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On Andrews' Partitions with Parts Separated by Parity

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Abstract: In this paper, we present a generalization of one of the theorems in Partitions with parts separated by parity introduced by George E. Andrews, and give its bijective proof. Further variations of related partition functions are studied resulting in a number of interesting identities.

Keywords: partitions; parity; generating functions; bijection

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1. Introduction, Definitions, Notation

Parity in partitions has played a useful role. A partition of an integer $n > 0$ is a representation $(\lambda_1, \lambda_2, \ldots, \ldots)$ where $\lambda_i \geq \lambda_{i+1}$ for all i and $\sum_{j\geq 1} \lambda_j = n$. The integer n is called the weight of the partition. However when further restrictions are imposed on the parts λ_i's, we get restricted partition functions. One such is the number of partitions into distinct parts. This means each part in a partition occurs only once. Parity of this partition function is known, and several authors, including Andrews [1] have delved into a broader subject, where parity affects parts of partitions. There are various resources on the theory of integer partitions, and the interested reader is referred to [2]. On this specific subject, one may consult [1], and citations listed in [3].

Definition 1. *Consider a partition λ of n. Suppose $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_\ell^{m_\ell})$ where m_i is the multiplicity of λ_i and $\lambda_1 > \lambda_2 > \ldots > \lambda_\ell$. Define another partition λ' whose jth part is given by*

$$\lambda'_j = \left(\sum_{i=1}^{\ell-j+1} m_i \right)^{\lambda_{\ell-j+1} - \lambda_{\ell-j+2}}, \qquad \text{where } \lambda_{\ell+1} := 0.$$

The partition λ' is called the conjugate of λ and has weight n.

Given two partitions λ and μ, we consider the union $\lambda \cup \mu$ to be the multiset union, and $\lambda + \mu$ is the sum of two partitions obtained via vector addition in which the i^{th} largest part of $\lambda + \mu$ is equal to the sum of the i^{th} largest parts in λ and μ. In finding the sum $\lambda + \mu$, the partition with smaller length must have zeros appended to it in order to match in length with the other partition. Similar rules apply to computing $\lambda - \mu$.

Suppose μ is a subpartition of λ. We define a new partition $sub(\lambda, \mu)$ to be a partition obtained by deleting μ from λ. For instance $sub((8, 7^2, 6^3, 2^3), (7, 6^3, 2)) = (8, 7, 2^2)$. Further, $L_k(\lambda)$ is the partition obtained by multiplying k to each part of λ whose multiplicity is divisible by k and dividing its multiplicity by k. On the other hand, $L_k^{-1}(\lambda)$ is obtained by dividing by k each part divisible by k and multiplying its multiplicity by k.

For q-series, we use the following standard notation:

$$(a;q)_n = \prod_{i=1}^{n-1}(1 - aq^i), \qquad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Some q-identities which will be useful are recalled as follows:

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty} (1-aq^{2n-1})(1+q^n), \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \prod_{n=1}^{\infty} \frac{(1+q^{8n-3})(1+q^{8n-5})(1-q^{8n})}{1-q^{2n}}, \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_n (z;q)_n}{(q;q)_n (az;q)_n} b^n, \quad |z|<1, |b|<1, |q|<1. \tag{3}$$

For proof of the above identities, see [2,4,5], respectively. Euler discovered the following theorem.

Theorem 1 (Euler, [2]). *The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.*

This theorem has an interesting bijective proof supplied by J .W. L Glaisher (see [6]). We shall denote Glaisher's map by ϕ. In fact ϕ converts a partition into odd parts to a partition into disctinct parts.

Let $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_r^{m_r})$ be a partition of n whose parts are odd. Note that the notation for λ implies $\lambda_1 > \lambda_2 > \ldots$ are parts with multiplicities $m_1, m_2, \ldots$, respectively.

