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On a Boundary Value Problem of Hybrid Functional Differential Inclusion with Nonlocal Integral Condition

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Abstract: In this work, we present a boundary value problem of hybrid functional differential inclusion with nonlocal condition. The boundary conditions of integral and infinite points will be deduced. The existence of solutions and its maximal and minimal will be proved. A sufficient condition for uniqueness of the solution is given. The continuous dependence of the unique solution will be studied.

Keywords: hybrid differential inclusion; boundary value problem; nonlocal condition; integral condition; infinite point boundary condition; existence of solutions; continuous dependence

MSC: 26A33; 34K45; 47G10

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1. Introduction

Models of hybrid functional differential and integral equations have many applications (see [1–14]).

Boundary value problems with nonlocal boundary conditions have been studied by some authors (see [15–18]).

Here, we assess the boundary value problem of hybrid nonlinear functional differential inclusion with nonlocal condition.

$$\frac{d}{dt}\left(\frac{x(t)-x(0)}{g(t,x(\phi_1(t)))}\right) \in F(t,x(\phi_2(t))), \quad t \in (0,1) \tag{1}$$

with the nonlocal boundary condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0,\ a_k > 0\ \ \tau_k \in [0,1]. \tag{2}$$

The existence of solutions $x \in C[0,1]$ will be proved. The maximal and minimal solutions will be studied. A sufficient condition for uniqueness of the solution will be given. The continuous dependence of the unique solution on x_o and on $\sum_{k=1}^{m} a_k$ will be proved.

Additionally, we deduce the same results for the boundary value problem of hybrid nonlinear functional differential inclusion (1) with a nonlocal integral condition

$$\int_0^1 x(s)dh(s) = x_0 \tag{3}$$

and infinite point boundary conditions

$$\sum_{k=1}^{\infty} a_k x(\tau_k) = x_0,\ a_k > 0\ \ \tau_k \in [0,1]. \tag{4}$$

The following assumptions will be needed for our goals:

(I) (i) The function $\phi_i : I \longrightarrow I, I = [0,1]$ is continuous and $\phi_i(t) \le t, i = 1,2$.
(ii) The set $F : I \times R \longrightarrow 2^R$ is nonempty, closed, and convex for all $(t,x) \in [0,1] \times R$.
(iii) $F(t,x)$ is measurable in $t \in [0,1]$ for every $x \in R$.
(iv) $F(t,x)$ is upper semicontinuous in x for every $t \in [0,1]$.
(v) There exists a bounded measurable function $a_2 : [0,1] \longrightarrow R$ and a positive constant K_2 such that

$$\begin{aligned} \|F(t,x)\| &= \sup\{|f| : f \in F(t,x)\} \\ &\le |a_2(t)| + K_2(|x|). \end{aligned}$$

Remark 1. *From assumptions (ii)–(iv), we can deduce that the set of selection S_F of F is nonempty (see [1,2,5]), that there exists $f \in F(t,x)$ such that*

(vi) $f : I \times R \longrightarrow R$ is measurable in t for every $x \in R$ and continuous in x for $t \in [0,1]$, there exists a bounded measurable function $a_2 : [0,1] \to R$ and a positive constant $K_2 > 0$ such that

$$|f(t,x)| \le |a_2(t)| + K_2(|x|),$$

and that the function f satisfies the differential equation

$$\frac{d}{dt}\left(\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))}\right) = f(t, x(\phi_2(t))),\ t \in (0,1). \tag{5}$$

Therefore, any solution of the nonlocal problem of the hybrid functional differential Equation (5) with any of the nonlocal boundary conditions (2)–(4) is a solution of the nonlocal problem of the hybrid nonlinear functional differential inclusion with any one of the nonlocal conditions (1)–(4).

(II) $g : I \times R \longrightarrow R$ is measurable in t for any $x \in R$ and Lipschitz in x for $t \in [0,1]$, and there exists a positive constant $K_1 > 0$ such that

$$|g(t,x) - g(t,y)| \le K_1|x - y|,\ \forall t \in I,\ and\ x, y \in R.$$

From assumption (II), we have

$$|g(t,y)| - |g(t,0)| \le |g(t,y) - g(t,0)| \le K_1|y|.$$

Then,

$$\begin{aligned} |g(t,x)| &\le K_1|x| + |g(t,0)| \\ &\le K_1|x| + |\, a_1(t) \end{aligned}$$

where $|a_1(t)| = \sup_{t\in I} |g(t,0)|$, $K = \max\{K_1, K_2\}$ and

$$|g(t,x) - g(t,0)| \le K_1|x| + |g(t,0)| \le K_1|x| + a_1(t).$$

(III) There exists a positive solution r of the algebraic equation.

$$A|x_0| + 2(Kr + a)^2 - r = 0$$

where $A = (\Sigma_{k=1}^{m} a_k)^{-1}$.

