}(q_1,q_3;x) \geq q_2 \Phi_{(s_1,s_2)}^{(s)}(q_1+1,q_3+1;x) \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x).$$

Hence proof of Theorem (5), is completed. \square

Theorem 6. For $\beta \in [0, 1]$, $x', y' > 0$, and $q_1, q_2 > 0$ the mapping

$$x' \mapsto \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x') \quad (23)$$

is logarithmically convex on $(-\infty, \infty)$ and the extended confluent hypergeometric function $\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x')$ satisfies the inequalities

$$\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;\beta x' + (1-\beta)y') \leq \left[\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x') \right]^\beta \left[\Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;x') \right]^{(1-\beta)}. \quad (24)$$

Proof of Theorem 6. By the integral representation of extended hypergeometric function (5), we have:

$$\begin{aligned} & \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;\beta x' + (1-\beta)y') = \\ & \frac{1}{B(q_1,q_2-q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} e^{(\beta x' + (1-\beta)y')t} E_{s_1,s_2}(-s(t(1-t))^{-1}) dt. \end{aligned} \quad (25)$$

After re-arranging the terms, we get:

$$\begin{aligned} & \Phi_{(s_1,s_2)}^{(s)}(q_1,q_2;\beta x' + (1-\beta)y') = \\ & \frac{1}{B(q_1,q_2-q_1)} \int_0^1 \left[\left(t^{q_1-1} (1-t)^{q_2-q_1-1} e^{x't} E_{s_1,s_2}(-s(t(1-t))^{-1}) \right)^\beta \times \right. \\ & \left. \times \left(t^{q_1-1} (1-t)^{q_2-q_1-1} e^{y't} E_{s_1,s_2}(-s(t(1-t))^{-1}) \right)^{(1-\beta)} \right] dt. \end{aligned} \quad (26)$$

Then, from Lemma (2), we have:

$$\begin{aligned} \Phi_{(s_1, s_2)}^{(s)}(s_1, s_2; \beta x' + (1 - \beta)y') &\leq \\ &\left[\frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} e^{x't} E_{s_1, s_2}(-s(t(1-t))^{-1}) dt \right]^\beta \times \quad (27) \\ &\times \left[\frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} e^{y't} E_{s_1, s_2}(-s(t(1-t))^{-1}) dt \right]^{(1-\beta)}. \end{aligned}$$

After using the integral representation of the extended confluent hypergeometric function (5), we get our desired result and from above observations, we conclude that the extended confluent hypergeometric function is logarithmically convex on $(-\infty, \infty)$.

$$\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; \beta x' + (1 - \beta)y') \leq \left[\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x') \right]^\beta \left[\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x') \right]^{(1-\beta)}. \quad (28)$$

Hence, the Theorem (6), is proved. \square

Theorem 7. For $x, q, q_2 > 0$, then the mapping

$$q_1 \mapsto \frac{B(q_1 + q, q_2 - q_1 - q) \Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1) \Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)} \quad (29)$$

is decreasing function on $(0, \infty)$.

Proof of Theorem 7. Consider $q'_1 \geq q_1$,

$$h'_1(t) = t^{q'_1-1} (1-t)^{q_2-q'_1-1} e^{xt} E_{s_1, s_2}(-s(t(1-t))^{-1})$$

$$f'_1(t) = (\frac{t}{1-t})^{q_1-q'_1} \text{ and } g'_1(t) = (\frac{t}{1-t})^q$$

Then, from integral representation of the extended confluent hypergeometric function (5), we have:

$$\begin{aligned} \Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x) &= \\ &\frac{1}{B(q_1, q_2 - q_1)} \int_0^1 t^{q_1-1} (1-t)^{q_2-q_1-1} e^{xt} E_{s_1, s_2}(-s(t(1-t))^{-1}) dt. \quad (30) \end{aligned}$$