Now, write m_i's in k-ary expansion, i.e.,

$$m_i = \sum_{j=0}^{l_i} a_{ij} 2^j \quad \text{where} \quad 0 \le a_{ij} \le 1.$$

We map $\lambda_i^{m_i}$ to $\bigcup_{j=0}^{l_i} (2^j \lambda_i)^{a_{ij}}$, where now $2^j \lambda_i$ is a part with multiplicity a_{ij}. The image of λ which we shall denote by $\phi(\lambda)$, is given by

$$\bigcup_{i=1}^{r} \bigcup_{j=0}^{l_i} (2^j \lambda_i)^{a_{ij}}.$$

Clearly, this is a partition of n with distinct parts.

On the other hand, assume that $\mu = (\mu_1^{f_1}, \mu_2^{f_2}, \ldots)$ is a partition of n into ditinct parts. Write $\mu_i = 2^{r_i} \cdot a_i$ where $2 \nmid a_i$ and then map $\mu_i^{f_i}$ to $(a_i)^{2^{r_i} f_i}$ for each i, where now a_i is a part with multiplicity $2^{r_i} f_i$. The inverse of ϕ is then given by

$$\phi^{-1}(\mu) = \bigcup_{i \ge 1} (a_i)^{2^{r_i} f_i}.$$

In the resulting partition, it is also clear that the parts are odd.

We also recall the following notation from [3].

$p_{eu}^{od}(n)$: the number of partitions of n in which odd parts are distinct and greater than even parts.

$\mathcal{O}_d(n)$: the number of partitions of n in which the odd parts are distinct and each odd integer smaller than the largest odd part must appear as a part. Theorem 2 of [3] is restated below.

Theorem 2 (Andrews, [3]). *For $n \ge 0$, we have*

$$p_{eu}^{od}(n) = \mathcal{O}_d(n).$$

In this paper, we generalise Theorem 2 and look at various variations.

2. A Generalisation of Theorem 2

Define $D(n,p,r)$ to be the number of partitions of n in which parts are congruent to $0, r \pmod p$, and each part congruent to $r \pmod p$ is distinct and greater than parts congruent to $0 \pmod p$. Our theorem is stated below.

Theorem 3. *Let $O(n,p,r)$ be the number of partitions of n in which parts are congruent to $0, r \pmod p$, parts $\equiv r \pmod p$ are distinct, and each integer congruent to $r \pmod p$ smaller than the largest part that is congruent to $r \pmod p$ must appear as a part. Then,*

$$D(n,p,r) = O(n,p,r).$$

Proof. Setting $p = 2, r = 1$ in Theorem 3 gives rise to Theorem 2. We give a desired bijective proof.

Let λ be enumerated by $O(n,p,r)$. We have the decomposition $\lambda = (\lambda_1, \lambda_2)$ where λ_1 is the subpartition of λ whose parts are $\equiv r \pmod p$, and λ_2 is the subpartition of λ whose parts are congruent to $0 \pmod p$. Then, the image is given by $\lambda_1 + \lambda_2$, i.e.,

$$\lambda \mapsto \lambda_1 + \lambda_2.$$

The inverse of the bijection is given as follows:

Let μ be a partition enumerated by $D(n,p,r)$. Then, decompose μ as $\mu = (\mu_1, \mu_2)$ where μ_1 is the subpartition with parts congruent to $r \pmod p$ and μ_2 is the subpartition with parts congruent to $0 \pmod p$. Construct μ_3 as

$$\mu_3 = (p\ell(\mu_1) - p + r, p\ell(\mu_1) - 2p + r, p\ell(\mu_1) - 3p + r, \ldots, r + 2p, r + p, r)$$

where $\ell(\mu_1)$ is the number of parts in μ_1.

Then the image of μ is given by

$$\mu \mapsto \mu_2 \cup \mu_3 \cup [\mu_1 - \mu_3].$$

□

Example 1. *Consider $p = 4, r = 1$ and an $O(190,4,1)$-partition*
$\lambda = (32, 32, 21, 17, 16, 13, 9, 8, 8, 8, 8, 5, 4, 4, 4, 1)$.

