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Chaos on Fuzzy Dynamical Systems

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Abstract: Given a continuous map $f : X \to X$ on a metric space, it induces the maps $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$, on the hyperspace of nonempty compact subspaces of X, and $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$, on the space of normal fuzzy sets, consisting of the upper semicontinuous functions $u : X \to [0,1]$ with compact support. Each of these spaces can be endowed with a respective metric. In this work, we studied the relationships among the dynamical systems (X, f), $(\mathcal{K}(X), \overline{f})$, and $(\mathcal{F}(X), \hat{f})$. In particular, we considered several dynamical properties related to chaos: Devaney chaos, $\mathcal{A}$-transitivity, Li–Yorke chaos, and distributional chaos, extending some results in work by Jardón, Sánchez and Sanchis (Mathematics 2020, 8, 1862) and work by Bernardes, Peris and Rodenas (Integr. Equ. Oper. Theory 2017, 88, 451–463). Especial attention is given to the dynamics of (continuous and linear) operators on metrizable topological vector spaces (linear dynamics).

Keywords: chaotic operators; hypercyclic operators; hyperspaces of compact sets; spaces of fuzzy sets; $\mathcal{A}$-transitivity

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1. Introduction and Basic Definitions

The interplay between individual dynamics (the action of the system on points of the phase space) and collective dynamics (the action of the system on subsets of the phase space) can be extended by including the dynamics of the fuzzy sets (the action of the system on functions from the phase space to the interval $[0,1]$).

Consider the action of a continuous map $f : X \to X$ on a metric space X. The most usual context for collective dynamics is that of the induced map $\overline{f}$ on the hyperspace of all nonempty compact subsets, endowed with the Vietoris topology. The first study about the connection between the dynamical properties of the dynamical system generated by the map f and the induced system generated by $\overline{f}$ on the hyperspace was given by Bauer and Sigmund [1] in 1975. Since this work, the subject of hyperspace dynamical systems has attracted the attention of many researchers (see for instance [2,3] and the references therein).

Recently, another type of *collective dynamics* has been considered. Namely, the dynamical system (X, f) induces a dynamical system, $(\mathcal{F}(X), \hat{f})$, on the space $\mathcal{F}(X)$ of normal fuzzy sets. The map $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ is called the Zadeh extension (or fuzzification) of f. In this context, Jardón et al. studied in [4] the relationship between some dynamical properties (mainly transitivity) of the systems (X, f) and $(\mathcal{F}(X), \hat{f})$. In this same context, we consider in this note several notions of chaos, such as the ones given by Devaney [5] and Li and Yorke [6].

Given a topological space X and a continuous map $f : X \to X$, we recall that f is said to be *topologically transitive* (respectively, *mixing*) if, for any pair $U, V \subset X$ of nonempty open sets, there exists $n \geq 0$ (respectively, $n_0 \geq 0$) such that $f^n(U) \cap V \neq \emptyset$ (respectively, for all $n \geq n_0$). Moreover, f is said to be *weakly mixing* if $f \times f$ is topologically transitive on $X \times X$.

There is no unified concept of chaos, and we study here three of the most usual definitions of chaos. The map f is said to be *Devaney chaotic* if it is topologically transitive and has a dense set of periodic points [5]. The set of periodic points of f will be denoted by $\mathrm{Per}(f)$.