After using the above definition, we have:

$$\begin{aligned} \frac{B(q_1 + q, q_2 - q_1 - q) \Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1) \Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)} - \\ \frac{B(q'_1 + q, q_2 - q'_1 - q) \Phi_{(s_1, s_2)}^{(s)}(q'_1 + q, q_2; x)}{B(q'_1, q_2 - q'_1) \Phi_{(s_1, s_2)}^{(s)}(q'_1, q_2; x)} &= \quad (31) \\ \frac{\int_0^1 f'_1(t) g'_1(t) h'_1(t) dt}{\int_0^1 f'_1(t) h'_1(t) dt} - \frac{\int_0^1 g'_1(t) h'_1(t) dt}{\int_0^1 h'_1(t) dt}. \end{aligned}$$

For the conditions $q \geq 0$ and $q'_1 - q_1 \geq 0$, we can easily determine that the function $f'_1(t)$ is decreasing and the function $g'_1(t)$ is increasing, since function $h'_1(t)$ is a non-negative mapping for $t \in [0, 1]$.

After by the using Lemma (3), we have:

$$\int_0^1 h'_1(t)f'_1(t)dt \cdot \int_0^1 g'_1(t)h'_1(t)dt \leq \int_0^1 h'_1(t)dt \cdot \int_0^1 h'_1(t)f'_1(t)g'_1(t)dt.$$

Then, after re-arranging the terms, we have:

$$\frac{\int_0^1 f'_1(t)g'_1(t)h'_1(t)dt}{\int_0^1 f'_1(t)h'_1(t)dt} \geq \frac{\int_0^1 g'_1(t)h'_1(t)dt}{\int_0^1 h'_1(t)dt}.$$

On some more calculations, we get:

$$\frac{\int_0^1 f'_1(t)g'_1(t)h'_1(t)dt}{\int_0^1 f'_1(t)h'_1(t)dt} - \frac{\int_0^1 g'_1(t)h'_1(t)dt}{\int_0^1 h'_1(t)dt} \geq 0.$$

On using above relation in Equation (31), we have:

$$\begin{aligned} & \frac{B(q_1 + q, q_2 - q_1 - q)\Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1)\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)} - \\ & \frac{B(q'_1 + q, q_2 - q'_1 - q)\Phi_{(s_1, s_2)}^{(s)}(q'_1 + q, q_2; x)}{B(q'_1, q_2 - q'_1)\Phi_{(s_1, s_2)}^{(s)}(q'_1, q_2; x)} \geq 0. \end{aligned} \quad (32)$$

After some calculations, we get:

$$\begin{aligned} & \frac{B(q_1 + q, q_2 - q_1 - q)\Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1)\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)} \geq \\ & \frac{B(q'_1 + q, q_2 - q'_1 - q)\Phi_{(s_1, s_2)}^{(s)}(q'_1 + q, q_2; x)}{B(q'_1, q_2 - q'_1)\Phi_{(s_1, s_2)}^{(s)}(q'_1, q_2; x)}. \end{aligned} \quad (33)$$

As $q'_1 \geq q_1$, we conclude that the mapping $q_1 \mapsto \frac{B(q_1 + q, q_2 - q_1 - q)\Phi_{(q_1, q_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1)\Phi_{(q_1, q_2)}^{(r)}(q_1, q_2; x)}$ is decreasing function on $(0, \infty)$.

Hence, proof of Theorem (7), is completed. \square

Remark 1. In particular, the following decreasing property of the function

$$q_1 \mapsto \frac{B(q_1 + q, q_2 - q_1 - q)\Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x)}{B(q_1, q_2 - q_1)\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)} \quad (34)$$

is equivalent to the inequality

$$\begin{aligned} & \left[\Phi_{(s_1, s_2)}^{(s)}(q_1 + q, q_2; x) \right]^2 \geq \\ & \frac{B(q_1 + 2q, q_2 - q_1 - 2q)B(q_1, q_2 - q_1)\Phi_{(s_1, s_2)}^{(s)}(q_1 + 2q, q_2; x)\Phi_{(s_1, s_2)}^{(s)}(q_1, q_2; x)}{\left[B(q_1 + q, q_2 - q_1 - q) \right]^2}. \end{aligned} \quad (35)$$

2.2. Inequalities of Extended Gauss Hypergeometric Function $F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x)$

Theorem 8. Let $q_1 \geq 0$ and $q_2, q_3 > 0$, then for $q_2 - q_3 \geq 0$, the mapping

$$x \mapsto \frac{F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x)}{F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_3; x)}, \quad (36)$$

is increasing on $(0, 1)$.