By our mapping, λ decomposes as follows:

$$\lambda = ((21, 17, 13, 9, 5, 1), (32, 32, 16, 8, 8, 8, 8, 4, 4, 4)).$$

The image is then given by

$$(21, 17, 13, 9, 5, 1, 0, 0, 0, 0) + (32, 32, 16, 8, 8, 8, 8, 4, 4, 4)$$

(we append zeros to the subpartition with smaller length), and addition is componentwise in the order demonstrated. Thus

$$\lambda \mapsto (53, 49, 29, 17, 13, 9, 8, 4, 4, 4)$$

which is a partition enumerated by $D(190,4,1)$.

To invert the process, starting with $\mu = (53, 49, 29, 17, 13, 9, 8, 4, 4, 4)$, enumerated by $D(190,4,1)$, we have the decomposition $\mu = (\mu_1, \mu_2) = ((53, 49, 29, 17, 13, 9), (8, 4, 4, 4))$ where $\mu_1 = (53, 49, 29, 17, 9)$ and $\mu_2 = (8, 4, 4, 4)$.

Note that $\ell(\mu_1) = 5$ so that $\mu_3 = (17, 13, 9, 5, 1)$. Hence, the image is

$$\begin{aligned}\mu_2 \cup \mu_3 \cup [\mu_1 - \mu_3] &= (8,4,4,4) \cup (21,17,13,9,5,1) \cup [(53,49,29,17,13,9) - (21,17,13,9,5,1)] \\ &= (8,4,4,4) \cup (21,17,13,9,5,1) \cup (32,32,16,8,8,8) \\ &= (32,32,21,17,16,13,9,8,8,8,8,5,4,4,4,1),\end{aligned}$$

which is enumerated by $O(186,4,1)$ and the λ we started with.

Corollary 1. *The number of partitions of n in which all parts $\not\equiv 0 \pmod{p}$ form an arithmetic progression with common difference p and the smallest part is less than p equals the number of partitions of n in which parts $\not\equiv 0 \pmod{p}$ are distinct, have the same residue modulo p and are greater than parts $\equiv 0 \pmod{p}$.*

Proof. By Theorem 3, we have $\sum_{r=1}^{p-1} O(n,p,r) = \sum_{r=1}^{p-1} D(n,p,r)$. □

3. Related Variations

In Theorem 2, if we reverse the roles of odd and even parts by letting any positive even integer less than the largest even part appear as a part and each odd part be greater than the largest even part, we obtain the following theorem.

Theorem 4. *Let $r = 1,3$ and $A(n,r)$ denote the number of partitions of n in which each even integer less than the largest even part appears as a part and the smallest odd part is at least $r +$ the largest even part. Then, $A(n,r)$ is equal to the number of partitions of n with parts $\equiv r,2 \pmod{4}$.*

Proof.

$$\sum_{n=0}^{\infty} A(n,r)q^n = \frac{1}{\prod_{j=0}^{\infty}(1-q^{2j+r})} + \sum_{n=1}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{\prod_{j=n}^{\infty}(1-q^{2j+r})}$$

$$\begin{aligned}&= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{\prod_{j=n}^{\infty}(1-q^{2j+r})} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{1}{(q^{2n+r};q^2)_\infty} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} \frac{(q^r;q^2)_n}{(q;q^2)_\infty} \\ &= \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q^2)_n} (q^r;q^2)_n \\ &= \frac{1}{(q;q^2)_\infty} \prod_{n=1}^{\infty} (1-q^{4n-r})(1+q^{2n}) \text{ (by (1))} \\ &= \frac{\prod_{n=1}^{\infty}(1-q^{4n-r})(1+q^{2n})(1-q^{2n})}{(q;q^2)_\infty (q^2;q^2)_\infty} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^{4n-r})(1-q^{4n})}{1-q^n} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{4n+r})(1-q^{4n+2})}.\end{aligned}$$

The Bijective Proof

Let μ be a partition enumerated by $A(n,r)$. Execute the following steps:

