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Soft ω_p-Open Sets and Soft ω_p-Continuity in Soft Topological Spaces

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Abstract: We define soft ω_p-openness as a strong form of soft pre-openness. We prove that the class of soft ω_p-open sets is closed under soft union and do not form a soft topology, in general. We prove that soft ω_p-open sets which are countable are soft open sets, and we prove that soft pre-open sets which are soft ω-open sets are soft ω_p-open sets. In addition, we give a decomposition of soft ω_p-open sets in terms of soft open sets and soft ω-dense sets. Moreover, we study the correspondence between the soft topology soft ω_p-open sets in a soft topological space and its generated topological spaces, and vice versa. In addition to these, we define soft ω_p-continuous functions as a new class of soft mappings which lies strictly between the classes of soft continuous functions and soft pre-continuous functions. We introduce several characterizations for soft pre-continuity and soft ω_p-continuity. Finally, we study several relationships related to soft ω_p-continuity.

Keywords: soft ω-open; soft pre-open sets; soft pre-continuity; generated soft topology; soft induced topological spaces

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1. Introduction and Preliminaries

In this work, we follow the notions and terminologies of [1,2]. TS and STS will denote topological space and soft topological space, respectively. A soft set defined by Molodtsov [3] in 1999 is a generic mathematical tool for dealing with uncertainty. The notion of STSs was initiated by Shabir and Naz [4] in 2011. Then, many topological concepts were modified to include soft topology. The concepts of soft topology and their applications is still a hot area of research (see for example [1,2,5–19]).

The generalizations of soft open sets play an effective role in the structure of soft topology by using them to redefine and investigate some soft topological concepts such as soft continuity, soft compactness, soft separation axioms, etc. As an important generalization of open sets, ω-open sets in TSs have been defined in [20]. Then, via ω-open sets, many research papers have appeared. In particular, via ω-open sets, the author in [21] introduced the notions of ω_p-open sets in TSs and ω_p-continuous functions between TSs. Authors in [2], defined soft ω-open sets in STSs as follows: Let (X, τ, A) be a STS and F be a soft set in (X, A), then F is called a soft ω-open set if for every soft point $a_x \widetilde{\in} F$, there exists $G \in \tau$ and a countable soft set K in (X, A) such that $a_x \widetilde{\in} G$ and $G - K$ is a countable soft set. In this work, we define soft ω_p-openness as a strong form of soft pre-openness. We prove that the class of soft ω_p-open sets is closed under soft union and does not form a soft topology, in general. We prove that soft ω_p-open sets which are countable are soft open sets, and we prove that soft pre-open sets which are soft ω-open sets are soft ω_p-open sets. We also give a decomposition of soft ω_p-open sets in terms of soft open sets and soft ω-dense sets. Moreover, we study the correspondence between the soft topology soft ω_p-open sets in a soft topological space and its generated topological spaces, and vise versa. In addition to these, we define soft ω_p-continuous functions as a new class of soft mappings which lies strictly between the classes of soft continuous functions and soft pre-continuous functions. We introduce several characterizations for soft pre-continuity and soft ω_p-continuity. Finally, we study several relationships related to soft ω_p-continuity.

Authors in [22,23] showed that soft sets are a class of special information systems. This constitutes a motivation to study the structures of soft sets for information systems. In addition, authors in [24] applied soft sets to a decision making problem. So, this paper not only can form the theoretical basis for further applications of soft topology such as soft continuity, soft ω_s-compactness, soft connectedness, soft separation axioms, and so on, but it also leads to the development of information systems and decision making problems.

Let (X, τ, A) be a STS, $(X, \Im)$ be a TS, $H \in SS(X, A)$, and $D \subseteq X$. Throughout this paper, $Cl_\tau(H)$, $int_\tau(H)$, and $Cl_\Im(D)$ will denote the soft closure of H in (X, τ, A), the soft interior of H in (X, τ, A), and the closure of D in $(X, \Im)$, respectively.

The following definitions and results will be used in the sequel:

Definition 1. *Let $(X, \Im)$ be a TS and let $D \subseteq X$. The D is said to be*

(a) [22] pre-open if there is $U \in \Im$ such that $U \subseteq D \subseteq Cl_\Im(U)$. The family of all pre-open sets in $(X, \Im)$ will be denoted by $PO(X, \Im)$.

(b) [21] ω_p-open if there is $U \in \Im$ such that $U \subseteq D \subseteq Cl_{\Im_\omega}(U)$. The family of all ω_p-open sets in $(X, \Im)$ will be denoted by $\omega_p(X, \Im)$.

Definition 2 ([25])**.** *Let (X, τ, A) be a STS and let $K \in SS(X, A)$. Then K is called a soft pre-open set if there exists $G \in \tau$ such that $K \widetilde{\subseteq} G \widetilde{\subseteq} Cl_\tau(K)$. The family of all soft pre-open sets in (X, τ, A) will be denoted by $PO(X, \tau, A)$.*

Definition 3. *Let X be a universal set and A be a set of parameters, and $K \in SS(X, A)$.*

(a) [1] If $K(b) = \begin{cases} Z & \text{if } b = e \\ \emptyset & \text{if } b \neq e \end{cases}$, then we will denote K by e_Z.

(b) [1] If $K(b) = Z$ for all $b \in A$, then we will denote K by C_Z.

(c) [26] If $K(b) = \begin{cases} \{x\} & \text{if } b = e \\ \emptyset & \text{if } b \neq e \end{cases}$, then we will denote K by e_x and we will call K a soft point.

We will denote the set of all soft points in $SS(X, A)$ by $SP(X, A)$.

Definition 4 ([26])**.** *Let $G \in SS(X, A)$ and $a_x \in SP(X, A)$. Then a_x is said to belong to F (notation: $a_x \widetilde{\in} G$) if $a_x \widetilde{\subseteq} G$ or equivalently: $a_x \widetilde{\in} G$ if and only if $x \in G(a)$.*

Definition 5. *A STS (X, τ, A) is called*

(a) [2] soft locally countable if for every $b_x \in SP(X, A)$, we find $K \in \tau$ such that $a_x \widetilde{\in} K$ and K is countable.

(b) [2] soft anti-locally countable for every $F \in \tau - \{0_A\}$, F is not a countable soft set.

(c) [5] soft ω-regular whenever S is soft closed and $a_x \widetilde{\in} 1_A - S$, then we find $K \in \tau$ and $N \in \tau_\omega$ such that $a_x \widetilde{\in} K$, $S \widetilde{\subseteq} N$, and $K \widetilde{\cap} N = 0_A$.

Theorem 1 ([4])**.** *Let (X, τ, A) be a STS. Then the collection $\{F(a) : F \in \tau\}$ defines a topology on X for every $a \in A$. This topology will be denoted by τ_a.*

Theorem 2 ([27])**.** *For any TS $(Y, \aleph)$ and any set of parameters B, the family*

$$\{G \in SS(Y, B) : G(b) \in \aleph \text{ for every } b \in B\}$$

is a soft topology on Y relative to B. We will denote this soft topology by $\tau(\aleph)$.

