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s-Sequences and Monomial Modules

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Abstract: In this paper we study a monomial module M generated by an s-sequence and the main algebraic and homological invariants of the symmetric algebra of M. We show that the first syzygy module of a finitely generated module M, over any commutative Noetherian ring with unit, has a specific initial module with respect to an admissible order, provided M is generated by an s-sequence. Significant examples complement the results.

Keywords: symmetric algebra; monomial modules; Gröbner bases

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1. Introduction

In this paper we consider finitely generated modules, over a Noetherian commutative ring with identity R, generated by an s-sequence, whose rank is greater or equal to one, that is the modules are not necessarily ideals.

In this direction, the modules that imitate the ideals are the direct sum modules $\oplus I_i e_i$, submodules of a free R-module with basis $\{e_i\}, i = 1, \dots, n$, and I_i ideals of R. Since the main idea in the use of Gröbner bases is to reduce all problems to questions of monomial ideals, we study the monomial submodules $\oplus I_i e_i$, where all I_i are monomial ideals. Monomial modules were defined in [1] and were studied by many authors (see [2–7]). The aim of this paper is to investigate the symmetric algebra of a monomial module $M = \oplus I_i e_i$, a submodule of R^n, $R = K[x_1, \dots, x_m]$, K a field, and $I_1, \dots, I_n$ monomial ideals of R, via the theory of s-sequences [8–10]. the In Section 2, we review basic concepts of the theory of s-sequences and results about the main algebraic and homological invariants of the symmetric algebra of a finitely generated graded R-module M, generated by an s-sequence, provided R is a standard graded K-algebra and the generators of M are homogeneous sequence, or R is a polynomial ring in the field K. Then we introduce monomial modules and we recall several results and examples. After introducing a term order on the free module $M = I_i e_i$, $I_i \subset K[x_1, \dots, x_m]$, which is induced by the order $x_1 < x_2 < \dots < x_m < e_1 < \dots < e_n$, we formulate sufficient conditions to be a monomial module M generated by an s-sequence. As an application, we consider the special class of squarefree monomial S-modules $M = \oplus I^{(i)} e_i$, where each $I^{(i)}$ is the $(t_i - 1)$-th squarefree Veronese ideal of the polynomial ring $S^{(i)} = K[x_1^{(i)}, \dots, x_{t_i}^{(i)}]$, $S = K[\underline{x}^{(1)}, \underline{x}^{(2)}, \dots, \underline{x}^{(n)}]$, $\underline{x}^i = \{x_1^{(i)}, x_2^{(i)}, \dots, x_{t_i}^{(i)}\}, 1 \le i \le n$. In Section 3, inspired by [8], we introduce an admissible term order on the free module R^n, with basis $\{e_i\}, i = 1, \dots, n$, such that $e_1 < e_2 < \dots < e_n$, R a Noetherian ring with unit. We prove a remarkable result for the feature of the initial module, with respect to $<$, of the first syzygy module of a finitely generated R-module M generated by an s-sequence. Finally, we give an application to the first syzygy module of the class of mixed product ideals in two sets of variables [11,12], generated by an s-sequence [13–15].

Although the theory of s-sequences is defined in any field K, $char(K) = p \geq 0$, p a prime natural number, we fix the field $K = \mathbb{Q}$ if we use software CoCoA ([16]) to compute the Gröbner basis of the relation ideal of the symmetric algebra of a finitely generated $K[x_1, \ldots, x_m]$-module and the related algebraic invariants.

2. *s*-Sequences and Monomial Modules

The notion of s-sequences was given first in [8]. Let R be a Noetherian ring and let M be a finitely generated R-module with generators $f_1, f_2, \cdots, f_n$. We denote by (a_{ij}), $i = 1, \ldots, t, j = 1, \ldots, n$, the presentation matrix of M and by $Sym_R(M) = \oplus_{i \geq 0} Sym_i(M)$ the symmetric algebra of M, $Sym_i(M)$ the i-th symmetric power of $Sym_R(M)$. Note that $Sym_R(M) = R[y_1, \ldots, y_n]/J$, where $J = (g_1, \ldots, g_t)$, and $g_i = \sum_{j=1}^{n} a_{ij} y_j$, $i = 1, \ldots, t$. We consider a graded ring $S = R[y_1, \ldots, y_n]$ by assigning to each variable y_i the degree 1 and to the elements of R the degree 0. Then J is a graded ideal of S and the natural epimorphism $S \to Sym_R(M)$ is a homomorphism of graded R-algebras. Now, we introduce a monomial order $<$ on the monomials in $y_1, \ldots, y_n$ which is induced by the order on the variables $y_1 < y_2 < \ldots < y_n$. We call such an order an admissible order. For any polynomial $f \in R[y_1, \ldots, y_n], f = \sum_\alpha a_\alpha y^\alpha$, we put $in(f) = a_\alpha y^\alpha$ where y^α is the largest monomial in f with $a_\alpha \neq 0$, and we set $in(J) = (in(f) : f \in J)$. For $i = 1, \ldots, n$, we set $M_i = \sum_{j=1}^{i} Rf_j$, and let I_i be the colon ideal $M_{i-1} :< f_i >$. For convenience we put $I_0 = (0)$.

The colon ideals I_i are called annihilator ideals of the sequence $f_1, \ldots, f_n$. It easy to see that $(I_1 y_1, I_2 y_2, \ldots, I_n y_n) \subseteq in(J)$ and the two ideals coincide in degree 1.

Definition 1. *The generators $f_1, \ldots, f_n$ of M are called an s-sequence (with respect to an* admissible *order $<$) if $in(J) = (I_1 y_1, I_2 y_2, \ldots, I_n y_n)$.*

If in addition $I_1 \subset I_2 \subset \cdots \subset I_n$, then $f_1, \ldots, f_n$ is called a strong s-sequence.

In the case M is generated by an s-sequence, the theory of s-sequences leads to computations of invariants of $Sym_R(M)$ quite efficiently, in particular the Krull dimension $\dim(Sym_R(M))$, the multiplicity $e(Sym_R(M))$, the Castelnuovo Mumford regularity $reg(Sym_R(M))$ and the $depth(Sym_R(M))$, with respect to the graded maximal ideal, in terms of the invariants of quotients of R by the annihilators ideals of M (for more details on the invariants, see [17]).