Definition 1. *x of the problem in Equations (2) and (5) is $x \in C[0,1]$ such that $\left(\frac{x(t)-x(0)}{g(t,x(\phi_1(t)))}\right)$ satisfies (5).*

Now, we have the following lemma.

Lemma 1. *If the solution of the problems in Equations (2) and (5) exists, then it can be expressed by the integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds] \\ &+ g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds. \end{aligned} \tag{6}$$

Proof. Let the boundary value problem in Equations (2) and (5) be satisfied; then, we can obtain

$$x(t) = x(0) + g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds. \tag{7}$$

Putting $t = \tau$ and multiplying both sides of (7) by a_k, we obtain

$$\Sigma_{k=1}^{m} a_k x(\tau_k) = \Sigma_{k=1}^{m} a_k x(0) + \Sigma_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds;$$

then

$$x_0 = \Sigma_{k=1}^{m} a_k x(0) + \Sigma_{k=1}^{m} a_k g(\tau_k, x((\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds$$

and

$$x(0) = \frac{1}{\Sigma_{k=1}^{m} a_k}[x_0 - \Sigma_{k=1}^{m} a_k g(\tau_k, x((\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]. \tag{8}$$

Substituting (8) in (7), we obtain (6). □

2. Existence of Solutions

Theorem 1. *Assume that assumptions (I)–(III) are valid. Then, the integral Equation (6) has at least one solution $x \in C[0,1]$.*

Proof. Define the set Q_r by

$$Q_r = \{x \in C[0,1] : \|x\| \leq r, r > 0\}.$$

Define the operator $\mathcal{F}$ by

$$\mathcal{F}x(t) = \frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds\right]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$

Let $x \in Q_r$; then

$$
\begin{aligned}
&|\mathcal{F}x(t)| = |\frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds] \\
&+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds| \\
&\leq \frac{|x_0|}{\sum_{k=1}^{m} a_k} \\
&+\frac{\sum_{k=1}^{m} a_k[K_1|x(\phi_1(\tau_k))| + |a_1(\tau_k)|] \int_0^{\tau_k}[K_2|x(\phi_2(s))| + |a_2(s)|]ds}{\sum_{k=1}^{m} a_k} \\
&+[K_1|x(\phi_1(t))| + |a_1(t)|] \int_0^t [K_2|x(\phi_2(s))| + |a_2(s)|]ds \\
&\leq \frac{|x_0|}{\sum_{k=1}^{m} a_k} + [K\|x\| + a][K\|x\| + a] \int_0^1 ds + [K\|x\| + a][K\|x\| + a] \int_0^1 ds \\
&\leq \frac{|x_0|}{\sum_{k=1}^{m} a_k} + 2[Kr + a][Kr + a] \\
&\leq A|x_0| + 2(Kr + a)^2 = r.
\end{aligned}
$$

Thus, the class of functions $\{\mathcal{F}x\}$ is uniformly bounded on Q_r and $\mathcal{F} : Q_r \to Q_r$. Let $x \in Q_r$ and $t_1, t_2 \in [0, 1]$ such that $|t_2 - t_1| < \delta$; then,

$$
\begin{aligned}
&|\mathcal{F}x(t_2) - \mathcal{F}x(t_1)| = |\frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds] \\
&+g(t_2, x(\phi_1(t_2))) \int_0^{t_2} f(s, x(\phi_2(s)))ds \\
&-(\frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds] \\
&+g(t_1, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds)| \\
&\leq |g(t_2, x(\phi_1(t_2))) \int_0^{t_2} f(s, x(\phi_2(s)))ds - g(t_1, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds| \\
&\leq |g(t_2, x(\phi_1(t_2))) \int_0^{t_2} f(s, x(\phi_2(s)))ds - g(t_2, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds \\
&+g(t_2, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds - g(t_1, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds| \\
&\leq |g(t_2, x(\phi_1(t_2))) - g(t_2, x(\phi_1(t_1)))| \int_{t_1}^{t_2} |f(s, x(\phi_2(s)))|ds \\
&+|g(t_2, x(\phi_1(t_1))) - g(t_1, x(\phi_1(t_1)))| \int_0^{t_1} |f(s, x(\phi_2(s)))|ds \\
&\leq K_1|x(\phi_1(t_2)) - x(\phi_1(t_1))| \int_{t_1}^{t_2} (|a_2(s)| + K_2|x(\phi_2(s))|)ds \\
&+|g(t_2, x(\phi_1(t_1))) - g(t_1, x(\phi_1(t_1)))| \int_0^{t_1} (|a_2(s)| + K_2|x(\phi_2(s))|)ds \\
&\leq K|x(\phi_1(t_1)) - x(\phi_1(t_2))| \int_{t_1}^{t_2} (a + K|x(\phi_2(s))|)ds \\
&+|g(t_2, x(\phi_1(t_1))) - g(t_1, x(\phi_1(t_1)))| \int_0^{t_1} (a + K|x(\phi_2(s))|)ds.
\end{aligned}
$$

Thus, the class of functions $\{\mathcal{F}x\}$ is equicontinuous on Q_r and $\{\mathcal{F}y\}$ is a compact operator by the Arzela–Ascoli Theorem [19].