We say that a collection of sets of non-negative integers $\mathcal{A} \subset 2^{\mathbb{Z}_+}$ is a *Furstenberg family* (or simply a family) if it is hereditarily upwards, that is when $A \in \mathcal{A}$, $B \subset \mathbb{Z}_+$, and $A \subset B$, then $B \in \mathcal{A}$. A family $\mathcal{A}$ is a *filter* if, in addition, for every $A, B \in \mathcal{A}$, we have that $A \cap B \in \mathcal{A}$. A family $\mathcal{A}$ is *proper* if $\emptyset \notin \mathcal{A}$. Given a dynamical system (X, f) and $U, V \subset X$, we set:

$$N_f(U, V) := \{n \in \mathbb{Z}_+ : f^n(U) \cap V \neq \emptyset\},$$

Therefore, a relevant family for the dynamical system is:

$$\mathcal{N}_f := \{A \subset \mathbb{Z}_+ : \exists U, V \subset X \text{ open and nonempty with } N_f(U, V) \subset A\}.$$

Reformulating previously defined concepts, (X, f) is topologically transitive if and only if $\mathcal{N}_f$ is a proper family, and the weak mixing property is equivalent to the fact that $\mathcal{N}_f$ is a proper filter by a classical result of Furstenberg [7]. Given a family $\mathcal{A}$, we say that (X, f) is *$\mathcal{A}$-transitive* if $\mathcal{N}_f \subset \mathcal{A}$ (that is, if $N_f(U, V) \in \mathcal{A}$ for each pair of nonempty open sets $U, V \subset X$). Within the framework of linear operators, $\mathcal{A}$-transitivity was recently studied for several families $\mathcal{A}$ in [8].

When $f : (X, d) \to (X, d)$ is a continuous map on a metric space, the concept of chaos introduced by Li and Yorke [6] is the following: a pair $(x, y) \in X \times X$ is called a *Li–Yorke pair* for f if:

$$\liminf_{n\to\infty} d(f^n x, f^n y) = 0 \text{ and } \limsup_{n\to\infty} d(f^n x, f^n y) > 0.$$

The map f is said to be *Li–Yorke chaotic* if there exists an uncountable set S (a *scrambled set* for f) such that (x, y) is a Li–Yorke pair for f whenever x and y are distinct points in S.

A step forward by taking into account the distribution of the orbits was introduced by Schweizer and Smital in [9] as a natural extension of Li–Yorke chaos. We considered only the definition of uniform distributional chaos, which is one of the strongest possibilities. Recall that, if $A \subset \mathbb{N}$, then its upper density is the number:

$$\overline{\mathrm{dens}}(A) = \limsup_{n\to\infty} \frac{1}{n} |\{i < n;\ i \in A\}|,$$

where $|S|$ denotes the cardinality of the set S. If there exists an uncountable set $D \subset X$ and $\varepsilon > 0$ such that for every $t > 0$ and every distinct $x, y \in D$, the following conditions hold:

$$\overline{\mathrm{dens}}\{i \in \mathbb{N};\ d(f^i(x), f^i(y)) \geq \varepsilon\} = 1,$$
$$\overline{\mathrm{dens}}\{i \in \mathbb{N};\ d(f^i(x), f^i(y)) < t\} = 1,$$

then we say that f exhibits *uniform distributional chaos*. The set D is called a *distributionally ε-scrambled set*. Within the framework of linear dynamics, there is recent and intensive research activity on Li–Yorke and distributional chaos (see, e.g., [10–12]). See the survey articles [13,14] for more details and notions of chaos. There are still natural questions in the topic, which will be a matter of future study, such as the comparison of the considered notions of chaos for fuzzy dynamical systems with entropy-based notions of chaos (see, e.g., [15]), as well as considering the possibilities of generalizing the notions of chaos based on Lyapunov exponents and dimension (see, e.g., [16]) for the case of fuzzy dynamical systems. We do not know yet if we will encounter examples in which chaos occurs for some of the concepts considered here, but not for the ones to be studied in the future, or vice versa, within the framework of fuzzy dynamics.