Proposition 1 ([8] (Proposition 2.4, Proposition 2.6))**.** *Let M be a graded R-module, R a standard graded algebra, generated by a homogeneous s-sequence $f_1, \ldots, f_n$, where $f_1, \ldots, f_n$ have the same degree, with annihilator graded ideals $I_1, \ldots, I_n$. Then*

$$d := \dim(Sym_R(M)) = \max_{\substack{0 \leq r \leq n, \\ 1 \leq i_1 < \ldots < i_r \leq n}} \{\dim(R/(I_{i_1} + \ldots + I_{i_r})) + r\};$$

$$e(Sym_R(M)) = \sum_{\substack{0 \leq r \leq n, \\ 1 \leq i_1 < \ldots < i_r \leq n, \\ \dim(R/(I_{i_1} + \ldots + I_{i_r})) = d - r}} e(R/(I_{i_1} + \ldots + I_{i_r})).$$

When $f_1, \ldots, f_n$ is a strong s-sequence, then

$$d = \max_{0 \leq r \leq n} \{\dim(R/I_r) + r\};$$

$$e(Sym_R(M)) = \sum_{\substack{0 \leq r \leq n, \\ \dim(R/I_r) = d - r}} e(R/I_r).$$

If $R = K[x_1, \ldots, x_m]$ and $f_1, f_2, \ldots, f_n$ is a strong s-sequence:

$$reg(Sym_R(M)) \leq \max\{reg(I_i) : i = 1, \ldots, n\};$$

$$depth(Sym_R(M)) \geq \min\{depth(R/I_i) + i : i = 0, 1, \ldots, n\}.$$

We recall fundamental results on monomial sequences.

Consider $R = K[x_1, x_2, \ldots, x_m]$, where K is a field, and let $I = (f_1, \ldots, f_n)$ be, where $f_1, \ldots, f_n$ are monomials. Set $f_{ij} = \frac{f_i}{\gcd(f_i, f_j)}$, $i \neq j$. Then J is generated by $g_{ij} := f_{ij}y_j - f_{ji}y_i, 1 \leq i < j \leq n$, and the annihilator ideals of the sequence $f_1, \ldots, f_n$ are the ideals $I_i = (f_{1i}, f_{2i}, \ldots, f_{(i-1)i})$. As a consequence, a monomial sequence is an s-sequence if and only if the set $\{g_{ij}, 1 \leq i < j \leq n\}$, is a Gröbner basis for J for any term order on the monomials of $R[y_1, \ldots, y_n]$ which extends an admissible term order on the monomials in the y_i. Let us now fix such a term order.

Proposition 2 ([8] (Proposition 1.7))**.** *Let $I = (f_1, \ldots, f_n) \subset K[x_1, x_2, \ldots, x_m]$ be a monomial ideal. Suppose that for all $i, j, k, l \in \{1, \ldots, n\}$, with $i < j$, $k < l$, $i \neq k$ and $j \neq l$, we have* $\gcd(f_{ij}, f_{kl}) = 1$. *Then $f_1, \ldots, f_n$ is an s-sequence.*

Now let $R = K[x_1, x_2, \ldots, x_m]$ be and let F be the finite free R-module $F = Re_1 \oplus \ldots \oplus Re_n$ with basis $e_1, \ldots, e_n$. We refer to [1] (Ch.15, 15.2) for definitions and results on monomial modules.

Definition 2. *An element $m \in F$ is a monomial if m has the form ue_i, for some i, where u is a monomial of R. A submodule $U \subset F$ is a monomial module if it is generated by monomials of F.*

One can observe that if U be a submodule of the free R-module $F = \oplus_{i=1}^n Re_i$, then U is a monomial module if and only if for each i there exists a monomial ideal I_i such that $U = I_1e_1 \oplus I_2e_2 \oplus \ldots \oplus I_ne_n$. In particular, U is finitely generated.

Theorem 1. *Let $M = \oplus_{i=1}^n I_ie_i$ be a monomial R-module, $M_i = I_ie_i$, $I_i = (m_{i1}, \ldots, m_{ir_i})$, a monomial ideal of $R = K[x_1, \ldots, x_n]$ then*

(i) $Syz_1(M_i) \cong Syz_1(I_i)$,

(ii) $Syz_1(M) \cong Syz_1(I_1) \oplus Syz_1(I_2) \oplus \ldots \oplus Syz_1(I_n)$,

Proof. (i) Write $M_i = \langle m_{i1}e_i, \ldots, m_{ir_i}e_i \rangle$ and let

$$0 \to Syz_1(M_i) \to \quad R^{r_i} \to M_i \quad \to 0 \tag{1}$$

be a presentation of M_i. Consider the R-linear homomorphism $R^{r_i} \to M_i$ such that $g_j \to m_{ij}e_i$, $R^{r_i} = Rg_1 \oplus \ldots \oplus Rg_{r_i}$, and a syzygy of M_i, $a \in R^{r_i}$, $a = (\lambda_{i1}, \ldots, \lambda_{ir_i})$. Then

$$\sum_{j=1}^{r_i} \lambda_{ij} m_{ij} = 0,$$

and a is a syzygy of I_i.

(ii) It follows by (i). □

Let M be a monomial R-module defined as in Theorem 1. We will prove a criterion for a monomial module to be generated by an s-sequence. Set

$$m_{ij,lk} = \frac{m_{ij}}{gcd(m_{ij}, m_{lk})}, \qquad m_{ij} \in I_i, m_{lk} \in I_l,$$

$$1 \leq i, j \leq n, \quad 1 \leq j \leq r_i, \quad 1 \leq k \leq r_l.$$

Theorem 2. *Let $M = \oplus_{i=1}^n I_ie_i$ be a monomial module, $I_i = (m_{i1}, \ldots, m_{ir_i})$, $i = 1, \ldots, n$. Suppose $gcd(m_{ij,ik}, m_{tu,tv}) = 1$, $j < k$, $u < v$, with $i = t$ and $j \neq u$, $k \neq v$ or with $i \neq t$ and $1 \leq j, k \leq r_i, 1 \leq u, v \leq r_t$. Then M is generated by the s-sequence $m_{11}e_1, \ldots, m_{1r_1}e_1, \ldots, m_{n1}e_n, \ldots, m_{nr_n}e_n$.*

Proof. For each $i = 1, \ldots, n$, $Syz_1(M_i)$ is generated by the binomials:

$$m_{ij,ik} g_{ik} - m_{ik,ij} g_{ij}$$

since i is fixed, $1 \le j, k \le r_i$, being g_{ik}, g_{ij} the free basis of R^{r_i}. Thanks to the hypothesis, we have $gcd(m_{ij,ik}, m_{iu,iv}) = 1$, $j < k$, $u < v$, $j \ne u$, $k \ne v$, $\forall\, i = 1, \ldots, n$, and we conclude, by Proposition 2, that M_i is generated by an s-sequence.