Now, we prove that $\mathcal{F}$ is a continuous operator. Let $x_n \subset Q_r$ be a convergent sequence such that $x_n \to x$; then,

$$\mathcal{F}x_n(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_n(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_n(\phi_2(s)))ds]$$
$$+g(t, x_n(\phi_1(t))) \int_0^t f(s, x_n(\phi_2(s)))ds.$$

Using Lebesgue-dominated convergence Theorem [19] and assumptions (iv)–(III), we have

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{F}x_n(t) = \lim_{n\to\infty}(\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_n(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_n(\phi_2(s)))ds] \\
&+g(t, x_n(\phi_1(t))) \int_0^t f(s, x_n(\phi_2(s)))ds) \\
&= \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, \lim_{n\to\infty} x_n(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, \lim_{n\to\infty} x_n(\phi_2(s)))ds] \\
&+g(t, \lim_{n\to\infty} x_n(\phi_1(t))) \int_0^t f(s, \lim_{n\to\infty} x_n(\phi_2(s)))ds. \\
&= \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds] \\
&+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds \\
&= \mathcal{F}x(t).
\end{aligned}$$

Then, $\mathcal{F} : Q_r \to Q_r$ is continuous, and by Schauder fixed point Theorem [19], there exists at least one solution $x \in C[0,1]$ of (6).

Now,

$$\begin{aligned}
&x(t) = x(0) + g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds \\
&\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))} = \int_0^t f(s, x(\phi_2(s)))ds \\
&\frac{d}{dt}(\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))}) = \frac{d}{dt}\int_0^t f(s, x(\phi_2(s)))ds \\
&\frac{d}{dt}(\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))}) = f(t, x(\phi_2(t))).
\end{aligned}$$

putting $t = \tau_k$ and multiplying by $\Sigma_{k=1}^m a_k$ in (6), we obtain

$$\sum_{k=1}^m a_k x(\tau_k) = \sum_{k=1}^m a_k(\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds).$$

Then,

$$\sum_{k=1}^m a_k x(\tau_k) = x_0, a_k > 0, \tau_k \in [0,1].$$

This proves the equivalence between the problem in Equations (2) and (5) and the integral Equation (6). Then, there exists at least one solution $x \in C[0,1]$ of the hybrid nonlinear functional differential Equation (5) with the nonlocal condition (2). Consequently, there exists at least one solution $x \in C[0,1]$ of the nonlocal problem of the hybrid nonlinear functional differential inclusion (1) with the nonlocal condition (2). $\square$

3. Maximal and Minimal Solutions

Here, we study the maximal and minimal solutions for the problem in Equations (2) and (5). Let $u(t)$ be a solution of (6); then, $u(t)$ is said to be a maximal solution of (6) if it satisfies the inequality

$$x(t) \leq u(t), \quad t \in [0,1].$$

A minimal solution $v(t)$ can be defined in a similar way by reversing the above inequality i.e.,

$$x(t) > v(t), \quad t \in [0,1].$$

Lemma 2. *Let the assumptions of Theorem 1 be satisfied. Assume that x and y are two continuous functions on $[0,1]$ satisfying.*

$$x(t) < \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds \;\; t \in [0,1].$$
$$y(t) > \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, y(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, y(\phi_2(s)))ds]$$
$$+g(t, y(\phi_1(t))) \int_0^t f(s, y(\phi_2(s)))ds \;\; t \in [0,1]$$

where one of them is strict.

Let the functions f and g be monotonically nondecreasing in x; then,

$$x(t) < y(t), \quad t > 0. \tag{9}$$

Proof. Let the conclusion (9) be untrue; then, there exists t_1 with

$$x(t_1) < y(t_1), \quad t_1 > 0 \quad \text{and } x(t) < y(t), \quad 0 < t < t_1.$$

If f and g are monotonic functions in x, we have

$$x(t_1) \leq \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t_1, x(\phi_1(t_1))) \int_0^{t_1} f(s, x(\phi_2(s)))ds \;\; t_1 \in [0,1].$$
$$\geq \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k g(\tau_k, y(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, y(\phi_2(s)))ds]$$
$$+g(t_1, y(\phi_1(t_1))) \int_0^{t_1} f(s, y(\phi_2(s)))ds \;\; t_1 \in [0,1] < y(t_1).$$

This contradicts the fact that $x(t_1) = y(t_1)$. This completes the proof. $\square$

For the continuous maximal and minimal solutions for (6), we have the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 hold. Moreover, if f and g are monotonically nondecreasing functions in x for each $t \in [0,1]$, then Equation (6) has maximal and minimal solutions.*