Definition 6 ([28])**.** *A function $p : (X, \Im) \longrightarrow (Y, \aleph)$ between the TSs $(X, \Im)$ and $(Y, \aleph)$ is said to be pre-continuous if $p^{-1}(V) \in PO(X, \Im)$ for every $V \in \aleph$.*

Definition 7 ([21])**.** *A function $p : (X, \Im) \longrightarrow (Y, \aleph)$ between the TSs $(X, \Im)$ and $(Y, \aleph)$ is said to be ω_p-continuous if $p^{-1}(V) \in \omega_p(X, \Im)$ for every $V \in \aleph$.*

Authors in [21] called in ω_p-continuous in Definition 1.10 as ω-almost continuous.

Lemma 1. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of TSs and let $\tau = \underset{a \in A}{\oplus} \Im_a$. Let $H \in SS(X, A)$, then for every $a \in A$, $Cl_{\tau_a}(H(a)) = (Cl_\tau(H))(a)$.*

Proof. Straightforward. □

Theorem 3. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of TSs and let $\tau = \underset{a \in A}{\oplus} \Im_a$ and let $F \in SS(X, A)$. Then $F \in PO(X, \tau, A)$ if and only if $F(a) \in PO(X, \tau_a)$ for every $a \in A$.*

Proof.

1. *Necessity*. Suppose that $F \in PO(X, \tau, A)$ and let $a \in A$. Then there there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_\tau(F)$. So, $F(a) \subseteq G(a) \subseteq (Cl_\tau(F))(a)$. Since $G(a) \in \tau_a$ and by Lemma 1 we have $(Cl_\tau(F))(a) = Cl_{\tau_a}(F(a))$, then $F(a) \in PO(X, \tau_a)$.
2. *Sufficiency*. Suppose that $G(a) \in PO(X, \tau_a)$ for every $a \in A$. Then for every $a \in A$, there exists $V_a \in \tau_a = \Im_a$ such that $F(a) \subseteq V_a \subseteq Cl_{\tau_a}(F(a))$. Let $G \in SS(X, A)$ with $G(a) = V_a \in \Im_a$ for every $a \in A$. Then $G \in \underset{a \in A}{\oplus} \Im_a = \tau$. Also, by Lemma 1, $(Cl_\tau(F))(a) = Cl_{\tau_a}(F(a))$ for all $a \in A$. Therefore, then $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_\tau(F)$. Hence, $F \in PO(X, \tau, A)$.

□

2. Soft ω_p-Open Sets

Definition 8. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then F is said to be a soft ω_p-open set in (X, τ, A) if there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. The family of all soft ω_p-open sets in (X, τ, A) will be denoted by $\omega_p(X, \tau, A)$.*

Theorem 4. *Let (X, τ, A) be a STS. Then $\tau \subseteq \omega_p(X, \tau, A) \subseteq PO(X, \tau, A)$.*

Proof. To see that $\tau \subseteq \omega_p(X, \tau, A)$, let $F \in \tau$. Take $G = F$. Then $G \in \tau$ and $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Therefore, $F \in \omega_p(X, \tau, A)$. To see that $\omega_p(X, \tau, A) \subseteq PO(X, \tau, A)$, let $F \in \omega_p(X, \tau, A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_\tau(F)$ which shows that $F \in PO(X, \tau, A)$. Therefore, $\omega_p(X, \tau, A) \subseteq PO(X, \tau, A)$. □

The following example shows that neither of the two inclusions in Theorem 4 is equal in general:

Example 1. *Let $X = \mathbb{R}$ and $A = \mathbb{Z}$. Let $\Im$ be the usual topology on $\mathbb{R}$ and $\tau = \{F \in SS(X, A) : F(a) \in \Im \text{ for all } a \in A\}$. Let $M, N \in SS(X, A)$ such that for every $a \in A, M(a) = \mathbb{Q}$ and $N(a) = \mathbb{Q}^c$. It is easy to see that $Cl_{\tau_\omega}(M) = M$, $Cl_\tau(M) = 1_A$, and $Cl_{\tau_\omega}(N) = 1_A$. Therefore, $M \in PO(X, \tau, A) - \omega_p(X, \tau, A)$ and $N \in \omega_p(X, \tau, A) - \tau$.*

Theorem 5. *For any STS (X, τ, A), $CSS(X, A) \cap \omega_p(X, \tau, A) \subseteq \tau$.*

Proof. Let $F \in CSS(X, A) \cap \omega_p(X, \tau, A)$. Since $F \in CSS(X, A)$, then by Corollary 5 of [2], $Cl_{\tau_\omega}(F) = F$. Since $F \in \omega_p(X, \tau, A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F) = F$. Therefore, $F = G$ and hence $F \in \tau$. □

Theorem 6. *If (X, τ, A) is soft locally countable, then $\omega_p(X, \tau, A) = \tau$.*

Proof. Suppose that (X, τ, A) is soft locally countable. To see that $\omega_p(X, \tau, A) \subseteq \tau$, let $F \in \omega_p(X, \tau, A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Since (X, τ, A) is soft

locally countable, then by by Corollary 5 of [2] , $Cl_{\tau_\omega}(F) = F$. Hence, $F = G$, and so $F \in \tau$. On the other hand, by Theorem 4, $\tau \subseteq \omega_p(X, \tau, A)$.
□

Corollary 1. *If (X, τ, A) is soft locally countable, then $\left(X, A, \omega_p(X, \tau, A)\right)$ is a STS.*

Remark 1. *In Corollary 1, the condition 'soft locally countable' cannot be dropped:*

Example 2. *Let $X = \mathbb{R}$, $A = \mathbb{Z}$, and*
$\tau = \{F \in SS(X, A) : F(a) \in \{\emptyset, X\}$ for all $a \in A\}$.
Let $M, N \in SS(X, A)$ defined by $M(a) = (-\infty, -1) \cup \{1\}$ and $N(a) = (2, \infty) \cup \{1\}$ for all $a \in A$. Then $\left(M \widetilde{\cap} N\right)(a) = \{1\}$ for all $a \in A$, $Cl_{\tau_\omega}(M) = Cl_{\tau_\omega}(N) = 1_A$. So, $M, N \in \omega_p(X, \tau, A)$. On the other hand, since $M \widetilde{\cap} N$ is a countable soft set and $M \widetilde{\cap} N \notin \tau$, then by Theorem 6, $M \widetilde{\cap} N \notin \omega_p(X, \tau, A)$.