Now, suppose $i < t$. If T_{ik} and T_{tv} are the variables that correspond to g_{ik} and g_{tv}, then $T_{ik} \ne T_{tv}$. We have $gcd(m_{ij,ik} T_{ik}, m_{tu,tv} T_{tv}) = gcd(m_{ij,ik}, m_{tu,tv}) = 1$by hypothesis. In conclusion, the S-pair $S(b_{ijk}, b_{tuv})$ reduces to zero, where $b_{ijk} = m_{ij,ik} T_{ik} - m_{ik,ij} T_{ij}$ and $b_{tuv} = m_{tu,tv} T_{tv} - m_{tv,tu} T_{tu}$. Then the assertion follows. □

Example 1. *Let $M = I_1 e_1 \oplus I_2 e_2$, $I_1 = (x^2, y^2, z)$ and $I_2 = (z^2, zy)$ be ideals of $K[x, y, z]$. We have $m_{11,12} = m_{11,13} = x^2, m_{12,13} = y^2, m_{21,22} = z$. Since $gcd(m_{11,12}, m_{12,13}) = gcd(m_{11,12}, m_{21,22}) = gcd(m_{11,13}, m_{21,22}) = 1$, then M is generated by the s-sequence $x^2 e_1, y^2 e_1, z e_1, z^2 e_2, zy e_2$.*

The next example considers a monomial module M not generated by an s-sequence, even if each addend is generated by an s-sequence.

Example 2. *Let $M = (x, y)e_1 \oplus (x, y)e_2$ be, $I_1 = I_2 = (x, y)$ ideals of $R = K[x, y]$. Write $Sym_R(M) = R[T_1, T_2, T_3, T_4]/J$, where $J = (yT_1 - xT_2, yT_3 - xT_4)$ We compute the S-pair $S(yT_1 - xT_2, yT_3 - xT_4) = -y(T_1 T_4 - T_2 T_3)$, with $T_4 > T_3 > T_2 > T_1$. If $T_1 T_4 > T_2 T_3$, $in_< J = (xT_2, xT_4, yT_1 T_4)$ and if $T_1 T_4 < T_2 T_3$, $in_< J = (xT_2, xT_4, yT_2 T_3)$. In any case, J does not have a Gröbner basis which is linear in the variables T_i.*

Now we quote a statement on computation of the annihilator ideals of $M = \oplus_{i=1}^n I_i e_i$, that is to say the annihilator ideals of the generating sequence of M

$$m_{11} e_1, m_{12} e_1, \ldots, m_{1 r_1} e_1, m_{21} e_2, \ldots, m_{2 r_2} e_2, \ldots, m_{n1} e_n, \ldots, m_{n r_n} e_n.$$

Proposition 3. *Let $K_{i1}, K_{i2}, \ldots, K_{i r_i}$ be the annihilator ideals of $M_i = I_i e_i$, Set $J_1, \ldots, J_{r_1}, J_{r_1+1}, J_{r_1+2}, \ldots, J_{r_1+r_2}, J_{r_1+r_2+1}, \ldots, J_{r_1+r_2+\ldots+r_n}$ the annihilator ideals of the sequence. Then we have:*

$$J_1 = K_{11} = (0), J_2 = K_{12}, \ldots, J_{r_1} = K_{1 r_1}, J_{r_1+1} = K_{21} = (0), J_{r_1+2} = K_{22}, \ldots,$$

$$J_{r_1+r_2} = K_{2 r_2}, \ldots, J_{r_1+r_2+\ldots+r_{n-1}+1} = K_{n1} = (0), J_{r_1+r_2+\ldots+r_{n-1}+2} = K_{n2},$$
$$\ldots, J_{r_1+r_2+\ldots+r_n} = K_{n r_n}.$$

Proof. An elementary computation gives:

$$\langle 0 \rangle : \langle m_{11} e_1 \rangle = K_{11} = (0)$$

$$\langle m_{11} e_1 \rangle : \langle m_{12} e_1 \rangle = K_{12}$$

$$\langle m_{11} e_1, m_{12} e_1 \rangle : \langle m_{13} e_1 \rangle = K_{13}$$

$$\cdots\cdots\cdots$$

$$\langle m_{11} e_1, m_{12} e_2, \ldots, m_{1 r_{1-1}} e_1 \rangle : \langle m_{1 r_1} e_1 \rangle = K_{1 r_1}$$

$$\begin{aligned} \langle m_{11} e_1, m_{12} e_1, \ldots, m_{1 r_{1-1}} e_1, m_{1 r_1} e_1 \rangle : \langle m_{21} e_2 \rangle &= I_1 e_1 : \langle m_{21} e_2 \rangle + (0) : \langle m_{21} e_2 \rangle = \\ &= (0) + K_{21} = (0) \\ \langle m_{11} e_1, m_{12} e_1, \ldots, m_{1 r_{1-1}} e_1, m_{1 r_1} e_1, m_{21} e_2 \rangle : \langle m_{22} e_2 \rangle &= \langle I_1 e_1, m_{21} e_2 \rangle : \langle m_{22} e_2 \rangle = \\ &= I_1 e_1 : \langle m_{22} e_2 \rangle + K_{22} = (0) + K_{22} = K_{22}. \end{aligned}$$