Proof. First, we must demonstrate the existence of the maximal solution of (6). Let $\epsilon > 0$ be given. Now, consider the integral equation

$$x_\epsilon(t) \leq \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_\epsilon(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_\epsilon(\phi_2(s)))ds]$$
$$+g(t, x_\epsilon(\phi_1(t))) \int_0^t f(s, x_\epsilon(\phi_2(s)))ds \;\; t \in [0,1]$$

where

$$f(t, x_\epsilon(\phi_2(t))) = f(s, x_\epsilon(\phi_2(t))) + \epsilon,$$
$$g(s, x_\epsilon(\phi_1(t))) = f(s, x_\epsilon(\phi_1(t))) + \epsilon.$$

Let ϵ_1, ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1$; then,

$$x_{\epsilon_2}(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_{\epsilon_1}(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_{\epsilon_2}(\phi_2(s)))ds]$$
$$+g(t, x_{\epsilon_2}(\phi_1(t))) \int_0^t f(s, x_{\epsilon_2}(\phi_2(s)))ds \;\; t \in [0,1].$$

$$x_{\epsilon_2}(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_{\epsilon_2}(\phi_1(\tau_k)) + \epsilon_2) \int_0^{\tau_k} (f(s, x_{\epsilon_2}(\phi_2(s))) + \epsilon_2)ds]$$
$$+g(t, x_{\epsilon_2}(\phi_1(t)) + \epsilon_2) \int_0^t (f(s, x_{\epsilon_2}(\phi_2(s))) + \epsilon_2)ds \;\; t \in [0,1].$$

Additionally,

$$x_{\epsilon_1}(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_{\epsilon_1}(\phi(\tau_k))) \int_0^{\tau_k} f(s, x_{\epsilon_1}(\phi_2(s)))ds]$$
$$+g(t, x_{\epsilon_1}(\phi_1(t))) \int_0^t f(s, x_{\epsilon_1}(\phi_2(s)))ds \;\; t \in [0,1].$$
$$x_{\epsilon_1}(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k (g(\tau_k, x_{\epsilon_1}(\phi_1(\tau_k)) + \epsilon_1) \int_0^{\tau_k} (f(s, x_{\epsilon_1}(\phi_2(s))) + \epsilon_1)ds]$$
$$+(g(t, x_{\epsilon_1}(\phi_1(t)) + \epsilon_1) \int_0^t f(s, x_{\epsilon_1}(\phi_2(s))) + \epsilon_1))ds \;\; t \in [0,1].$$

Then,

$$x_{\epsilon_1} > \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_{\epsilon_2}(\phi_1(\tau_k))) \int_0^{\tau_k} (f(s, x_{\epsilon_2}(\phi_2(s))))ds]$$
$$+g(t, x_{\epsilon_2}(\phi_1(t))) \int_0^t (f(s, x_{\epsilon_2}(\phi_2(s))))ds \;\; t \in [0,1].$$

Applying Lemma 2, we obtain

$$x_{\epsilon_2} < x_{\epsilon_1}, \; t \in [0,1].$$

As shown before, the family of function $x_\epsilon(t)$ is equi-continuous and uniformly bounded; then, by the Arzela Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon_0 \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0,1]$, and denote this limit by $u(t)$. From the continuity of the functions, $f_\epsilon(t, x_\epsilon(\phi_2(t)))$, we get

$$f_\epsilon(t, x_\epsilon(\phi_2(t))) \longrightarrow f(t, x(\phi_2(t))) \;\; as \;\; n \to \infty,$$

$$g_\epsilon(t, x_\epsilon(\phi_2(t))) \longrightarrow g(t, x(\phi_2(t))) \;\; as \;\; n \to \infty$$

and

$$u(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, u(\phi_1(\tau_k))) \int_0^{\tau_k} (f(s, u(\phi_2(s)))))ds]$$
$$+g(t, u(\phi_1(t))) \int_0^t (f(s, u(\phi_2(s))))ds \;\; t \in [0,1].$$

Now, we prove that $u(t)$ is the maximal solution of (6). To do this, let $x(t)$ be any solution of (6); then,

$$x(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds \;\; t \in [0,1]$$

and

$$x_\epsilon(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_\epsilon(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_\epsilon(\phi_2(s)))ds]$$
$$+g(t, x_\epsilon(\phi_1(t))) \int_0^t f(s, x_\epsilon(\phi_2((s))ds \;\; t \in [0,1].$$
$$= \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_\epsilon(\phi_1(\tau_k)) + \epsilon) \int_0^{\tau_k} (f(s, x_\epsilon(\phi_2(s)) + \epsilon)ds]$$
$$+g(t, x_\epsilon(\phi_1(t)) + \epsilon) \int_0^t f(s, x_\epsilon(\phi_2(s)) + \epsilon)ds \;\; t \in [0,1].$$

Then,

$$x_\epsilon(t) > \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_\epsilon(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_\epsilon(\phi_2(s)))ds]$$
$$+g(t, x_\epsilon(\phi_1(t))) \int_0^t f(s, x_\epsilon(\phi_2(s)))ds \;\; t \in [0,1].$$

Applying Lemma 2, we obtain

$$x(t) < x_\epsilon(t), \;\; t \in [0,1].$$

From the uniqueness of the maximal solution, it clear that $x_\epsilon(t)$ tends to $u(t)$ uniformly in $[0,1]$ as $\epsilon \to 0$ in a similar way as above and we can prove the existence of the minimal solution. □

4. Uniqueness of the Solution

Here, we study a sufficient condition for the uniqueness of the solution $x \in C[0,1]$ of the problem in Equations (2) and (5).