Theorem 7. *Let (X, τ, A) be a STS. If $\{F_\lambda : \lambda \in \Gamma\} \subseteq \omega_p(X, \tau, A)$, then $\widetilde{\bigcup\limits_{\lambda \in \Gamma}} F_\lambda \in \omega_p(X, \tau, A)$.*

Proof. Let $\{F_\lambda : \lambda \in \Gamma\} \subseteq \omega_p(X, \tau, A)$, then for every $\lambda \in \Gamma$, there exists $G_\lambda \in \tau$ such that $F_\lambda \widetilde{\subseteq} G_\lambda \widetilde{\subseteq} Cl_{\tau_\omega}(F_\lambda)$. So, $\widetilde{\bigcup\limits_{\lambda \in \Gamma}} G_\lambda \in \in \tau$ and $\widetilde{\bigcup\limits_{\lambda \in \Gamma}} F_\lambda \widetilde{\subseteq} \widetilde{\bigcup\limits_{\lambda \in \Gamma}} G_\lambda \widetilde{\subseteq} \widetilde{\bigcup\limits_{\alpha \in \Delta}} Cl_{\tau_\omega}(F_\lambda) \widetilde{\subseteq} Cl_{\tau_\omega}\left(\widetilde{\bigcup\limits_{\lambda \in \Gamma}} F_\lambda\right)$. Hence, $\widetilde{\bigcup\limits_{\lambda \in \Gamma}} F_\lambda \in \omega_p(X, \tau, A)$. □

Theorem 8. *For any STS (X, τ, A), $PO(X, \tau_\omega, A) = \omega_p(X, \tau_\omega, A)$.*

Proof. Let (X, τ, A) be STS. By Theorem 4, we have $\omega_p(X, \tau_\omega, A) \subseteq PO(X, \tau_\omega, A)$. To see that $PO(X, \tau_\omega, A) \subseteq \omega_p(X, \tau_\omega, A)$, let $F \in PO(X, \tau_\omega, A)$, then there is $G \in \tau_\omega$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{(\tau_\omega)_\omega}(F)$. By Theorem 5 of [2] , $(\tau_\omega)_\omega = \tau_\omega$, and so $Cl_{(\tau_\omega)_\omega}(F) = Cl_{\tau_\omega}(F)$. It follows that $F \in \omega_p(X, \tau_\omega, A)$. □

Theorem 9. *For any soft anti-locally countable STS (X, τ, A), $\tau_\omega \cap PO(X, \tau, A) \subseteq \omega_p(X, \tau, A)$.*

Proof. Let (X, τ, A) be soft anti-locally countable. Let $F \in \tau_\omega \cap PO(X, \tau, A)$. Since (X, τ, A) is soft anti-locally countable and $F \in \tau_\omega$, then by Theorem 14 of [2], $Cl_{\tau_\omega}(F) = Cl_\tau(F)$. Since $F \in PO(X, \tau, A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_\tau(F) = Cl_{\tau_\omega}(F)$. Hence, $F \in \omega_p(X, \tau, A)$. □

Remark 2. *The concepts soft ω-open sets and soft ω_p-open sets are independent of each other:*

Example 3. *Let $X = \mathbb{Q}$ and $A = \mathbb{N}$. Let $\Im = \{X\} \cup \{U \subseteq X : 0 \notin U\}$ and let $\tau = \{F \in SS(X, A) : F(a) \in \Im$ for all $a \in A\}$. Let $F \in SS(X, A)$ defined by $F(a) = \{0\}$ for all $a \in A$. Then F is soft ω-open but not soft ω_p-open.*

Example 4. *Let (X, τ, A) be as in Example 2. Let $F \in SS(X, A)$ defined by $F(a) = (5, 7)$ for all $a \in A$. Then F is soft ω_p-open but not soft ω-open.*

Theorem 10. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then $F \in \omega_p(X, \tau, A)$ if and only if $F \widetilde{\subseteq} int_\tau(Cl_{\tau_\omega}(F))$.*

Proof.

1. *Necessity.* Suppose that $F \in \omega_p(X, \tau, A)$. Then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Therefore, $F \widetilde{\subseteq} int_\tau(Cl_{\tau_\omega}(F))$.

2. *Sufficiency.* Suppose that $F \tilde{\subseteq} int_\tau(Cl_{\tau_\omega}(F))$. Let $G = int_\tau(Cl_{\tau_\omega}(F))$. Then $G \in \tau$ with $F \tilde{\subseteq} G \tilde{\subseteq} int_\tau(Cl_{\tau_\omega}(F)) \tilde{\subseteq} Cl_{\tau_\omega}(F)$. Hence, $F \in \omega_p(X, \tau, A)$.

□

Definition 9. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then F is said to be soft ω-dense if $Cl_{\tau_\omega}(F) = 1_A$.*

Proposition 1. *Every soft ω-dense set in a STS is soft dense.*

Proof. Straightforward. □

Remark 3. *The converse of Proposition 1 is not true in general:*

Example 5. *Let $X = \mathbb{R}$ and $A = \mathbb{N}$. Let $\Im$ be the usual topology on $\mathbb{R}$ and let $\tau = \{F \in SS(X, A) : F(a) \in \Im$ for all $a \in A\}$. Let $F \in SS(X, A)$ defined by $F(a) = \mathbb{Q}$ for all $a \in A$. Then F is soft dense but not soft ω-dense.*

Theorem 11. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then $F \in \omega_p(X, \tau, A)$ if and only if F is a soft intersection of a soft open set and a soft ω-dense set.*

Proof.

1. *Necessity.* Suppose that $F \in \omega_p(X, \tau, A)$, then by Theorem 10, $F \tilde{\subseteq} int_\tau(Cl_{\tau_\omega}(F))$. Put $G = int_\tau(Cl_{\tau_\omega}(F))$ and $H = (1_A - G)\tilde{\cup}F$. Then G is soft open and $F = G\tilde{\cap}H$. In addition,

$$\begin{aligned} Cl_{\tau_\omega}(H) &= Cl_{\tau_\omega}((1_A - int_\tau(Cl_{\tau_\omega}(F)))\tilde{\cup}F) \\ &= Cl_{\tau_\omega}(1_A - int_\tau(Cl_{\tau_\omega}(F)))\tilde{\cup}Cl_{\tau_\omega}(F) \\ &= Cl_{\tau_\omega}(Cl_\tau(1_A - Cl_{\tau_\omega}(F)))\tilde{\cup}Cl_{\tau_\omega}(F) \\ &= Cl_{\tau_\omega}(Cl_\tau(int_{\tau_\omega}(1_A - F)))\tilde{\cup}Cl_{\tau_\omega}(F) \\ &\tilde{\supseteq} int_{\tau_\omega}(1_A - F)\tilde{\cup}Cl_{\tau_\omega}(F) \\ &= 1_A. \end{aligned}$$

and so, H soft ω-dense.

2. *Sufficiency*. Suppose that $F = G\tilde{\cap}H$ with $G \in \tau$ and H is soft ω-dense. To show that $G\tilde{\subseteq}Cl_{\tau_\omega}(F)$, suppose to the contrary that there exists $a_x\tilde{\in}G - Cl_{\tau_\omega}(F)$. Since $a_x\tilde{\in}1_A - Cl_{\tau_\omega}(F)$, then there exists $M \in \tau_\omega$ such that $a_x\tilde{\in}M$ and $M\tilde{\cap}F = 0_A$. Since $a_x\tilde{\in}M\tilde{\cap}G \in \tau_\omega$ and H is soft ω-dense, then $M\tilde{\cap}G\tilde{\cap}H = M\tilde{\cap}F \neq 0_A$, a contradiction. Therefore, $F \in \omega_p(X, \tau, A)$.