The proof goes on by a routine computation. □

Example 3. *Let $M = I_1e_1 \oplus I_2e_2$ be a monomial module on $R = K[x,y,z]$, where $I_1 = (x^2, y^2, xy)$, $I_2 = (z^2, zy)$.Then M is generated by the s-sequence $x^2e_1, y^2e_1, xye_1, z^2e_2, zye_2$ with $x < y < z < e_1 < e_2$. The s-sequence has the following annihilator ideals:*

$$\begin{aligned} J_1 &= \langle 0\rangle : \langle x^2e_1\rangle = K_{11} = (0) \\ J_2 &= \langle x^2e_1\rangle : \langle y^2e_1\rangle = K_{12} = (x^2) \\ J_3 &= \langle x^2e_1, y^2e_1\rangle : \langle xye_1\rangle = K_{13} = (x,y) \\ J_4 &= \langle x^2e_1, y^2e_1, xye_1\rangle : \langle z^2e_2\rangle = K_{21} = (0) \\ J_5 &= \langle x^2e_1, y^2e_1, xye_1, z^2e_2\rangle : \langle zye_2\rangle = (0) + K_{22} = (z) \end{aligned}$$

By Proposition 1, we have $\dim(Sym_R(M)) = 5$. The maximum of the dimensions is obtained by $\dim(R/(J_1+J_2+J_3+J_4+J_5)) + 5 = \dim(R/((x^2)+(x,y)+(z)) + 5 = 5$. For the multiplicity, we have $e(Sym_R(M)) = e(R/(J_1+J_4)) + e(R/(J_1+J_2+J_3+J_4 +J_5)) = 1$, since $e(R/(J_1+J_4)) = e(K[x,y,z]) = 1$ and $e(R/(J_1+J_2+J_3+J_4+J_5)) = e(K) = 0$. Concerning the depth and the Castelnuovo regularity, since it results $Sym_R(M) = R[T_1,T_2,T_3,T_4,T_5]/J = R[T_1,T_2,T_3,T_4,T_5]/(xT_2 - yT_3, yT_1 - xT_3, yT_4 - zT_5)$, we compute $depth(Sym_R(M)) = 5$ and $reg(Sym_R(M)) = 3$ using software CoCoA ([16]).

We conclude the section yielding a class of monomial modules that would be of large interest in combinatorics, considering that they involve monomial squarefree ideals. Let $S = K[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}]$ be a polynomial ring in n sets of variables $\underline{x}^i = \{x_1^{(i)}, x_2^{(i)}, \ldots, x_{t_i}^{(i)}\}$, $1 \le i \le n$. Let I_s be the monomial ideal of S generated by all squarefree monomials of degree s (the s-th squarefree Veronese ideal of S). Consider the squarefree monomial ideal $I^{(i)}_{t_i-1}$, $i = 1,\ldots,n$, of $S^{(i)} = K[\underline{x}^{(i)}]$ generated by all squarefree monomials of degree $t_i - 1$ (the (t_i-1)-th squarefree Veronese ideal) as a monomial ideal of S.

Theorem 3. *The monomial module $M = \oplus_{i=1}^n I^{(i)}_{t_i-1}e_i$ on $S = K[\underline{x}^{(1)}, \underline{x}^{(2)}, \ldots, \underline{x}^{(n)}]$ is generated by an s-sequence.*

Proof. It is known that for each i, $I^{(i)}_{t_i-1}$ is generated by an s-sequence ([14] (Theorem 2.3)), being generated by t_i squarefree monomials in $t_i - 1$ variables in the polynomial ring in t_i variables and that condition 1) of [14] (Theorem 1.3.2.) is satisfied. The ideals $I^{(i)}_{t_i-1}$ and $I^{(j)}_{t_j-1}$, for any $i \ne j$, $i,j = 1,\ldots,n$, are generated in 2 disjoint sets of variables, then the condition of Theorem 2 is easily verified. □

The invariants of $Sym_S(M)$ depend on the invariants of each addend of M.

Theorem 4. *Let $M = \oplus_{i=1}^n I^{(i)}_{t_i-1}e_i$ be and let $Sym_S(M)$ be its symmetric algebra. Then:*

(1) $\dim_S(Sym_S(M)) = \sum_{i=1}^n t_i + n = \sum_{i=1}^n \dim_{S^{(i)}}(Sym_{S^{(i)}}(M_i))$
(2) $depth(Sym_S(M)) = \sum_{i=1}^n t_i + n = \sum_{i=1}^n depth_{S^{(i)}}(Sym_{S^{(i)}}(M_i))$
(3) $e(Sym_S(M)) = \sum_{j=1}^{\sum t_i - n - 1} \binom{\sum t_i - n}{j} + 2$
(4) $reg(Sym_S(M)) = \sum_{i=1}^n t_i - n$

Proof. We consider an admissible term order on the monomials of $S[T_1^{(1)}, \ldots, T_{t_n}^{(n)}]$ such that $x_j^l < T_1^{(1)} < T_2^{(1)} < \ldots < T_{t_n}^{(n)}$.

(1) The annihilators ideals of the module $M_i = I^{(i)}_{t_i-1}e_i$ are the annihilators ideals $J_j^{(i)}$ of the sequence generating $I^{(i)}_{t_i-1}$, in the lexicographic order, for each $i = 1,\ldots,n$, $j = 1,\ldots,t_i$.

By [14] (Proposition 3.1), we have $J_1^{(i)} = (0), J_2^{(i)} = (x_{t_i-1}^{(i)}), J_3^{(i)} = (x_{t_i-2}^{(i)}), \ldots, J_{t_i}^{(i)} = (x_1^{(i)})$. Then, if J is the relation ideal of $Sym_S(M)$, we have:

$$in_<(J) = (x_{t_1-1}^{(1)}T_2^{(1)}, x_{t_1-2}^{(1)}T_3^{(1)}, \ldots, x_1^{(1)}T_{t_1}^{(1)}, \ldots, x_{t_n-1}^{(n)}T_2^{(n)},$$

$$x_{t_n-2}^{(n)}T_3^{(n)}, \ldots, x_1^{(n)}T_{t_n}^{(n)})$$

and it is generated by a regular sequence. We obtain

$$\dim_S(Sym_S(M)) = \sum_{i=1}^n t_i + \sum_{i=1}^n t_i - \left(\sum_{i=1}^n t_i - n\right) = \sum_{i=1}^n t_i + n.$$

(2) Since $depth(Sym_S(M)) \leq \dim_S(Sym_S(M)) = \sum_{i=1}^n t_i + n$ and $depth(Sym_S(M)) \geq depth(S[T_1^{(1)}, \ldots, T_{t_1}^{(1)}, \ldots, T_1^{(n)}, \ldots, T_{t_n}^{(n)}]/in_<(J)) = \sum_{i=1}^n t_i + n$, the equality follows.