Consider the following assumptions:

(I^*) (i) The function $\phi_i : I \longrightarrow I$ is continuous, and $\phi_i(t) \leq t, \quad i = 1,2$.

(ii) The set $F(t,x)$ is nonempty, compact, and convex for all $(t,x) \in [0,1] \times R$.

(iii) $F(t,x)$ is measurable in $t \in [0,1]$ for every $x \in R$ and satisfies the Lipschitz condition with a positive constant K_2 such that

$$H(F(t,x), F(t,y)| \leq K_2(|x-y|)$$

where $H(A,B)$ is the Hausdorff metric between the two subsets $A, B \in I \times E$ (see [16]).

Remark 2. *From this assumptions, we can deduce that there exists a function $f \in F(t,x)$ such that*

(iv) $f : I \times R \to R$ *is measurable in $t \in [0,1]$ for every $x \in R$ and satisfies the Lipschitz condition with a positive constant K_1 such that (see [19–21])*

$$|f(t,x) - f(t,y)| \leq K_2(|x-y|).$$

$(II^*) g : I \times R \longrightarrow R$ is continuous and satisfies the Lipschitz condition with positive constant K_1 such that

$$|g(t,x) - g(t,y)| \leq K_1|x-y|.$$

From the assumption (I^*), we have

$$|f(t,x)| - |f(t,0)| \leq |f(t,x) - f(t,0)| \leq K_2(|x|).$$

Then,

$$\begin{aligned} |f(t,x)| &\leq K_1(|x|) + |f(t,0)| \\ &\leq K_2(|x|) + | a_2(t), \end{aligned}$$

where $|a_2(t)| = \sup_{t \in I} |f(t,0)|$.

From assumption (II^*), we have

$$|g(t,y)| - |g(t,0)| \leq |g(t,y) - g(t,0)| \leq K_1|y|.$$

Then,

$$\begin{aligned} |g(t,x)| &\leq K_1|x| + |g(t,0)| \\ &\leq K_1|x| + | a_1(t), \end{aligned}$$

where $|a_1(t)| = \sup_{t \in I} |g(t,0)|, \quad K = \max\{K_1, K_2\}$

Theorem 3. *Let the assumptions (I^*)–(II^*) be satisfied. If $(1 - 4K^2 r - 4aK) \leq 1$, then the solution of the problem in Equations (2) and (5) is unique.*

Proof. Let x_1 and x_2 be two solutions of the problem in Equations (2) and (5); then,

$$
\begin{aligned}
&|x_1(t) - x_2(t))| = |\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_1(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_1(\phi_2(s)))ds] \\
&+g(t, x_1(\phi_1(t))) \int_0^t f(s, x_1(\phi_2((s))ds \\
&-(\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x_2(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_2(\phi_2(s)))ds] \\
&+g(t, x_2(\phi_1(t))) \int_0^t f(s, x_2(\phi_2(s)))ds)| \\
\leq &\frac{|\sum_{k=1}^m a_k(g(\tau_k, x_1(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_1(\phi_2(s)))ds - g(\tau_k), x_2(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_2(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
+&g(t, x_1(\phi_1(t)) \int_0^t f(s, x_1(\phi_2(s)))ds - g(\tau_k, x_2(\phi_1(t))) \int_0^t f(s, x_2(\phi_2(s)))ds| \\
\leq &\frac{|\sum_{k=1}^m a_k(g(\tau_k, x_1(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_1(\phi_2(s)))ds - g(\tau_k), x_2(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_1(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
\leq &\frac{|\sum_{k=1}^m a_k(g(\tau_k, x_2(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_1(\phi_2(s)))ds - g(\tau_k, x_2(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x_2(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
&+|g(t, x_1(\phi_1(t))) \int_0^t f(s, x_1(\phi_2(s)))ds - g(t, x_2(\phi_1(t))) \int_0^t f(s, x_1(\phi_2(s)))ds| \\
&+|g(t, x_2(\phi_1(t))) \int_0^t f(s, x_1(\phi_2(s)))ds - g(t, x_2(\phi_1(t))) \int_0^t f(s, x_2(\phi_2(s)))ds| \\
&\leq \frac{\sum_{k=1}^m a_k K\|x_1 - x_2\|(K\|x_1\| + a)}{\sum_{k=1}^m a_k} \\
&+\frac{\sum_{k=1}^m a_k K\|x_1 - x_2\|(K\|x_2\| + a)}{\sum_{k=1}^m a_k} \\
&+K\|x_1 - x_2\|(K\|x_1\| + a) + K\|x_1 - x_2\|(K\|x_2\| + a) \\
&\leq 2K\|x_1 - x_2\|(Kr + a) + 2K\|x_1 - x_2\|(Kr + a) \\
&\leq (4K^2 r + 4aK)\|x_1 - x_2\|.
\end{aligned}
$$