1. Conjugate μ, obtaining μ'.
2. If μ' has no part with odd multiplicity, set $\bar{\alpha} := \mu'$ and go to step 4. Otherwise, decompose $\mu' = (\alpha, \beta)$ where
   β is the subpartition of μ' consisting of all parts less than or equal to the largest part that has odd multiplicity and α is the subpartition $sub(\mu', \beta)$. Recall that β can be written as
   $$\beta = \langle \beta_1^{m_{\beta_1}(\beta)} \beta_2^{m_{\beta_2}(\beta)} \dots \beta_m^{m_{\beta_m}(\beta)} \rangle$$
   where $\beta_1 > \beta_2 > \ldots > \beta_m$. We use this notation of β in the next step.
3. a. If $m_{\beta_1}(\beta) \not\equiv r \pmod 4$, then update α and β as follows:
   $$\beta := sub(\beta, \langle \beta_1^2 \rangle) \ , \alpha := \alpha \cup \langle \beta_1^2 \rangle.$$
   b. For $j = 2, 3, \ldots, m$, if $m_{\beta_j}(\beta) \not\equiv 0 \pmod 4$, then update α and β as follows:
   $$\beta := sub(\beta, \langle \beta_j^2 \rangle) \ , \alpha := \alpha \cup \langle \beta_j^2 \rangle.$$
   Now call the new updated α and β, $\bar{\alpha}$ and $\bar{\beta}$, respectively. Observe that $\mu' = \bar{\alpha} \cup \bar{\beta}$.
4. Compute
   $$\gamma = L_2(\phi(L_2(\bar{\alpha}))).$$
   Note that
   $$\lambda = \bar{\beta}' \cup \gamma$$
   is a partition into parts $\equiv r, 2 \pmod 4$.

Before giving the inverse mapping, let us look at an example.

The inverse

Let λ be a partition of n into parts $\equiv r, 2 \pmod 4$. Decompose λ as follows $\lambda = \lambda_1 \cup \lambda_r$ where λ_1 is the subpartition of λ with parts $\equiv 2 \pmod 4$ and λ_r is the subpartition with parts $\equiv r \pmod 4$. Compute

$$h = L_2^{-1}(\phi^{-1}(L_2^{-1}(\lambda_1))).$$

Then

$$\mu = \lambda_r + h'$$

is a partition in $A(n,r)$.

Example

Let $r = 1$ with $\mu = \langle 23^2 17^1 11^3 8^1 6^3 4^1 2^4 \rangle \in A(124, 1)$. Then, $\mu' = \langle 15^2 11^2 10^2 7^2 6^3 3^6 2^6 \rangle$. Thus $\alpha = 15^2 11^2 10^2 7^2$ and $\beta = 6^3 3^6 2^6$. Updating α and β yields: $\bar{\alpha} = \langle 15^2 11^2 10^2 7^2 6^2 3^2 2^2 \rangle$ and $\bar{\beta} = \langle 6^1 3^4 2^4 \rangle$.

Now we have $L_2(\bar{\alpha}) = \langle 30, 22, 20, 14, 12, 6, 4 \rangle$ so that $\phi(L_2(\bar{\alpha})) = \langle 15^2 11^2 7^2 5^4 3^6 1^4 \rangle$. Thus

$$\gamma = L_2(\phi(L_2(\bar{\alpha}))) = \langle 30^1 22^1 14^1 10^2 6^3 2^2 \rangle.$$

Since $\bar{\beta}' = \langle 9^2 5^1 1^3 \rangle$, the image is

$$\lambda = \bar{\beta}' \cup \gamma = \langle 30^1 22^1 14^1 10^2 9^2 6^3 5^1 2^2 1^3 \rangle.$$

To find the inverse, consider $\lambda = \langle 30^1 22^1 14^1 10^2 9^2 6^3 5^1 2^2 1^3 \rangle$ in the example above ($r = 1$). Then, $\lambda_1 = \langle 30^1 22^1 14^1 10^2 6^3 2^2 \rangle$ and $\lambda_r = \langle 9^2 5^1 1^3 \rangle$.

Now $L_2^{-1}(\lambda_1) = \langle 15^2 11^2 7^2 5^4 3^6 1^4 \rangle$ so that $\phi^{-1}(L_2^{-1}(\lambda_1)) = \langle 30, 22, 20, 14, 12, 6, 4 \rangle$.