Proof of Theorem 8. By the similar procedure as used in the proof of Theorem (4), with some computation, we get our desired result of Theorem (8). \square

Theorem 9. For $q_2 - q_3 \geq 0$, the extended Gauss hypergeometric function $F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x)$ satisfied the inequality

$$\begin{aligned} q_3 F_{(s_1,s_2)}^{(s)}(q_0 + 1, q_1 + 1, q_2 + 1; x) F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_3; x) \geq \\ q_2 F_{(s_1,s_2)}^{(s)}(q_0 + 1, q_1 + 1, q_3 + 1; x) F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x). \end{aligned} \quad (37)$$

Proof of Theorem 9. On the same parallel lines as used in the proof of Theorem (5), after some computation, we get our desired result of Theorem (9). \square

Theorem 10. For $x \in (0, 1)$, the mapping

$$q_0 \mapsto F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x) \quad (38)$$

is logarithmically convex on $(0, \infty)$

Proof of Theorem 10. To prove above result, applying Theorem (3), to the extended Gauss hypergeometric function $F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; x)$.

From definition of the extended Gauss hypergeometric function (2) we have:

$$F_{(s_1,s_2)}^{(s)}(q_0, q_1, q_2; z) = \sum_{k=0}^{\infty} \frac{B_{(s_1,s_2)}^{(s)}(q_1 + k, q_2 - p_1)}{B(q_1, q_2 - q_1)} (q_0)_k \frac{z^k}{k!}.$$

Now consider $f'_k = \frac{B_{(s_1,s_2)}^{(s)}(q_1 + k, q_2 - q_1)}{B(q_1, q_2)}$, to prove our result it sufficient to show that the sequence $d_k = (\frac{f'_k}{f'_{k-1}})$ is increasing. Clearly,

$$d_k - d_{k-1} = \frac{B_{(s_1,s_2)}^{(s)}(q_1 + k, q_2 - q_1)}{B_{(s_1,s_2)}^{(s)}(q_1 + k - 1, q_2 - q_1)} - \frac{B_{(s_1,s_2)}^{(s)}(q_1 + k - 1, q_2 - q_1)}{B_{(s_1,s_2)}^{(s)}(q_1 + k - 2, q_2 - q_1)}. \quad (39)$$

On using the Theorem (2), and letting $x'_1 = q_1 + k, x'_2 = q_1 + k - 2, y'_1 = y'_2 = q_2 - q_1$, we have:

$$\begin{aligned} [B_{(s_1,s_2)}^{(s)}(q_1 + k - 1, q_2 - q_1)]^2 \\ \leq B_{(s_1,s_2)}^{(s)}(q_1 + k, q_2 - q_1) \cdot B_{(s_1,s_2)}^{(s)}(q_1 + k - 2, q_2 - q_1). \end{aligned} \quad (40)$$

After re-arranging the terms, we get:

$$\frac{B_{(s_1,s_2)}^{(s)}(q_1 + k, q_2 - q_1)}{B_{(s_1,s_2)}^{(s)}(q_1 + k - 1, q_2 - q_1)} \geq \frac{B_{(s_1,s_2)}^{(s)}(q_1 + k - 1, q_2 - q_1)}{B_{(s_1,s_2)}^{(s)}(q_1 + k - 2, q_2 - q_1)}. \quad (41)$$

Then, by using above result in Equation (39), we have: $d_k \geq d_{k-1}$, which implies (d_k) sequence is increasing. Now, we conclude by the using Theorem (3), that the mapping $q_0 \mapsto F_{(s_1, s_2)}^{(s)}(q_0, q_1, q_2; x)$ is logarithmically convex on $(0, \infty)$. Hence, proof of the Theorem (10), is completed. \square

3. Concluding Remark

We conclude our investigation by remarking that here, we describe some (presumably) new inequalities including the extended type Gauss hypergeometric function and confluent hypergeometric function, respectively. These inequalities are important for the approximation of extended confluent hypergeometric function, extended Gauss hypergeometric function, generalized Appell and Lauricella hypergeometric functions. We hope our investigation is capable of providing potential directions for future research in the approximation theory and applications of special functions.

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