Then,

$$\|x_1 - x_2\|(1 - 4K^2 r - 4aK) \leq 0,$$

Since $(1 - 4K^2 r - 4aK) \leq 1$, $x_1(t) = x_2(t)$ and the solution of (5) and (2) is unique. □

5. Continuous Dependence of the Solution

Definition 2. *The unique solution of the problem in Equations (2) and (5) depends continuously on initial data x_0, if $\epsilon > 0$, $\exists \delta > 0$, such that*

$$|x_0 - x_0^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of the integral equation

$$
\begin{aligned}
&x^*(t) = \frac{1}{\sum_{k=1}^m a_k}[x_0^* - \sum_{k=1}^m a_k g(\tau_k, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds] \\
&+g(t, x^*(\phi_1(t)) \int_0^t f(s, x^*(\phi_2(s))ds.
\end{aligned}
$$

Theorem 4. *Let the assumptions (I^*)–(II^*) be satisfied; then, the unique solution of (5) and (2) depends continuously on x_0*

Proof. Let x and x^* be the solutions of the problem in Equations (2) and (5); then,

$$
\begin{aligned}
&|x(t) - x^*(t)| = |\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s))))ds] \\
&+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s))))ds \\
&-\frac{1}{\sum_{k=1}^m a_k}[x_0^* - \sum_{k=1}^m a_k g(\tau_k, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s))))ds] \\
&-g(t, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s))))ds)| \\
&\leq \frac{|x_0 - x_0^*|}{\sum_{k=1}^m a_k} \\
&\leq \frac{|\sum_{k=1}^m a_k(g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, (\phi_2 x(s)))ds - g(\tau_k), x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
&+g(t, x(\phi_1(t)) \int_0^t f(s, x(\phi_2(s)))ds - g(\tau_k, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s)))ds| \\
&\leq \delta\ A \\
&+\frac{|\sum_{k=1}^m a_k(g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds - g(\tau_k), x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
&+\frac{|\sum_{k=1}^m a_k(g(\tau_k, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds - g(\tau_k, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds|}{\sum_{k=1}^m a_k} \\
&+|g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds - g(t, x^*(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds| \\
&+|g(t, x^*(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds - g(t, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s)))ds| \\
&\leq \delta\ A \\
&+\frac{\sum_{k=1}^m a_k K\|x - x^*\|(K\|x\| + a)}{\sum_{k=1}^m a_k} \\
&+\frac{\sum_{k=1}^m a_k K\|x - x^*\|(K\|x^*\| + a)}{\sum_{k=1}^m a_k} \\
&+K\|x - x^*\|(K\|x\| + a) + K\|x - x^*\|(K\|x^*\| + a) \\
&\leq \delta\ A \\
&+2K\|x - x^*\|(Kr + a) + 2K\|x - x^*\|(Kr + a) \\
&+2K\|x - x^*\|(Kr + a) + 2K\|x - x^*\|(Kr + a) \\
&\leq \delta A + (4K^2 r + 4aK)\|x - x^*\|.
\end{aligned}
$$

Then,

$$\|x - x^*\| \leq \delta A(1 - 4K^2 r - 4aK)^{-1} \leq \epsilon.$$

□

Definition 3. *The unique solution of the problem in Equations (2) and (5) depends continuously on the initial data a_k if $\epsilon > 0$, $\exists \delta > 0$ such that*

$$\sum_{k=1}^m |a_k - a_k^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of the integral equation

$$x^*(t) = \frac{1}{\sum_{k=1}^m a_k^*}[x_0 - \sum_{k=1}^m a_k^* g(\tau_k, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds]$$
$$+g(t, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s)))ds.$$

Theorem 5. *Let the assumptions $(I^*) - (II^*)$ be satisfied; then, the unique solution of (5) and (2) depends continuously on $\sum_{k=1}^m a_k$. Then,*

Proof.