Thus $h = L_2^{-1}(\langle 30, 22, 20, 14, 12, 6, 4 \rangle) = \langle 15^2 11^2 10^2 7^2 6^2 3^2 2^2 \rangle$ and that $h' = \langle 14^2 12^1 10^3 8^1 6^3 4^1 2^4 \rangle$. Hence,

$$\mu = \langle 9^2 5^1 1^3 \rangle + \langle 14^2 12^1 10^3 8^1 6^3 4^1 2^4 \rangle = \langle 23^2 17^1 11^3 8^1 6^3 4^1 2^4 \rangle.$$

□

Theorem 5. *Let $C(n)$ be the number of partitions where if $2j$ occurs, then all even integers less than $2j$ occur as parts and any part greater than $2j$ is odd. Then, $C(n) \equiv 1 \pmod 2$ if and only if $n = \frac{j(j+1)}{2}$ for some $j \geq 0$.*

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} C(n) q^n &= \sum_{n=0}^{\infty} \frac{q^{2+4+6+\ldots+2n}}{(1-q)(1-q^2)\ldots(1-q^{2n})(1-q^{2n+1})(1-q^{2n+3})\ldots} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}(q^{2n+1};q^2)_{\infty}} \\
&= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} \frac{(q;q^2)_n}{(q;q^2)_{\infty}} \\
&= \frac{1}{(q;q^2)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2;q^2)_n} \\
&= \frac{1}{(q;q^2)_{\infty}} \prod_{n=1}^{\infty} (1+q^{2n}) \text{ (by (1), } a = 0, q := q^2) \\
&= \prod_{n=1}^{\infty} \frac{1-q^{4n}}{1-q^n} \\
&\equiv \prod_{n=1}^{\infty} (1-q^n)^3 \pmod 2 \\
&\equiv \sum_{n=0}^{\infty} q^{n(n+1)/2} \pmod 2.
\end{aligned}$$

□

Remark 1. *It is clearly observable from line 6 of the proof that $C(n)$ is equal to the number of partitions of n into parts not divisible by 4. To prove this partition identity combinatorially, decompose $\lambda \in C(n)$ into (λ_o, λ_e) where λ_o is the subpartition consisting of odd parts, and λ_e is the subpartition consisting of even parts. Then compute $\phi(\lambda_o)$ and conjugate λ_e. Split each part of λ_e' into two identical parts, obtaining μ. Then,*

$$\phi(\phi(\lambda_o) \cup \mu)$$

is a partition in which parts are not divisible by 4. This transformation is invertible.

Theorem 6. *Let $B(n)$ be the number of partitions of n in which either (a) all parts are even and distinct or (b) 1 must appear and odd parts appear without gaps, even parts are distinct and each is greater than or equal to 3 + the largest odd part. Denote by $B_e(n)$ (resp. $B_o(n)$), the number of $B(n)$-partitions with an even (resp. odd) number of even parts. Then*

$$B_e(n) - B_o(n) = \begin{cases} 1 & \text{if } n = m(4m \pm 1), m \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Note that the generating function for the sequence $B(0), B(1), B(2), \ldots$ is

$$\frac{1}{(q^2;q^2)_\infty} + \sum_{n=1}^{\infty} \frac{q^{2(1+3+5+\ldots+2n-1)}}{(q;q^2)_n}(-q^{2n+2};q^2)_\infty$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty}(B_e(n) - B_o(n))q^n &= (q^2;q^2)_\infty + \sum_{n=1}^{\infty} \frac{q^{2(1+3+5+\ldots+2n-1)}}{(q;q^2)_n}(q^{2n+2};q^2)_\infty \\ &= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n} \frac{(q^2;q^2)_\infty}{(q^2;q^2)_n} \\ &= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} \\ &= \prod_{n=1}^{\infty}(1+q^{8n-3})(1+q^{8n-5})(1-q^{8n}) \text{ by (2)} \\ &= \sum_{n=-\infty}^{\infty} q^{4n^2+n}, \end{aligned}$$

and the result follows.