(3) In the following, we often use methods and tools of [14] (Theorem 3.6). For each i, $1 \leq i \leq n$, with $S^{(i)} = K[\underline{x}^{(i)}]$, we have

$$e(Sym_{S^{(i)}}(I_{t_i-1}^{(i)}e_i)) = \sum_{1 \leq i_1 < \ldots < i_r \leq t_i} e\left(S^{(i)}/(J_{i_1}^{(i)}, \ldots, J_{i_r}^{(i)})\right)$$

with $\dim\left(S^{(i)}/(J_{i_1}^{(i)}, \ldots, J_{i_r}^{(i)})\right) = d - r$, $d = \dim(Sym_{S^{(i)}}(I_{t_i-1}^{(i)}e_i)) = t_i + 1$ and $1 \leq r \leq t_i$, being $J_{i_1}^{(i)}, \ldots, J_{t_i}^{(i)}$ the annihilators ideals of $I_{t_i-1}^{(i)}$. It results, by the structure of the annihilators ideals, $H^{(i)} = (J_{i_1}^{(i)}, \ldots, J_{i_r}^{(i)}) = (x_{i_1}^{(i)}, \ldots, x_{i_r}^{(i)})$. Put $H = (H^{(1)}, H^{(2)}, \ldots, H^{(n)}) = (x_{i_1}^{(1)}, \ldots, x_{i_r}^{(1)}, x_{i_1}^{(2)}, \ldots, x_{i_r}^{(2)}, \ldots, x_{i_1}^{(n)}, \ldots, x_{i_r}^{(n)})$. Then $e(S/H) = 1$ since S/H is a polinomial ring on a field k. Let

$$d' = \dim(S/(J_{i_1}^{(i)}, \ldots, J_{i_r}^{(i)})) = \sum_{i=1}^n t_i + n - r, 1 \leq i \leq n, 1 \leq r \leq \sum_{i=1}^n t_i,$$

then $e(Sym_S(M))$ is given by the sum of the following addends:

$$e(S/(0)) = 1$$

for $r = 1,\ d' = \sum_{i=1}^n t_i + n - 1$.

$$\sum_{j=2}^{\sum t_i} e(S/J_j^{(i)}) = \underbrace{1 + \ldots + 1}_{\sum t_i - n}$$

for $r = 2,\ d' = \sum_{i=1}^n t_i + n - 2$.

$$\sum_{2 \leq k_1 \leq t_k, 2 \leq l_1 \leq t_l} e(S/(J_{k_1}^{(k)}, J_{l_1}^{(l)})) = \underbrace{1 + \ldots + 1}_{\binom{\sum t_i - n}{2}}$$

for $r = 3,\ d' = \sum_{i=1}^n t_i + n - 3, 1 \leq k, l \leq n$

$$\sum_{2 \leq k_1 \leq t_k, 2 \leq l_1 \leq t_l, 2 \leq m_1 \leq t_m} e(S/(J_{k_1}^{(k)}, J_{l_1}^{(l)}, J_{m_1}^{(m)})) = \underbrace{1 + \ldots + 1}_{\binom{\sum t_i - n}{3}}$$

for $r = 4,\ d' = \sum t_i + n - 4, 1 \leq k, l, m \leq n$

$$\vdots$$

$$\sum_{2\le u_1<\cdots<u_r\le t_1,\ldots,2\le s_1<\cdots<s_r\le t_n} e(S/(J^{(1)}_{u_1},\ldots,J^{(1)}_{u_r},\ldots,J^{(n)}_{s_1},\ldots,J^{(n)}_{s_r}) = \underbrace{1+\ldots+1}_{\binom{\sum t_i-n}{\sum t_i-n-1}}$$

for $r=\sum t_i-1,\ d'=n+1$.

$$e\Big(S/(J^{(1)}_2,\ldots,J^{(1)}_{t_1},J^{(2)}_2,\ldots,J^{(2)}_{t_2},J^{(n)}_2,\ldots,J^{(2)}_{t_n})\Big)=1$$

for $r=\sum_{i=1}^n t_i,\ d'=n$. Thus,

$$e(Sym_S(M))=\sum_{j=1}^{\sum t_i-n-1}\binom{\sum t_i-n}{j}+2.$$

(4) $reg(Sym_S(M))=reg(S[\underline{T}^{(1)},\ldots,\underline{T}^{(n)}]/J)\le reg(S[\underline{T}^{(1)},\ldots,\underline{T}^{(n)}]/in_<(J)),\ \underline{T}^{(i)}=\{T^{(i)}_1\ldots T^{(i)}_{t_i}\}$, for $1\le i\le n$. The ideal

$$in_<(J)=(x^{(1)}_{t_1-1}T^{(1)}_2,\ldots,x^{(1)}_1T^{(1)}_{t_1},x^{(2)}_{t_2-1}T^{(2)}_2,\ldots,x^{(2)}_1T^{(1)}_{t_2},\ldots x^{(n)}_{t_n-1}T^{(n)}_2,\ldots,x^{(n)}_1T^{(n)}_{t_n})$$

is generated by a regular sequence of length $\sum t_i-n$ of monomials of degree 2. The ring $S[\underline{T}^{(1)},\ldots,\underline{T}^{(n)}]/in_<(J)$ has a resolution of length $\sum_{i=1}^n t_i-n$, equal to the number of generators of $in_<(J)$, given by the Koszul complex of $in_<(J)$. Then $reg(Sym_S(M))\le\sum_{i=1}^n t_i-n$. Since J is Cohen-Macaulay and

$$\dim(Sym_S(M))=\sum_{i=1}^n t_i+n, \dim S[\underline{T}^{(1)},\ldots,\underline{T}^{(n)}]/J=\sum_{i=1}^n t_i+\sum_{i=1}^n t_i-ht(J),$$

then $ht(J)=grad(J)=2\sum_{i=1}^n t_i-(\sum_{i=1}^n t_i+n)=\sum_{i=1}^n t_i-n$. Since J is a graded ideal [17] (Proposition 1.5.12), we can choose the regular sequence in J inside the binomials of degree two generating J. So the Koszul complex on the regular sequence gives a 2-linear resolution of J. It follows

$$reg(S[\underline{T}^{(1)},\ldots,\underline{T}^{(n)}]/J)\ge 2\left(\sum_{i=1}^n t_i-n\right)-\left(\sum_{i=1}^n t_i-n\right)=\sum_{i=1}^n t_i-n.$$