Let us now describe the framework for collective dynamics. We begin with the dynamics on hyperspaces. Given a topological space X, we denote by $\mathcal{K}(X)$ the hyperspace of all nonempty compact subsets of X endowed with the Vietoris topology, that is the topology whose basic open sets are the sets of the form:

$$\mathcal{V}(U_1,\ldots,U_r) := \left\{ K \in \mathcal{K}(X) : K \subset \bigcup_{i=1}^{r} U_i \text{ and } K \cap U_i \neq \varnothing \text{ for all } i = 1,\ldots,r \right\},$$

where $r \geq 1$ and $U_1,\ldots,U_r$ are nonempty open subsets of X. When the topology of X is induced by a metric d, the Vietoris topology of $\mathcal{K}(X)$ is induced by the Hausdorff metric associated with d, namely:

$$d_H(K_1,K_2) := \max\left\{ \max_{x_1 \in K_1} d(x_1,K_2), \max_{x_2 \in K_2} d(x_2,K_1) \right\}.$$

Given $K \in \mathcal{K}(X)$ and $\varepsilon > 0$, then $B_H(K,\varepsilon)$ denotes the open ball of radius ε centered at K, with respect to d_H. If $f : X \to X$ is a continuous map, then $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ denotes the induced map defined by:

$$\overline{f}(K) := f(K) \text{ for } K \in \mathcal{K}(X),$$

where $f(K) := \{f(x) : x \in K\}$ as usual. Note that $\overline{f}$ is also continuous. We refer the reader to [17] for a detailed study of hyperspaces.

To set the more recent framework where the dynamics of the fuzzification of a map is studied, we need some basic facts for fuzzy sets. A fuzzy set u on the space X is a function $u : X \to [0,1]$. Given a fuzzy set u, let (u_α) with $\alpha \in [0,1]$ be the family of sets defined by:

$$u_\alpha = \{x \in X : u(x) \geq \alpha\},\ \alpha \in]0,1] \text{ and } u_0 = \overline{\cup\{u_\alpha : \alpha \in]0,1]\}}.$$

Let us denote by $\mathcal{F}(X)$ the family of all upper semicontinuous fuzzy sets with compact support on X such that u_1 is nonempty, which becomes a metric space with the metric:

$$d_\infty(u,v) = \sup_{\alpha \in [0,1]} \{d_H(u_\alpha, v_\alpha)\}.$$

The metric space $(\mathcal{F}(X), d_\infty)$ is denoted by $\mathcal{F}_\infty(X)$.

Another natural metric can be considered on $\mathcal{F}(X)$. Let $\xi : [0,1] \to [0,1]$ be a strictly increasing homeomorphism; the function $d_0 : \mathcal{F}(X) \to \mathcal{F}(X)$ given by:

$$d_0(u,v) = \inf\{\varepsilon : \sup_{\alpha \in [0,1]} |\xi(\alpha) - \alpha| \leq \varepsilon \text{ and } d_\infty(u, \xi v) \leq \varepsilon\}$$

defines a metric on $\mathcal{F}(X)$ called Skorokhod's metric. In general, it is fulfilled that $d_0 \leq d_\infty$, which means that the topology induced in $\mathcal{F}(X)$ by d_0 is weaker than the one induced by d_∞, i.e., $\tau_0 \subset \tau_\infty$, where τ_0 and τ_∞ denote the respective topologies. The metric space $(\mathcal{F}(X), d_0)$ is denoted by $\mathcal{F}_0(X)$. Given $u \in \mathcal{F}(X)$ and $\varepsilon > 0$, then $B_\infty(u,\varepsilon)$ and $B_0(u,\varepsilon)$ denote, respectively, the open ball of radius ε centered at u, with respect to d_∞ and d_0.

A continuous map $f : X \to X$ induces a function $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ called Zadeh's extension (fuzzification) defined as:

$$\hat{f}(u)(x) = \begin{cases} \sup\{u(z) : z \in f^{-1}(x)\} & \text{if } f^{-1}(x) \neq \varnothing \\ 0 & \text{if } f^{-1}(x) = \varnothing \end{cases}$$

We also recall that the hyperspace $\mathcal{K}(X)$ is a natural subspace of $\mathcal{F}(X)$ under the injection $K \mapsto \chi_K$, where χ_K denotes the characteristic function of K.