□

Proposition 2. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then for every $G \in \tau$, $Cl_\tau(G\tilde{\cap}F) = Cl_\tau(G\tilde{\cap}Cl_\tau(F))$.*

Proof. Let $F \in SS(X, A)$ and $G \in \tau$. Since $F \tilde{\subseteq} Cl_\tau(F)$, then $G\tilde{\cap}F\tilde{\subseteq}G\tilde{\cap}Cl_\tau(F)$ and so $Cl_\tau(G\tilde{\cap}F)\tilde{\subseteq}Cl_\tau(G\tilde{\cap}Cl_\tau(F))$. To see that

$Cl_\tau(G\tilde{\cap}Cl_\tau(F))\tilde{\subseteq}Cl_\tau(G\tilde{\cap}F)$, let $a_x\tilde{\in}Cl_\tau(G\tilde{\cap}Cl_\tau(F))$ and let $H \in \tau$ such that $a_x\tilde{\in}H$, then $(G\tilde{\cap}Cl_\tau(F))\tilde{\cap}H \neq 0_A$. Choose $b_y\tilde{\in}(G\tilde{\cap}Cl_\tau(F))\tilde{\cap}H$, then $b_y\tilde{\in}Cl_\tau(F)$ with $b_y\tilde{\in}G\tilde{\cap}H \in \tau$, and hence $(G\tilde{\cap}H)\tilde{\cap}F = (G\tilde{\cap}F)\tilde{\cap}H \neq 0_A$. It follows that $a_x\tilde{\in}Cl_\tau(G\tilde{\cap}F)$. □

Theorem 12. *Let (X, τ, A) be a STS and let $F \in SS(X, A)$. Then $F \in \omega_p(X, \tau, A)$ if and only if $F\tilde{\cap}G \in \omega_p(X, \tau, A)$ for every $G \in \tau$.*

Proof.

1. *Necessity.* Suppose that $F \in \omega_p(X, \tau, A)$ and let $G \in \tau$. Since $F \in \omega_p(X, \tau, A)$, then there exists $H \in \tau$ such that $F \widetilde{\subseteq} H \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. So, $F \widetilde{\cap} G \widetilde{\subseteq} H \widetilde{\cap} G \widetilde{\subseteq} G \widetilde{\cap} Cl_{\tau_\omega}(F)$. Since, $G \in \tau \subseteq \tau_\omega$, then by Proposition 2.18, $Cl_{\tau_\omega}(G \widetilde{\cap} Cl_{\tau_\omega}(F)) = Cl_{\tau_\omega}(G \widetilde{\cap} F)$ and so $G \widetilde{\cap} Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_{\tau_\omega}(G \widetilde{\cap} F)$. Thus, we have $H \widetilde{\cap} G \in \tau$ with $F \widetilde{\cap} G \widetilde{\subseteq} H \widetilde{\cap} G \widetilde{\subseteq} G \widetilde{\cap} Cl_{\tau_\omega}(F) \widetilde{\subseteq} Cl_{\tau_\omega}$
$(G \widetilde{\cap} F)$. Hence, $F \widetilde{\cap} G \in \omega_p(X, \tau, A)$.
2. *Sufficiency.* Suppose that $F \widetilde{\cap} G \in \omega_p(X, \tau, A)$ for every $G \in \tau$. Since $1_A \in \tau$, then $F \widetilde{\cap} 1_A = F \in \omega_p(X, \tau, A)$.

□

Corollary 2. *In any STS, then soft intersection of a soft open set and a soft ω_p-open set is soft ω_p-open.*

Theorem 13. *Let (X, τ, A) and let $F, H \in SS(X, A)$. If $F \in \omega_p(X, \tau, A)$ and $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_{\tau_\omega}(H)$, then $H \in \omega_p(X, \tau, A)$.*

Proof. Suppose that $F \in \omega_p(X, \tau, A)$ and $H \widetilde{\subseteq} F \widetilde{\subseteq} Cl_{\tau_\omega}(H)$. Since $F \in \omega_p(X, \tau, A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. This implies that, $H \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(H)$. Hence, $H \in \omega_p(X, \tau, A)$. □

Theorem 14. *Let (X, τ, A) be a soft locally countable and let $F \in \omega_p(X, \tau, A)$. Then $F(a) \in \omega_p(X, \tau_a)$ for every $a \in A$.*

Proof. Let $a \in A$. Since (X, τ, A) is soft locally countable and $F \in \omega_p(X, \tau, A)$, then by Theorem 6, $F \in \tau$ and so $F(a) \in \tau_a \subseteq \omega_p(X, \tau_a)$. □

Corollary 3. *Let $(X, \Im)$ be a TS and A be a set of parameters. Then $F \in PO(X, \tau(\Im), A)$ if and only if $F(a) \in PO(X, \Im)$ for all $a \in A$.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a \in A}{\oplus} \Im_a$. So by Theorem 3, we get the result. □

Lemma 2. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of topological spaces and let $\tau = \underset{a \in A}{\oplus} \Im_a$. Let $H \in SS(X, A)$, then for every $a \in A$, $Cl_{(\tau_a)_\omega}(H(a)) = (Cl_{\tau_\omega}(H))(a)$.*

Proof. Straightforward. □

Theorem 15. *Let $\{(X, \Im_a) : a \in A\}$ be an indexed family of TSs and let $\tau = \underset{a \in A}{\oplus} \Im_a$. Let $F \in SS(X, A)$. Then $F \in \omega_p(X, \tau, A)$ if and only if $F(a) \in \omega_p(X, \tau_a)$ for every $a \in A$.*

Proof.

1. *Necessity*. Suppose that $F \in \omega_p(X, \tau, A)$ and let $a \in A$. Then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. So, $F(a) \subseteq G(a) \subseteq (Cl_{\tau_\omega}(F))(a)$. Since $G(a) \in \tau_a$ and by Lemma 2 we have $(Cl_{\tau_\omega}(F))(a) = Cl_{(\tau_a)_\omega}(F(a))$, then $F(a) \in \omega_p(X, \tau_a)$.
2. *Sufficiency.* Suppose that $F(a) \in \omega_p(X, \tau_a)$ for every $a \in A$. Then for every $a \in A$, there exists $V_a \in \tau_a = \Im_a$ such that $F(a) \subseteq V_a \subseteq Cl_{(\tau_a)_\omega}(F(a))$. Let $G \in SS(X, A)$ with $G(a) = V_a \in \Im_a$ for every $a \in A$. Then $G \in \underset{a \in A}{\oplus} \Im_a = \tau$. In addition, by Lemma 2, $(Cl_{\tau_\omega}(F))(a) = Cl_{(\tau_a)_\omega}(F(a))$ for all $a \in A$. Thus, we have $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Therefore, $F \in \omega_p(X, \tau, A)$.