Corollary 2. *For all $n \geq 0$, $B(n)$ is odd if and only if $n = m(4m \pm 1)$ for some integer $m \geq 0$.*

Finally, consider the partition function;

$\tau(n)$: the number of partitions of n in which even parts are distinct or if an even part is repeated, it is the smallest and occurs exactly twice and all other even parts are distinct.

Let $\tau_e(n)$ (resp. $\tau_o(n)$) denote the number of $\tau(n)$-partitions with an even (resp. odd) number of distinct even parts. Then, the following identity follows:

Theorem 7. *For all non-negative integers n, we have*

$$\tau_e(n) - \tau_o(n) = \begin{cases} 1, & \text{if } n = 3m, 3m+1, m \geq 0; \\ 0, & \text{otherwise} \end{cases}$$

where $\tau_e(0) - \tau_o(0) := 1$.

Proof. Note that

$$\begin{aligned} \sum_{n=0}^{\infty} \tau(n)q^n &= \frac{(-q^2;q^2)_\infty}{(q;q^\infty)_\infty} + \sum_{n=1}^{\infty} q^{2n+2n}(-q^{2n+2};q^2)_\infty(q^{2n+1};q^2)_\infty^{-1} \\ &= \sum_{n=0}^{\infty} q^{2n+2n}(-q^{2n+2};q^2)_\infty(q^{2n+1};q^2)_\infty^{-1} \end{aligned}$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty}(\tau_e(n)-\tau_o(n))q^n &= \sum_{n=0}^{\infty} q^{2n+2n}(q^{2n+2};q^2)_\infty(q^{2n+1};q^2)_\infty^{-1} \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{4n}}{(q^2;q^2)_n}(q;q^2)_n \\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{(q;q^2)_\infty}{(q^4;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^n}{(q^2;q^2)_n}(q^4;q^2)_n \\
&\quad \Big(\text{by (3)}, a=c=0, b=q, t=q^4\Big). \\
&= (1-q^2)\sum_{n=0}^{\infty}\frac{q^n}{(q^2;q^2)_n}(q^4;q^2)_n \\
&= (1-q^2)\sum_{n=0}^{\infty}\frac{(1-q^{2n+2})q^n}{1-q^2} \\
&= \sum_{n=0}^{\infty}(1-q^{2n+2})q^n \\
&= \sum_{n=0}^{\infty}q^n - \sum_{n=0}^{\infty}q^{3n+2} \\
&= \sum_{n=0}^{\infty}q^{3n} + \sum_{n=0}^{\infty}q^{3n+1}.
\end{aligned}$$

□

Example 2. *Consider* $n = 8$.

The $\tau(8)$-partitions are:

$$(8),(7,1),(6,2),(5,3),(5,1,1,1),(4,4),(4,2,1,1),(3,3,1,1),(3,1,1,1,1,1),$$
$$(6,1,1),(5,2,1),(4,3,1),(4,2,2),(4,1,1,1,1),(3,3,2),(3,2,1,1,1),$$
$$(2,1,1,1,1,1,1),(1,1,1,1,1,1,1,1).$$

The $\tau_e(8)$-partitions are:

$$(7,1),(6,2),(5,3),(5,1,1,1),(4,4),(4,2,1,1),(3,3,1,1),(3,1,1,1,1,1),(1,1,1,1,1,1,1,1),$$

and $\tau_o(8)$-partitions are:

$$(8),(6,1,1),(5,2,1),(4,3,1),(4,2,2),(4,1,1,1,1),(3,3,2),(3,2,1,1,1),(2,1,1,1,1,1,1),$$

Indeed $\tau_e(8) - \tau_o(8) = 0$.

The above theorem can be used to determine the parity of $\tau(n)$. We write down this as a consequence in the corollary below.

Corollary 3. *For all* $n \geq 0$, $\tau(n)$ *is odd if and only if* $n \equiv 0, 1 \pmod 3$.

4. Conclusions

Much as we could not generalize Theorem 2 via generating functions, we supplied a generalization via a bijective construction. Various partition functions that are related to the theorem were studied. Our investigation included deriving parity formulas and establishing new partition identities. Of particular interest was Theorem 7 whose combinatorial proof we seek.

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