Some dynamical properties of $\hat{f}$ on the metric spaces $\mathcal{F}_\infty(X)$ and $\mathcal{F}_0(X)$ were studied by Jardón et al. in [4] in connection with the dynamics of f on the space X, and it is our aim to extend this study to some notions of chaos.

In the next section, we use the following properties of fuzzy sets on the spaces $\mathcal{F}_\infty(X)$ and $\mathcal{F}_0(X)$ (see [4,18,19] for the details).

Proposition 1. *Let $f : (X, d) \to (X, d)$ be a continuous function on a metric space, $u \in \mathcal{F}(X)$, $\alpha \in [0, 1]$, $n \in \mathbb{N}$, and $K \in \mathcal{K}(X)$. The following properties hold:*

1. $\left[\hat{f}(u)\right]_\alpha = f(u_\alpha)$;
2. $(\hat{f})^n = \hat{f^n}$;
3. $\hat{f}(\chi_K) = \chi_{\overline{f}(K)}$;
4. $d_0(u, \chi_K) = d_\infty(u, \chi_K)$.

2. Periodic Points and Devaney Chaos

The main results in this section are the equivalence between the Devaney chaos of $\overline{f}$ in $\mathcal{K}(X)$ and of $\hat{f}$ in $\mathcal{F}(X)$ and, as a consequence, the equivalence of Devaney chaos for a continuous linear operator T on a metrizable and complete locally convex space X, for its Zadeh extension $\hat{T}$ defined on the space of normal fuzzy sets $\mathcal{F}(X)$ and for the induced hyperspace map $\overline{T}$ on $\mathcal{K}(X)$. This extends previous results of D. Jardón, I. Sánchez, and M. Sanchis about the transitivity in fuzzy metric spaces [4] (see also [20]) and another result of N. Bernardes, A. Peris, and F. Rodenas [2] about the linear Devaney chaos of locally convex spaces.

We recall that Banks [21], Liao, Wang, and Zhang [22], and Peris [23] independently characterized the topological transitivity for $(\mathcal{K}(X), \overline{f})$ in terms of the weak mixing property for (X, f). Concerning the space of fuzzy sets, in [4], the authors showed (Theorem 3) the equivalences of the weak mixing property of f on X with the transitivity of $\hat{f}$ on $\mathcal{F}_\infty(X)$ or on $\mathcal{F}_0(X)$. They also considered the fuzzy space $\mathcal{F}(X)$ endowed with the sendograph metric and the endograph metric. Here, our attention is focused on the interplay between the dynamical systems (X, f), $(\mathcal{K}(X), \overline{f})$ and $(\mathcal{F}(X), \hat{f})$, where $\mathcal{F}(X)$ is equipped with the supremum metric d_∞ or Skorokhod's metric d_0. On the other hand, it is a well-known fact that the topologies induced by the endograph and the sendographs metrics, respectively, are included in the topology induced by d_∞, then some results can be extended as direct consequence of this fact.

On the other hand, it was shown in [2] (Theorem 2.2), in the setting of the dynamics of a continuous linear operator T on a complete locally convex space X, the equivalence of Devaney's chaos of T on X and of $\overline{T}$ on $\mathcal{K}(X)$.

Let us recall a couple of well-known properties of the Hausdorff metric, which will be useful in the sequel. Given any A, B, C, and D in $\mathcal{K}(X)$:

$$d_H(A \cup B, C \cup D) \leq max\{d_H(A, C)\, , d_H(B, D)\}\, , \tag{1}$$

$$\text{If } A \subseteq B \subseteq C\, , \quad \text{then} \quad d_H(A, B) \leq d_H(A, C) \quad \text{and} \quad d_H(B, C) \leq d_H(A, C)\, . \tag{2}$$