$$|x(t) - x^*(t)| = \frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t)))) \int_0^t f(s, x(\phi_2(s))))ds - \frac{1}{\sum_{k=1}^m a_k^*}[x_0$$
$$- \sum_{k=1}^m a_k^* g(t, x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds] - g(t, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s))))ds|$$
$$\leq \frac{|x_0|| \sum_{k=1}^m a_k - \sum_{k=1}^m a^*|}{\sum_{k=1}^m ak - \sum_{k=1}^m a^*}$$
$$+|\frac{g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds}{\sum_{k=1}^m a_k} - \frac{g(\tau_k), x^*(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x^*(\phi_2(s)))ds}{\sum_{k=1}^m a_k^*}|$$
$$+|g(t, x(\phi_1(t)) \int_0^t f(s, x(\phi_2(s)))ds - g(\tau_k, x^*(\phi_1(t))) \int_0^t f(s, x^*(\phi_2(s)))ds|$$
$$\leq \frac{|x_0|\delta}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*}$$
$$+\frac{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^* |g(\tau_k, x(\phi_1(\tau_k))) - g(\tau_k), x^*(\phi_1(\tau_k)))| \int_0^{\tau_k} |f(s, x(\phi_2(s)))|ds}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*}$$
$$+\frac{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^* |g(\tau_k, x^*(\phi_1(\tau_k)))| \int_0^{\tau_k} |f(s, x(\phi_2(s))) - f(s, x^*(\phi_2(s)))|ds}{\sum_{k=1}^m a_k \sum_{k=1}^m a_k^*}$$
$$+|g(t, x(\phi_1(t))) - g(t, x^*(\phi_1(t)))| \int_0^t |f(s, x(\phi_2(s)))|ds$$
$$+|g(t, x^*(\phi_1(t)))| \int_0^t |f(s, x(\phi_2(s))) - f(s, x^*(\phi_2(s)))|ds$$
$$\leq |x_0|\delta A A^*$$
$$+K\|x - x^*\|(K\|x\| + a)$$
$$+K\|x - x^*\|(K\|x^*\| + a)$$
$$+K\|x - x^*\|(K\|x\| + a) + K\|x - x^*\|(K\|x^*\| + a)$$
$$\leq |x_0|\delta A A^* + (4K^2 r + 4aK)\|x - x^*\|.$$

Then,

$$\|x - x^*\| \leq |x_0|\delta A A^*(1 - 4K^2 r - 4aK)^{-1} \leq \epsilon.$$

□

6. Riemann–Stieltjes Integral Condition

Let $x \in C[0,1]$ be the solution of the nonlocal boundary value problem in Equations (2) and (5).

Let $a_k = (h(t_k) - h(t_{k-1}))$, where h is an increasing function, $\tau_k \in (t_{k-1} - t_k)$, and $0 = t_0 < t_1 < t_2, \ldots < t_m = 1$; then, the nonlocal condition (2) is

$$\sum_{k=1}^{m} (h(t_k) - h(t_{k-1}))x(\tau_k) = x_0$$

and the limit implies

$$\lim_{m\to\infty} \sum_{k=1}^{m} (h(t_k) - h(t_{k-1}))x(\tau_k) = \int_0^1 x(s)dh(s) = x_0.$$

Theorem 6. *Let the assumptions (I)–(III) be satisfied; then, the nonlocal boundary value problem of (5) and (3) has at least one solution given by*

$$x(t) = \frac{1}{h(1)-h(0)}[x_0 - \int_0^1 g(s, x(\phi_1(s))) \int_0^s f(\theta, x(\phi_2(\theta))))d\theta dh(s)]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$

Proof. Let $h : [0,1] \to [0,1]$, and let $a_k = h(t_k) - h(t_{k-1})$ be an increasing function. Then, the solution of (5) and (3) is given by

$$x(t) = (h(1) - h(0))^{-1}x_0$$
$$-(h(1) - h(0))^{-1} \sum_{k=1}^{m} (h(\tau_k) - h(\tau_{k-1}))g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$

As $m \to \infty$, we obtain

$$\lim_{m\to\infty} x(t) = (h(1) - h(0))^{-1}x_0$$
$$-(h(1) - h(0))^{-1} \lim_{m\to\infty} \sum_{k=1}^{m} (h(\tau_k) - h(\tau_{k-1}))g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$
$$\frac{1}{h(1)-h(0)}[x_0 - \int_0^1 g(s, x(\phi_1(s))) \int_0^s f(\theta, x(\phi_2(\theta)))d\theta dh(s)]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$

□

7. Infinite-Point Boundary Condition

Theorem 7. *Let the assumptions (I)–(III) be satisfied; then, the nonlocal boundary value problem of (5) and (4) has at least one solution given by*

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k}[x_0 - \sum_{k=1}^{\infty} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t))) \int_0^t f(s, x(\phi_2(s)))ds.$$

Proof. Let the assumptions of Theorem 1 be satisfied. Let $\sum_{k=1}^{m} a_k$ be convergent; then, take the limit of (5). We have

$$\lim_{m\to\infty} x(t) = \lim_{m\to\infty} \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \lim_{m\to\infty} \sum_{k=1}^{m} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t))) \int_0^{t} f(s, x(\phi_2(s)))ds.$$

Now,

$$|a_k g(\tau_k, x(\phi_1(\tau_k)))| \leq K|a_k|x(\phi_1(\tau_k))|,$$
$$\leq K|a_k|\|x\| \leq K|a_k|r.$$

Then, by a comparison test, the series $\sum_{k=1}^{\infty} a_k g(\tau_k, x(\phi_1(\tau_k)))$ is convergent and