□

Corollary 4. *Let $(X,\Im)$ be a TS and A be a set of parameters. Then $F \in \omega_p(X,\tau(\Im),A)$ if and only if $F(a) \in \omega_p(X,\Im)$ for all $a \in A$.*

Proof. For each $a \in A$, put $\Im_a = \Im$. Then $\tau(\Im) = \underset{a\in A}{\oplus} \Im_a$. So by Theorem 15, we get the result. □

Theorem 16. *For any STS (X,τ,A), $\tau = \{int_\tau(F) : F \in \omega_p(X,\tau,A)\}$.*

Proof. Since $int_\tau(F) \in \tau$ for every $F \in \omega_p(X,\tau,A)$, then $\{int_\tau(F) : F \in \omega_p(X,\tau,A)\} \subseteq \tau$. Conversely, let $F \in \tau$, then by Theorem 4, $F \in \omega_p(X,\tau,A)$. On the other hand, since $F \in \tau$, then $int_\tau(F) = F$. Therefore, $F \in \{int_\tau(F) : F \in \omega_p(X,\tau,A)\}$. It follows that $\tau \subseteq \{int_\tau(F) : F \in \omega_p(X,\tau,A)\}$. □

Theorem 17. *Let $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ be a soft open function such that $f_{pu} : (X,\tau_\omega,A) \longrightarrow (Y,\sigma_\omega,B)$ is soft continuous, then $f_{pu}(F) \in \omega_p(Y,\sigma,B)$ for all $F \in \omega_p(X,\tau,A)$.*

Proof. Let $F \in \omega_p(X,\tau,A)$, then there exists $G \in \tau$ such that $F \widetilde{\subseteq} G \widetilde{\subseteq} Cl_{\tau_\omega}(F)$. Thus we have $f_{pu}(F) \widetilde{\subseteq} f_{pu}(G) \widetilde{\subseteq} f_{pu}(Cl_{\tau_\omega}(F))$. Since $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ is soft open, then $f_{pu}(G) \in \sigma$. Since $f_{pu} : (X,\tau_\omega,A) \longrightarrow (Y,\sigma_\omega,B)$ is soft continuous, then $f_{pu}(Cl_{\tau_\omega}(F)) \widetilde{\subseteq} Cl_{\sigma_\omega}(f_{pu}(F))$. Therefore, $f_{pu}(F) \in \omega_p(Y,\sigma,B)$. □

3. Soft ω_p-Continuity

Definition 10 ([29]). *A soft function $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$ is said to be soft pre-continuous if $f_{pu}^{-1}(G) \in PO(X,\tau,A)$ for every $G \in \sigma$.*

Theorem 18. *The following statements are equivalent for a soft function $f_{pu} : (X,\tau,A) \longrightarrow (Y,\sigma,B)$.*

(a) f_{pu} is soft pre-continuous.
(b) $f_{pu}(Cl_\tau(int_\tau(M)))\widetilde{\subseteq}Cl_\sigma(f_{pu}(M))$ for each $M \in SS(X,A)$.
(c) $f_{pu}(Cl_\tau(N)))\widetilde{\subseteq}Cl_\sigma(f_{pu}(N))$ for each $N \in SO(X,\tau,A)$.
(d) $f_{pu}(Cl_\tau(G)))\widetilde{\subseteq}Cl_\sigma(f_{pu}(G))$ for each $G \in \tau$.

Proof.

1. (a) $\Longrightarrow$ (b): Suppose that f_{pu} is soft pre-continuous and let $M \in SS(X,A)$. Let $b_y\widetilde{\in}f_{pu}(Cl_\tau(int_\tau(M)))$ and let $K \in \sigma$ such that $b_y\widetilde{\in}K$. We are going to show that $f_{pu}(M)\widetilde{\cap}K \neq 0_B$. Choose $a_x\widetilde{\in}Cl_\tau(int_\tau(M))$ such that $b_y = f_{pu}(a_x)$. Since f_{pu} is soft pre-continuous, then $f_{pu}^{-1}(K) \in PO(X,\tau,A)$ and so $f_{pu}^{-1}(K)\widetilde{\subseteq}int_\tau(Cl_\tau(f_{pu}^{-1}(K)))$. Since $a_x\widetilde{\in}f_{pu}^{-1}(K)$, then $a_x\widetilde{\in}int_\tau(Cl_\tau(f_{pu}^{-1}(K))) \in \tau$. Since $a_x\widetilde{\in}Cl_\tau(int_\tau(M))$, then $int_\tau(M)\widetilde{\cap}int_\tau(Cl_\tau(f_{pu}^{-1}(K))) \neq 0_A$. Consequently, $int_\tau(M)\widetilde{\cap}Cl_\tau(f_{pu}^{-1}(K)) \neq 0_A$ and $M\widetilde{\cap}f_{pu}^{-1}(K) \neq 0_A$. Choose $c_z\widetilde{\in}M$ such that $f_{pu}(c_z)\widetilde{\in}K$. Therefore, $f_{pu}(c_z)\widetilde{\in}f_{pu}(M)\widetilde{\cap}K$ and hence $f_{pu}(M)\widetilde{\cap}K \neq 0_B$.
2. (b) $\Longrightarrow$ (c): Suppose that $f_{pu}(Cl_\tau(int_\tau(M)))\widetilde{\subseteq}Cl_\sigma(f_{pu}(M))$ for each $M \in SS(X,A)$ and let $N \in SO(X,\tau,A)$. Then $N\widetilde{\subseteq}Cl_\tau(int_\tau(N))$ and so $Cl_\tau(N)\widetilde{\subseteq}Cl_\tau(int_\tau(N))$. Thus by assumption, $f_{pu}(Cl_\tau(N)))\widetilde{\subseteq}f_{pu}((Cl_\tau(int_\tau(N))))\widetilde{\subseteq}Cl_\sigma(f_{pu}(N))$.
3. (c) $\Longrightarrow$ (d): Obvious.
4. (d) $\Longrightarrow$ (a): Suppose that $f_{pu}(Cl_\tau(G)))\widetilde{\subseteq}Cl_\sigma(f_{pu}(G))$ for each $G \in \tau$ and let $K \in \sigma$. To show that $f_{pu}^{-1}(K)\widetilde{\subseteq}int_\tau(Cl_\tau(f_{pu}^{-1}(K)))$, let $a_x\widetilde{\in}f_{pu}^{-1}(K)$. Since $1_A - Cl_\tau(f_{pu}^{-1}(K)) \in \tau$, then by assumption, $f_{pu}(Cl_\tau(1_A - Cl_\tau(f_{pu}^{-1}(K)))))\widetilde{\subseteq}Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))$ and thus, $Cl_\tau(1_A - Cl_\tau(f_{pu}^{-1}(K)))\widetilde{\subseteq}f_{pu}^{-1}\left(Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))\right)$. Hence, $1_A -$

f_{pu}^{-1}
$\Big(Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))\Big)\widetilde{\subseteq}1_A - Cl_\tau(1_A - Cl_\tau(f_{pu}^{-1}(K))) = int_\tau(Cl_\tau(f_{pu}^{-1}(K)))$.
We shall show that $a_x\widetilde{\in}1_A - f_{pu}^{-1}\Big(Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))\Big)$. To do this suppose on the contrary that $a_x\widetilde{\in}f_{pu}^{-1}\Big(Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))\Big)$. Then $f_{pu}(a_x)\widetilde{\in}Cl_\sigma(f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K))))$. Since $f_{pu}(a_x)\widetilde{\in}K \in \sigma$, then $f_{pu}(1_A - Cl_\tau(f_{pu}^{-1}(K)))\widetilde{\cap}K \neq 0_B$. Choose $c_z\widetilde{\in}1_A - Cl_\tau(f_{pu}^{-1}(K))$ such that $f_{pu}(c_z) \in K$. Since $c_z\widetilde{\in}1_A - Cl_\tau(f_{pu}^{-1}(K))$, then there exists $H \in \tau$ such that $c_z\widetilde{\in}H$ and $f_{pu}^{-1}(K)\widetilde{\cap}H = 0_A$. But $c_z\widetilde{\in}f_{pu}^{-1}(K)\widetilde{\cap}H$, a contradiction.