The equality follows. □

3. Groebner Bases of Syzygy Modules and *s*-Sequences

Let R be a Noetherian commutative ring with unit. Let N be a finitely generated R-module submodule of a free R-module $R^n=Re_1\oplus\ldots\oplus Re_n$, $N=Rg_1+\ldots+Rg_m$, $g_i=a_{i1}e_1+\ldots a_{in}e_n$, $i=1,\ldots,m$. Consider an order on the standard vectors $e_1,\ldots,e_n$ of R^n such that $e_n>\ldots>e_1$. We may view N as a graded module by assigning to each vector e_i the degree 1 and to the elements of R the degree 0. For any vector $h\in Re_1+\ldots Re_n$, $h=\sum_{i=1}^n a_ie_i$, we put $in(h)=a_je_j$, where e_j is the largest vector in h with $a_j\neq 0$. Such an order will be called admissible. Set $in(N)=<in(h),h\in N>$. We say that $g_1,\ldots,g_m$ is a initial basis for N if $in(N)=<K_1e_1,\ldots,K_ne_n>=\oplus K_ie_i$, where K_j are ideals of R.

Take $N=Syz_1(M)$ the first syzygy module of a finitely generated R-module M. We have:

Theorem 5. *Let M be a finitely R-module generated by an s-sequence $f_1,\ldots,f_n$ and let $N=Syz_1(M)$. Then $in(N)=<I_1e_1,\ldots,I_ne_n>$, where $I_1,\ldots,I_n$ are the annihilator ideals of the sequence $f_1,\ldots,f_n$.*

Proof. Let us introduce an admissible order in $R^n=\oplus_{i=1}^n Re_i$, with $e_1<e_2<\ldots<e_n$. Then $in_<(N)=<in_<(f),f\in N>=<K_1e_1,\ldots,K_ne_n>$, with K_j ideals of R. Passing to the symmetric algebras $Sym_R(M)$, the relation ideal J is generated linearly in the variables

T_j, $j = 1, \ldots, n$, corresponding to the vectors $e_1 < e_2 < \ldots < e_n$, with the order $T_1 < T_2 < \ldots < T_n$, and $in_<(J) = (I_1T_1, \ldots, I_nT_n)$. Let $G(J)$ be the finite set of linear forms in $T_1, T_2, \ldots, T_n$, which generate J and such that $in_<(J) = (in_< f, f \in G(J))$ and let $\tilde{G}(J) = G(N)$ be the set of generators $\tilde{f}$ of $N = Syz_1(M)$ corresponding to f under the substitution $T_i \to e_i$, $i = 1, \ldots, n$. Then we have $in_<(N) =< in_<(\tilde{f}), \tilde{f} \in G(N) >$. We deduce that $K_j = I_j$ for $j = 1, \ldots, n$. Hence the assertion follows. $\square$

Example 4. *Let $I = (X^2, Y^2, XY)$ be an ideal of $R = K[X, Y]$. The relation ideal J of $Sym_R(I)$ is $J = (XT_3 - YT_1, YT_3 - XT_2)$. The Gröbner basis of J is $G(J) = \{XT_3 - YT_1, YT_3 - XT_2, X^2T_2 - Y^2T_1\}$ which is linear in the variables T_1, T_2, T_3 and I is generated by the s-sequence X^2, Y^2, XY. Consider $Syz_1(I) =< Xe_3 - Ye_1, Ye_3 - Xe_2 >$. Then $\tilde{G}(J) = G(N) = \{Xe_3 - Ye_1, Ye_3 - Xe_2, X^2e_2 - Y^2e_1\}$ and $in_< J = ((X^2)T_2, (X, Y)T_3)$, $in_<(N) =< (X^2)e_2, (X, Y)e_3 >$.*

Notice that X^2, XY, Y^2 is not an s-sequence for I. In fact, in such case, the relation ideal is $J = (XT_2 - YT_2, YT_2 - XT_3)$ and $G(J) = \{XT_2 - YT_1, YT_2 - XT_3, X^2T_3 - Y^2T_1, T_2^2 - T_1T_3\}$ not linear in the variables T_1, T_2, T_3, in both cases $T^2 > T_1T_3$ or $T_1T_3 > T^2$. We have $G(N) = \{Xe_2 - Ye_1, Ye_2 - Xe_3, X^2e_3 - Y^2e_1\}$, but the generators of $G(N)$ are not obtained by the substitution of T_i with e_i, in the elements of the Gröbner basis of J.

Now, let $R = K[X_1, \ldots, X_t]$ be a polynomial ring over the field K, and let $<$ be a term order on the monomials of $R^n = K[X_1, \ldots, X_t]e_1 \oplus \ldots \oplus K[X_1, \ldots, X_t]e_n$ with $e_1 < \ldots < e_n$ and $X_j < e_i$, for all i and j. The excellent book of D. Eisenbud ([1] (Ch.15,15.2)) covers all background for free modules on polynomial rings and Gröbner bases for their submodules. It is easy to prove:

1. For any Gröbner basis G of N (with respect to the order $<$) that exists finite, we have $in(N) =< in(f), f \in G >$.
2. If M is a monomial module, $in_<(M) = in(M)$.