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k}[x_0 - \sum_{k=1}^{\infty} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(t, x(\phi_1(t))) \int_0^{t} f(s, x(\phi_2(s)))ds.$$

Furthermore, from (10), we have

$$\sum_{k=1}^{\infty} a_k x(t) = \sum_{k=1}^{\infty} a_k \frac{1}{\sum_{k=1}^{\infty} a_k}[x_0 - \sum_{k=1}^{\infty} a_k g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds]$$
$$+g(\tau_k, x(\phi_1(\tau_k))) \int_0^{\tau_k} f(s, x(\phi_2(s)))ds = x_0.$$

□

Example 1. *Consider the boundary value problem, with nonlocal*

$$\frac{d}{dt}\left(\frac{x(t) - x(0)}{\frac{1}{4}(\frac{\cos(t)}{t+1} + x(t))}\right) = \frac{1}{4}(t^2 - t + e^t x(t)), \tag{10}$$

with the nonlocal boundary condition

$$\sum_{k=1}^{m} \left[\frac{1}{k} - \frac{1}{k+1}\right] x(\tau_k) = x_0,\ a_k > 0\ \ \tau_k \in [0,1] \tag{11}$$

putting

$$f(t, x(t)) = \frac{1}{4}\left(t^2 - 2t + e^t x(t)\right)$$
$$g(t, x(t)) = \frac{1}{4}\left(\frac{\cos(t)}{t+1} + x(t)\right)$$

and

$$|g(t, x(t)) - g(t, y(t))| \leq \frac{1}{4}|x(t) - y(t)|;$$

then,

$$|\frac{1}{4}(\frac{\cos(t)}{t+1} + x(t))| \leq \frac{1}{4}|(\frac{1}{t+1} + x(t))|$$
$$|\frac{1}{4}(t^2 - 2t + e^t x(t))| \leq \frac{1}{4}|(t^2 - t + x(t))$$

It is clear that assumptions (I)–(II) of Theorem 1 are satisfied with $a_1(t) = \frac{1}{4}|\frac{1}{t+1}| \in L^1[0,1]$; *if* $t = 0$, *then* $a_1(0) = \frac{1}{4}$; *if* $t = 1$, *then* $a_1(1) = 0.12375$ *and* $a_2(t) = \frac{1}{4}|t^2| \in L^1[0,1]$; *if* $t = 0$, *then* $a_2(0) = 0$; *and if* $t = 1$, *then* $a_2(1) = 0$.

Then, $a = \sup\{0, \frac{1}{4}\} = \frac{1}{4}$, $K = \sup\{\frac{1}{4}, \frac{1}{4}\} = \frac{1}{4}$, $x_0 = 1$, *and* $\sum_{k=1}^{m} a_k = 1$; *then,* $A|x_0| + 2(Kr + a)^2 - r = 0 = A|x_0| + 2K^2r^2 + 4kar + 2a^2 - r = 0$.

By applying Theorem 1, the nonlocal problem in Equations (10) and (11), has a continuous solution.

Example 2. *Consider the boundary value problem with nonlocal*

$$\frac{d}{dt}\left(\frac{x(t) - x(0)}{\frac{1+2t}{20} + e^{-t}\frac{1}{30} + \sqrt{\frac{2}{3}}\, x(t)}\right) = \frac{1}{7}\ln(\frac{20}{1+t}) + \frac{1}{3}x(t), \tag{12}$$

with the nonlocal boundary condition (11).

Set

$$\begin{aligned} f(t,y) &= \frac{1}{7}\ln(\frac{20}{1+t}) + \frac{1}{3}y, \\ g(t,x) &= \frac{1+2t}{20} + e^{-t}\frac{1}{30} + \sqrt{\frac{2}{3}}\, x. \end{aligned}$$

We can easily deduce the following:

$$|f(t,x) - f(s,y)| \leq \frac{1}{3}|x - y|$$

and

$$|g(t,x) - g(s,y)| \leq \sqrt{\frac{2}{3}}\, |x(t) - y(t)|.$$

Easily, we can verify the existence of a unique solution of the problem in Equations (11) and (12).

8. Conclusions

In this work, we proved the existence of at least one solution $x \in C[0,1]$ and its maximum and minimum of the nonlocal problem for the boundary value problem of hybrid functional differential inclusion (1)

$$\frac{d}{dt}\left(\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))}\right) \in F(t, x(\phi_2(t))), \quad t \in (0,1)$$

with the nonlocal boundary condition (2)

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \; a_k > 0 \;\; \tau_k \in [0,1].$$

The uniqueness of the solution $x \in C[0,1]$ of the nonlocal problem for the boundary value problem of hybrid functional differential Equation (5)

$$\frac{d}{dt}\left(\frac{x(t) - x(0)}{g(t, x(\phi_1(t)))}\right) = f(t, x(\phi_2(t))), \quad t \in (0,1)$$

and its continuous dependence on x_o and initial data a_k is proven.

The results have been generalized for problems with the nonlocal conditions (3) and (4).

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