□

Theorem 19. *Let $p : (X, \Im) \longrightarrow (Y, \aleph)$ be a function between two TSs and let $u : A \longrightarrow B$ be a function between two sets of parameters. Then $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft pre-continuous if and only if $p : (X, \Im) \longrightarrow (Y, \aleph)$ is pre-continuous.*

Proof.

1. *Necessity*. Suppose that $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft pre-continuous. Let $V \in \aleph$. Choose $a \in A$, then $u(a)_V \in \tau(\aleph)$. Since $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft pre-continuous, then $f_{pu}^{-1}(u(a)_V) \in PO(X, \tau(\Im), A)$. So, by Corollary 3, $\Big(f_{pu}^{-1}(u(a)_V)\Big)(a) \in PO(X, \Im)$. But $\Big(f_{pu}^{-1}(u(a)_V)\Big)(a) = p^{-1}((u(a)_V)(u(a))) = p^{-1}(V)$. Therefore, $p : (X, \Im) \longrightarrow (Y, \aleph)$ is pre-continuous.
2. *Sufficiency*. Suppose that $p : (X, \Im) \longrightarrow (Y, \aleph)$ is pre-continuous. Let $G \in \tau(\aleph)$. By Corollary 4 it is sufficient to show that $\Big(f_{pu}^{-1}(G)\Big)(a) \in PO(X, \Im)$ for all $a \in A$. Let $a \in A$, then $G(u(a)) \in \aleph$. Since $p : (X, \Im) \longrightarrow (Y, \aleph)$ is pre-continuous, then $p^{-1}(G(u(a))) \in PO(X, \Im)$. But $p^{-1}(G(u(a))) = \Big(f_{pu}^{-1}(G)\Big)(a)$.

□

Definition 11. *A soft function $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is said to be soft ω_p-continuous if $f_{pu}^{-1}(G) \in \omega_p(X, \tau, A)$ for every $G \in \sigma$.*

Theorem 20. *The following statements are equivalent for a soft function $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$.*
(a) f_{pu} is soft ω_p-continuous.
(b) $f_{pu}(Cl_\tau(G))\widetilde{\subseteq}Cl_\sigma(f_{pu}(G))$ for each $G \in \tau_\omega$.

Proof.

1. (a) $\Longrightarrow$ (b): Suppose that f_{pu} is soft ω_p-continuous and let $G \in \tau_\omega$. To see that $f_{pu}(Cl_\tau(G))\widetilde{\subseteq}Cl_\sigma(f_{pu}(G))$, let $b_y\widetilde{\in}f_{pu}(Cl_\tau(G))$ and let $K \in \sigma$ such that $b_y\widetilde{\in}K$. We are going to show that $f_{pu}(G)\widetilde{\cap}K \neq 0_B$. Choose $a_x\widetilde{\in}Cl_\tau(G)$ such that $b_y = f_{pu}(a_x)$. Since f_{pu} is soft ω_p-continuous, then $f_{pu}^{-1}(K) \in \omega_p(X, \tau, A)$ and so $f_{pu}^{-1}(K)$ $\widetilde{\subseteq}int_\tau(Cl_{\tau_\omega}(f_{pu}^{-1}(K)))$. Since $a_x\widetilde{\in}f_{pu}^{-1}(K)$, then $a_x\widetilde{\in}int_\tau(Cl_{\tau_\omega}(f_{pu}^{-1}(K))) \in \tau$. Since $a_x\widetilde{\in}Cl_\tau(G)$, then $G\widetilde{\cap}int_\tau(Cl_{\tau_\omega}(f_{pu}^{-1}(K))) \neq 0_A$ and so $G\widetilde{\cap}Cl_{\tau_\omega}(f_{pu}^{-1}(K)) \neq 0_A$. Choose $d_w\widetilde{\in}G\widetilde{\cap}Cl_{\tau_\omega}(f_{pu}^{-1}(K))$. Since we have $d_w\widetilde{\in}G \in \tau_\omega$ and $d_w\widetilde{\in}Cl_{\tau_\omega}(f_{pu}^{-1}(K))$, then $G\widetilde{\cap}f_{pu}^{-1}(K) \neq 0_A$. Choose $c_z\widetilde{\in}G$ such that $f_{pu}(c_z)\widetilde{\in}K$. Then, $f_{pu}(c_z)\widetilde{\in}f_{pu}(G)\widetilde{\cap}K$, and hence $f_{pu}(G)\widetilde{\cap}K \neq 0_B$.
2. (b) $\Longrightarrow$ (a): Suppose that $f_{pu}(Cl_\tau(G))\widetilde{\subseteq}Cl_\sigma(f_{pu}(G))$ for each $G \in \tau_\omega$ and let $K \in \sigma$. To show that $f_{pu}^{-1}(K)\widetilde{\subseteq}int_\tau(Cl_{\tau_\omega}(f_{pu}^{-1}(K)))$, let $a_x\widetilde{\in}f_{pu}^{-1}(K)$. Since $1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K)) \in \tau_\omega$, then by assumption, $f_{pu}(Cl_\tau(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))\widetilde{\subseteq}Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))$

and thus, $Cl_\tau(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))) \widetilde{\subseteq} f_{pu}^{-1}\left(Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))\right)$. Hence, $1_A - f_{pu}^{-1}\left(Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))\right) \widetilde{\subseteq} 1_A - Cl_\tau(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))) = int_\tau(Cl_{\tau_\omega}(f_{pu}^{-1}(K)))$. We shall show that $a_x \widetilde{\in} 1_A - f_{pu}^{-1}\left(Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))\right)$. To do this suppose on the contrary that $a_x \widetilde{\in} f_{pu}^{-1}\left(Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))\right)$. Then $f_{pu}(a_x) \widetilde{\in} Cl_\sigma(f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))))$. Since $f_{pu}(a_x) \widetilde{\in} K \in \sigma$, then $f_{pu}(1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))) \widetilde{\cap} K \neq 0_B$. Choose $c_z \widetilde{\in} 1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))$ such that $f_{pu}(c_z) \in K$. Since $c_z \widetilde{\in} 1_A - Cl_{\tau_\omega}(f_{pu}^{-1}(K))$, then there exists $H \in \tau_\omega$ such that $c_z \widetilde{\in} H$ and $f_{pu}^{-1}(K) \widetilde{\cap} H = 0_A$. But $c_z \widetilde{\in} f_{pu}^{-1}(K) \widetilde{\cap} H$, a contradiction.