Now we recall the definition of monomial mixed product ideals which were first introduced in [11], since some classes of such ideals are generated by an s-sequence. To be precise, in the polynomial ring $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$ in two set of variables on a field K, the squarefree monomial ideals $I_kJ_r + I_sJ_t$, with $k + r = s + t$, are called ideals of mixed products, where I_k (resp. J_r) is the squarefree ideal of $K[X_1, \ldots, X_n]$ (resp. $K[Y_1, \ldots, Y_m]$) generated by all squarefree monomials of degree k(resp. degree r). In the same way I_s and J_t are defined. Setting $I_0 = J_0 = R$, in [14] we find the following classification:

1. $I_k + J_k, 1 \leq k \leq \inf\{n, m\}$
2. $I_kJ_r, 1 \leq k \leq n, 1 \leq r \leq m$
3. $I_kJ_r + I_{k+1}J_{r-1}, 1 \leq k \leq n, 2 \leq r \leq m$
4. $J_r + I_sJ_t$, with $r = s + t, 1 \leq s \leq n, 1 \leq r \leq m, t \geq 1$
5. $I_kJ_r + I_sJ_t$, with $k + r = s + t, 1 \leq k \leq n, 1 \leq r \leq m$

Theorem 6 ([14] (Theorem 2.8, Theorem 2.11, Theorem 2.14))**.** *Let the ideal L_i be one of the following mixed product ideals*

1. $L_1 = I_{n-1}J_m$
2. $L_2 = I_nJ_{m-1}$
3. $L_3 = I_1J_m$
4. $L_4 = I_nJ_1$
5. $L_5 = I_nJ_{m-1} + I_{n-1}J_m$
6. $L_6 = I_nJ_1 + J_m, n + 1 = m.$

Then L_i is generated by an s-sequence.

We premise the following:

Proposition 4. *Let I_{n-1} be the Veronese squarefree $(n-1)$-th ideal of $R = K[X_1, \ldots, X_n]$. Let $N = Syz(I_{n-1})$ and G be the Gröbner basis of N. Then*

1. $G = \{X_n e_1 - X_{n-1}e_2, X_{n-1}e_2 - X_{n-2}e_3, \ldots, X_2 e_{n-1} - X_1 e_n\}$
2. $in_< N = (X_{n-1})e_2 \oplus (X_{n-2})e_3 \oplus \ldots \oplus (X_1)e_n \cong \underbrace{R(n) \oplus R(n) \oplus \ldots \oplus R(n)}_{(n-1)-times}$ *as graded R-modules.*
3. $in_< N$ *is generated by a s-sequence.*

Proof. Let $<$ be an admissible term order introduced on the monomials of $R^n = \oplus Re_i$, with $X_1 < X_2 < \ldots < X_n < e_1 < e_2 < \ldots < e_n$, $R = K[X_1, \ldots, X_n]$. The ideal $I_{n-1} = (X_1 \cdots X_{n-1}, \ldots, X_2 \cdots X_{n-1}X_n)$ is generated by an s-sequence ([14] (Theorem 2.3)), then

$$in_<(J) = (K_2T_2, \ldots, K_nT_n),$$

where J is the relation ideal of $Sym_R(I_{n-1})$ and $K_i = (X_{n-i+1}), i = 2, \ldots, n$, are the annihilator ideals of I_{n-1} (See [14] (Proposition 3.1)). Let $N = Syz_1(I_{n-1})$ be. Then $N =< X_n e_1 - X_{n-1}e_2, X_{n-1}e_2 - X_{n-2}e_3, \ldots, X_2 e_{n-1}, X_2 e_{n-1} - X_1 e_n >$ is generated by a Gröbner basis, being J generated by a Gröbner basis, $J = (X_nT_1 - X_{n-1}T_2, X_{n-1}T_2 - X_{n-2}T_3, \ldots, X_2T_{n-1}, X_2T_{n-1} - X_1T_n)$, with $X_1 < X_2 < \ldots < X_n < T_1 < T_2 < \ldots < T_n$ ([13] (Theorem 2.13)) and

$$in_< N =< (X_{n-1})e_2, (X_{n-2})e_3, \ldots, (X_1)e_n >$$

and it is trivially generated by an s-sequence or it follows by Theorem 2. □

For each L_i, $i = 1, \ldots, 6$, as in **T**heorem 6, we assume that $f_1 < f_2 < \ldots < f_{s_i}$ in the lexicografic order and $X_1 < X_2 < \ldots < X_n < Y_1 < Y_2 < \ldots < Y_m$ in the ring $R = K[X_1, \ldots X_n; Y_1, \ldots, Y_m]$.

Theorem 7. *Let $N_i = Syz(L_i)$ be the first syzygy module of L_i defined in Theorem 6 and let $G(N_i)$ be the Gröbner basis of N_i. Then we have:*

1. $G(N_1) = \{X_n e_1 - X_{n-1}e_2, X_{n-1}e_2 - X_{n-2}e_3, \ldots, X_2 e_{n-1} - X_1 e_n\}$ *and*
$$in_<(N_1) = K_2e_2 \oplus \ldots \oplus K_ne_n,\ K_i = (X_{n-i+1}), i = 2, \ldots, n$$
2. $G(N_2) = \{Y_m e_1 - Y_{m-1}e_2, Y_{m-1}e_2 - Y_{m-2}e_3, \ldots, Y_2 e_{m-1} - Y_1 e_m\}$ *and*
$$in_<(N_2) = K_2e_2 \oplus \ldots \oplus K_me_m,\ K_i = (Y_{m-i+1}), i = 2, \ldots, m$$
3. $G(N_3) = \{X_1e_2 - X_2e_1, X_2e_3 - X_3e_2, \ldots, X_{n-1}e_n - X_ne_{n-1}\}$ *and*
$$in_<(N_3) = K_2e_2 \oplus \ldots \oplus K_ne_n,\ K_i = (X_1, \ldots, X_{i-1}), i = 2, \ldots, n$$
4. $G(N_4) = \{Y_1e_2 - Y_2e_1, Y_2e_3 - Y_3e_2, \ldots, Y_{m-1}e_m - Y_me_{m-1}\}$
$$in_<(N_4) = K_2e_2 \oplus \ldots \oplus K_me_m,\ K_i = (Y_1, \ldots, Y_{i-1}), i = 2, \ldots, m$$
5. $G(N_5) = \{Y_me_1 - Y_{m-1}e_2, \ldots, Y_2e_{m-1} - Y_1e_m, Y_1e_m - X_ne_{m+1}, X_ne_{m+1} - X_{n-1}e_{m+2}, \ldots, X_2e_{m+n-1} - X_1e_{m+n}\}$
$$\text{and } in_<(N_5) = K_2e_2 \oplus \ldots \oplus K_me_m \oplus K_{m+1}e_{m+1} \oplus \ldots \oplus K_{m+n}e_{m+n}$$
with $K_i = (Y_{m-i+1}), i = 2, \ldots, m$, *and* $K_i = (X_{n+m-i+1}),\ i = m+1, \ldots, m+n$
6. $G(N_6) = \{Y_1e_2 - Y_2e_1, Y_2e_3 - Y_3e_2, \ldots, Y_{m-1}e_m - Y_me_{m-1}, (X_1 \cdots X_n)e_{m+1} - (Y_2 \cdots Y_m)e_1\}$ *and*
$$in_<(N_6) = K_2e_2 \oplus \ldots \oplus K_me_m \oplus (X_1 \cdots X_n)e_{m+1},\ K_i = (Y_1, \ldots, Y_{i-1}), i = 2, \ldots, m.$$