□

Theorem 21. *Let $p : (X, \Im) \longrightarrow (Y, \aleph)$ be a function between two TSs and let $u : A \longrightarrow B$ be a function between two sets of parameters. Then $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous if and only if $p : (X, \Im) \longrightarrow (Y, \aleph)$ is ω_p-continuous.*

Proof. *Necessity*. Suppose that $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous. Let $V \in \aleph$. Choose $a \in A$, then $u(a)_V \in \tau(\aleph)$. Since $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous, then $f_{pu}^{-1}(u(a)_V) \in \omega_p(X, \tau(\Im), A)$. So, by Corollary 4 , $\left(f_{pu}^{-1}(u(a)_V)\right)(a) \in \omega_p(X, \Im)$. However, $\left(f_{pu}^{-1}(u(a)_V)\right)(a) = p^{-1}((u(a)_V)(u(a))) = p^{-1}(V)$. Therefore, $p : (X, \Im) \longrightarrow (Y, \aleph)$ is ω_p-continuous.

Sufficiency. Suppose that $p : (X, \Im) \longrightarrow (Y, \aleph)$ is ω_p-continuous. Let $G \in \tau(\aleph)$. By Corollary 4 it is sufficient to show that $\left(f_{pu}^{-1}(G)\right)(a) \in \omega_p(X, \Im)$ for all $a \in A$. Let $a \in A$, then $G(u(a)) \in \aleph$. Since $p : (X, \Im) \longrightarrow (Y, \aleph)$ is ω_p-continuous, then $p^{-1}(G(u(a))) \in \omega_p(X, \Im)$. But $p^{-1}(G(u(a))) = \left(f_{pu}^{-1}(G)\right)(a)$. □

Theorem 22. *Every soft continuous function is soft ω_p-continuous.*

Proof. Follows from the definitions and Theorem 4. □

Remark 4. *The converse of Theorem 22 is not true, in general:*

Example 6. *Let $X = Y = \mathbb{R}$, and $A = B = \mathbb{Z}$. Let $\Im$ be the indiscrete topology on $\mathbb{R}$ and $\aleph$ be the usual topology on $\mathbb{R}$. Define $p : (X, \Im) \longrightarrow (Y, \aleph)$ and $u : A \longrightarrow B$ by $f(x) = x$ and $u(a) = a$ for all $x \in X$ and $a \in A$. Then p is ω_p-continuous but not continuous. So, by Theorem 21 and Theorem 5.31 of [1], $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous but not soft continuous.*

Theorem 23. *Every soft ω_p-continuous function is soft pre-continuous.*

Proof. Follows from the definitions and Theorem 4. □

Remark 5. *The converse of Theorem 23 is not true, in general:*

Example 7. *Let $X = Y = \mathbb{R}$, and $A = B = \mathbb{Z}$. Let $\Im$ be the indiscrete topology on $\mathbb{R}$ and $\aleph$ be the discrete topology on $\mathbb{R}$. Define $p : (X, \Im) \longrightarrow (Y, \aleph)$ and $u : A \longrightarrow B$ by $f(x) = x$ and $u(a) = a$ for all $x \in X$ and $a \in A$. Then p is pre-continuous but not ω_p-continuous. So, by Theorems 19 and 21, $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous but not soft continuous.*

Theorem 24. *If $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous, then $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous.*

Proof. Suppose that $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous. To show that $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous, we apply Theorem 18(b). Let $G \in (\tau_\omega)_\omega$. By Theorem 5 of [2], $(\tau_\omega)_\omega = \tau_\omega$ and so $G \in \tau_\omega$. Since $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous, then $f_{pu}(Cl_\tau(G)) \widetilde{\subseteq} Cl_\sigma(f_{pu}(G))$. Since $Cl_{\tau_\omega}(G) \widetilde{\subseteq} Cl_\tau(G)$, then $f_{pu}(Cl_{\tau_\omega}(G)) \widetilde{\subseteq} f_{pu}(Cl_\tau(G)) \widetilde{\subseteq} Cl_\sigma(f_{pu}(G))$. Therefore, $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous. □

Remark 6. *The converse of Theorem 24 is not true, in general:*

Example 8. *Let $X = Y = \mathbb{N}$, and $A = B = \mathbb{R}$. Let $\Im = \{\mathbb{N}\} \cup \{U \subseteq \mathbb{N} : 1 \notin \mathbb{N}\}$ and let $\aleph$ be the discrete topology on $\mathbb{N}$. Define $p : (X, \Im) \longrightarrow (Y, \aleph)$ and $u : A \longrightarrow B$ by $f(x) = x$ and $u(a) = a$ for all $x \in X$ and $a \in A$. Then $p : (X, \Im_\omega) \longrightarrow (Y, \aleph)$ is ω_p-continuous but $p : (X, \Im) \longrightarrow (Y, \aleph)$ is not ω_p-continuous. So by Theorem 21, $f_{pu} : (X, \tau(\Im_\omega), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous but $f_{pu} : (X, \tau(\Im), A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous. Since $\tau(\Im_\omega) = (\tau(\Im))_\omega$, then $f_{pu} : (X, (\tau(\Im))_\omega, A) \longrightarrow (Y, \tau(\aleph), B)$ is soft ω_p-continuous.*

Theorem 25. *Let $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ be a soft function with (X, τ, A) is soft anti-locally countable. Then $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous if and only if $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous.*

Proof.

1. *Necessity*. Follows from Theorem 24.
2. *Sufficiency*. Suppose that $f_{pu} : (X, \tau_\omega, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous. Let $G \in \tau_\omega$, then by Theorem 20, $f_{pu}(Cl_{\tau_\omega}(G)) \widetilde{\subseteq} Cl_\sigma(f_{pu}(G))$. Since (X, τ, A) is soft anti-locally countable, then by Theorem 14 of [2], $Cl_{\tau_\omega}(G) = Cl_\tau(G)$. Thus, $f_{pu}(Cl_\tau(G)) \widetilde{\subseteq} Cl_\sigma(f_{pu}(G))$. Hence, by Theorem 18, $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ is soft ω_p-continuous.

□

Theorem 26. *Let $f_{pu} : (X, \tau, A) \longrightarrow (Y, \sigma, B)$ be a soft function with (Y, σ, B) is soft ω-regular. Then f_{pu} is soft continuous if and only if f_{pu} is soft ω_p-continuous.*

Proof.