Proof. For each $i = 1, \ldots, 6$, the relation ideal J_i of $Sym_R(L_i)$ is generated by a Gröbner basis $G(J)$, then we apply Theorem 5 and we obtain the Gröbner basis $G(N_i)$, by the substitution of the vector e_i to the variable T_i in the forms of the set $G(J_i)$. For the structure of $in_<(N_i)$, $i = 1, \ldots, 6$, we have:

1. The ideal $I_{n-1}J_m$ has annihilator ideals $K_i = (X_{n-i+1})$, $i = 2, \ldots, n$ (See [14] (Proposition 3.3)). Then
$$in_< N_1 =< (X_{n-1})e_2, (X_{n-2})e_3, \ldots, (X_1)e_n >= (X_{n-1})e_2 \oplus (X_{n-2})e_3 \oplus \ldots \oplus (X_1)e_n \cong$$
$$\cong \underbrace{R(m+n) \oplus \ldots \oplus R(m+n)}_{(n-1)-\text{times}}$$
as graded R-modules.
2. In this case the the annihilator ideals of I_nJ_{m-1} are $K_i = (Y_{m-i+1}), i = 2, \ldots, m$. The proof is analogue to the case of $I_{n-1}J_m$.
3. The ideal $I_1J_m = (X_1, \ldots, X_n)(Y_1 \cdots Y_m)$ is generated by an s-sequence and $in_<(J) = (K_2T_2, \ldots, K_nT_n)$, where $K_i = (X_1, \ldots, X_{i-1}), i = 2, \ldots, n$, are the annihilator ideals (See [13] (Proposition 3.7)). Let $N_3 = Syz_1(I_1J_m)$ be. Then
$$in_< N_3 =< (X_1)e_2, (X_1, X_2)e_3, \ldots, (X_1, \ldots, X_{n-1})e_n >\cong \bigoplus_{i=2}^{n} K_i(m+2)$$
as graded R-modules.
4. The annihilator ideals of I_nJ_1 are $K_i = (Y_1, \ldots, Y_{i-1}), i = 2, \ldots, m$ (See [13] (Proposition 3.7)). The proof is analogue to the case of I_1J_m and $in_< N_4 \cong \bigoplus_{i=2}^{m} K_i(n+1)$ as graded R-modules.
5. The annihilator ideals of $I_nJ_{m-1} + I_{n-1}J_m$ are $K_i = (Y_{m-i+1})$ for $i = 2, \ldots, m$ and $K_i = (X_{n+m-i+1})$ for $i = m+1, \ldots, m+n$ by [13] (Proposition 3.11). The assertion follows and we have
$$in_< N_5 = \bigoplus_{i=2}^{m+n} K_ie_i \cong \underbrace{R(m+n-1) \oplus \ldots \oplus R(m+n)}_{(m+n)-\text{times}}$$
as graded R-modules.
6. The annihilator ideals of $I_nJ_1 + J_m$ are $K_i = (Y_1, \ldots, Y_{i-1}), i = 2, \ldots, m$ (See [13] (Proposition 3.7)) and $K_{m+1} = (X_1X_2 \cdots X_n)$, generated by the monomial $X_1X_2 \cdots X_n$. The assertion follows and we have
$$in_< N_6 \cong \bigoplus_{i=2}^{m} K_ie_i \oplus (X_1 \cdots X_n)e_{m+1} \cong \bigoplus_{i=2}^{m} K_i(n+2) \oplus R(m+n)$$
as graded R-modules.

□

Proposition 5. *The modules $in_< N_1, in_< N_2, in_< N_5$ are generated by an s-sequence.*

Proof. The assertion follows by Theorem 2. □

Theorem 8. *The modules $in_< N_3$, $in_< N_4$ and $in_< N_6$ are not generated by an s-sequence.*

Proof. Let $in_< N_3 =< (X_1)e_2, (X_1, X_2)e_3, \ldots, (X_1, \ldots, X_{n-1})e_n >$ be and with generating sequence $X_1e_2, X_1e_3, X_2e_3, X_1e_4, X_2e_4, X_3e_4, \ldots, X_{n-2}e_n, X_{n-1}e_n$. The corresponding symmetric algebra is

$$Sym_R(in_<N_3) = R[T_{12}, T_{13}, T_{23}, T_{14}, T_{24}, T_{34}, \ldots T_{(n-2)n}, T_{(n-1)n}]/J,$$

with $T_{12} < T_{13} < T_{23} < T_{14} < T_{24} < T_{34} < \ldots < T_{(n-2)n} < T_{(n-1)n}$. Consider the relations $g_1 = X_1T_{23} - X_2T_{13}$, $g_2 = X_1T_{24} - X_2T_{14}$ and the S-pair $S(g_1, g_2) = -X_2(T_{23}T_{14} - T_{24}T_{13})$. Then we have:

$$in_<J = (X_1T_{23}, X_1T_{24}, X_2T_{23}T_{14}, \ldots) \quad \text{if } T_{23}T_{14} > T_{24}T_{13}$$

or

$$in_<J = (X_1T_{23}, X_1T_{24}, X_2T_{24}T_{13}, \ldots) \quad \text{if } T_{23}T_{14} < T_{24}T_{13},$$

where $<$ is a term order on all monomials in the variables X_i, T_{jk}.

Since all initial terms of J are of the form X_1T_{2j}, $3 \leq j \leq n$, the Gröbner basis of J is never linear in the variables T_{jk}.

The same argument can be applied to $in_<N_4$ and $in_<N_6$. $\square$

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