1. *Necessity*. Follows from Theorem 22.
2. *Sufficiency*. Suppose that f_{pu} is soft ω_p-continuous. Let $a_x \in SP(X, A)$ and let $H \in \sigma$ such that $f_{pu}(a_x) \widetilde{\in} H$. Since (Y, σ, B) is soft ω-regular, then there exists $M \in \sigma$ such that $f_{pu}(a_x) \widetilde{\in} M \widetilde{\subseteq} Cl_{\sigma_\omega}(M) \widetilde{\subseteq} H$. Since f_{pu} is soft ω_p-continuous, then $f_{pu}^{-1}(M) \in \omega_p(X, \tau, A)$ and so there exists $N \in \tau$ such that $f_{pu}^{-1}(M) \widetilde{\subseteq} int_\tau\left(Cl_{\tau_\omega}(f_{pu}^{-1}(M))\right)$ $\widetilde{\subseteq} int_\tau(f_{pu}^{-1}(Cl_{\tau_\omega}(M))) \widetilde{\subseteq} f_{pu}^{-1}(Cl_{\tau_\omega}(M))$. Thus, we have $a_x \widetilde{\in} int_\tau\left(Cl_{\tau_\omega}(f_{pu}^{-1}(M))\right) \in \tau$ and $f_{pu}(int_\tau\left(Cl_{\tau_\omega}(f_{pu}^{-1}(M))\right)) \widetilde{\subseteq} f_{pu}\left(f_{pu}^{-1}(Cl_{\tau_\omega}(M))\right) \widetilde{\subseteq} Cl_{\tau_\omega}(M) \widetilde{\subseteq} H$.

□

4. Conclusions

The class of soft ω_p-open sets as a new class of soft sets which lies strictly between the classes of soft open sets and soft pre-open sets is introduced. It is proved that the family of soft ω_p-open sets form a supra soft topology. In addition, it is proved that a countable soft ω_p-open set is a soft set. Moreover, the correspondence between the soft topology soft ω_p-open sets in a STS and its generated topological spaces and vice versa are studied. In addition to these, via soft ω_p-open sets, the class of soft ω_p-continuous functions as a new class of soft functions which lies strictly between the classes of soft continuous functions and soft pre-continuous functions is defined and investigated. Several characterizations, relationships, and examples are given. The following topics could be considered in future studies: (1) define soft ω_p-open functions; (2) define soft separation axioms via soft ω_p-open sets; (3) define soft ω_p-compactness; (4) improve some known soft topological results.

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References

1. Al Ghour, S.; Bin-Saadon, A. On some generated soft topological spaces and soft homogeneity. *Heliyon* **2019**, *5*, e02061.
2. Al Ghour; S. Hamed, W. On two classes of soft sets in soft topological spaces. *Symmetry* **2020**, *12*, 265.
3. Molodtsov, D. Soft set theory—First results. *Comput. Math. Appl.* **1999**, *37*, 19–31.
4. Shabir, M.; Naz, M. On soft topological spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799.
5. Al Ghour, S. Weaker forms of soft regular and soft T_2 soft topological spaces. *Mathematics* **2021**, *9*, 2153.
6. Al Ghour, S. Some modifications of pairwise soft sets and some of their related concepts. *Mathematics* **2021**, *9*, 1781.
7. Mousarezaei, R.; Davvaz, B. On Soft Topological Polygroups and Their Examples. *Int. J. Fuzzy Log. And Intelligent Syst.* **2021**, *21*, 29–37.
8. Oztunc, S.; Aslan, S.; Dutta, H. Categorical structures of soft groups. *Soft Comput.* **2021**, *25*, 3059–3064.
9. Al-Shami, T.M.; Abo-Tabl, E.S.A. Soft α-separation axioms and α-fixed soft points. *AIMS Math.* **2021**, *6*, 5675–5694.
10. Al-Shami, T.M. On soft separation axioms and their applications on decision-making problem. *Math. Probl. Eng.* **2021**, 8876978, doi:10.1155/2021/8876978.
11. Al-Shami, T.M. Bipolar soft sets: Relations between them and ordinary points and their applications. *Complexity* **2021**, 6621854, doi:10.1155/2021/6621854.
12. Al-shami, T.M.; Alshammari, I.; Asaad, B.A. Soft maps via soft somewhere dense sets. *Filomat* **2020**, *34*, 3429–3440.
13. Min, W.K. On soft generalized closed sets in a soft topological space with a soft weak structure. *Int. J. Fuzzy Logic Intell. Syst.* **2020**, *20*, 119–123.
14. Çetkin, V.; Güner, E.; Aygün, H. On 2S-metric spaces. *Soft Comput.* **2020**, *24*, 12731–12742.
15. El-Shafei, M.E.; Al-shami, T.M. Applications of partial belong and total non-belong relations on soft separation axioms and decision-making problem. *Comput. Appl. Math.* **2020**, *39*, 138.
16. Bahredar, A.A.; Kouhestani, N. On ε-soft topological semigroups. *Soft Comput.* **2020**, *24*, 7035–7046.
17. Al-shami, T.M.; El-Shafei, M.E. Partial belong relation on soft separation axioms and decision-making problem, two birds with one stone. *Soft Comput.* **2020**, *24*, 5377–5387.
18. Al-shami, T.M. Soft somewhere dense sets on soft topological spaces. *Commun. Korean Math. Soc.* **2018**, *33*, 1341–1356.
19. Al-shami, T.M. New soft structure:Infra soft topological spaces. *Math. Probl. Eng.* **2021**, 3361604, doi:10.1155/2021/3361604.
20. Hdeib, H.Z. ω-closed mappings. *Rev. ColombianaMat.* **1982**, *16*, 65–78.
21. Darwesh, H.M. Between preopen and open sets in topological spaces. *Thai J. Math.* **2013**, *11*, 143–155.
22. Xiao, Z.; Chen, L.; Zhong, B.; Ye, S. Recognition for soft information based on the theory of soft sets. In Proceedings of the International Conference on Services Systems and Services Management, Chongquing, China, 13–15 June 2005; pp. 1104–1106.
23. Pei, D.; Miao, D. From soft sets to information systems. *IEEE Int. Conf. Granul. Comput.* **2005**, *2*, 617–621.
24. Maji, P.K.; Roy, A.R.; Biswas, R. An application of soft sets in a decision making problem. *Comput. Math. Appl.* **2002**, *44*, 1077–1083.
25. Arockiarani, I.; Arokialancy, A. Generalized soft $g\beta$-closed sets and soft $gs\beta$-closed sets in soft topological spaces. *Int. J. Math. Arch.* **2013**, *4*, 1–7.
26. Das, S.; Samanta, S.K. Soft metric. *Ann. Fuzzy Math. Inform.* **2013**, *6*, 77–94.
27. Nazmul, Sk.; Samanta, S.K. Neigbourhood properties of soft topological spaces. *Ann. Fuzzy Math. Informatics* **2013**, *6*, 1–15.
28. Mashhour, A.; Abd El-Monsef, M.; El-Deep, S. On precontinuous and weak precontinuous mappings. *Proc. Math. Phys. Soc. Egypt* **1982**, *53*, 47–53.
29. Akdag, M.; Ozkan, A. Soft α-open sets and soft α-continuous functions. *Abstr. Appl. Anal.* **2014**, 891341, doi:10.1155/2014/891341.