Lemma 1. *Let f be a continuous map on a topological space X. A nonempty compact set $K \in \mathcal{K}(X)$ is a periodic point of $\overline{f}$ if and only if its characteristic function $\chi_K \in \mathcal{F}(X)$ is $\hat{f}$-periodic. The periods of K and χ_K are the same.*

Proof. Let us assume that $K \in \mathcal{K}(X)$ is a periodic point such that $(\overline{f})^n(K) = \overline{f^n}(K) = K$, then:

$$(\hat{f})^n(\chi_K) = \hat{f^n}(\chi_K) = \chi_{\overline{f^n}(K)} = \chi_K \;\Rightarrow\; \chi_K \in Per(\hat{f}).$$

Now, we assume that χ_K is periodic, $(\hat{f})^n(\chi_K) = \chi_K$. Since, $\hat{f^n}(\chi_K) = \chi_{\overline{f^n}(K)}$ is fulfilled for every $n \in \mathbb{N}_+$, we obtain that:

$$\chi_{\overline{f}^n(K)} = \hat{f}^n(\chi_K) = \chi_K \quad \Rightarrow \quad \overline{f}^n(K) = K \quad \Rightarrow \quad K \in Per(\overline{f})\, .$$

Finally, it is obvious that periods of K and χ_K must be the same. $\square$

The following lemma was extracted from [4], and we included its proof for the sake of completeness.

Lemma 2. *Let (X, d) be a metric space. For any $u \in \mathcal{F}(X)$ and $\varepsilon > 0$, there exist numbers $0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_m = 1$ such that:*

$$d_H(u_\alpha, u_{\alpha_{i+1}}) < \varepsilon\,, \quad \text{for each } \alpha \in]\alpha_i, \alpha_{i+1}] \text{ and } \; i = 0, 1, 2..., m-1\,. \tag{3}$$

Proof. From Lemma 1 in [4], there exists a partition of the interval $[0, 1]$ given by numbers $0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_m = 1$, which satisfies:

$$d_H(u_{\alpha_i^+}, u_{\alpha_{i+1}}) < \varepsilon \text{ for each } \; i = 0, 1, 2..., m-1 \tag{4}$$

where $u_{\alpha^+} := \lim_{\lambda \to \alpha^+} L(\lambda)$, $L : [0, 1] \to \mathcal{K}(X)$ being defined by $L(\alpha) = u_\alpha$.

Since $u_{\alpha_{i+1}} \subseteq u_\alpha \subseteq u_{\alpha_i^+}$, for each $\alpha \in]\alpha_i, \alpha_{i+1}]$, $i = 0, 1, 2..., m-1$, Equation (3) is obtained as a direct consequence of the property (2) of Hausdorff's metric. □

The equivalence of (i) and (ii) in the following result was obtained by Kupka ([24], Theorem 1), with a slightly different notation. We included the proof for the sake of completeness and following the notation of the present paper.

Proposition 2. *Let f be a continuous map on a metric space X. The following assertions are equivalent:*

(i) The set of periodic points $Per(\overline{f})$ is dense in $\mathcal{K}(X)$;
(ii) The set of periodic points $Per(\hat{f})$ is dense in $\mathcal{F}_\infty(X)$;
(iii) The set of periodic points $Per(\hat{f})$ is dense in $\mathcal{F}_0(X)$.

Proof. (i) ⇒ (ii): Given a fuzzy set $u \in \mathcal{F}_\infty(X)$ and $\varepsilon > 0$, let us consider the compact sets:

$$u_\alpha = \{x \in X : u(x) \geq \alpha\}\,, \alpha \in]0, 1] \; \text{ and } \; u_0 = \overline{\bigcup_{\alpha \in]0,1]} u_\alpha}\,.$$

By Lemma 2, there exist numbers $0 = \alpha_0 < \alpha_1 < \alpha_2 < ... < \alpha_m = 1$ such that:

$$d_H(u_\alpha, u_{\alpha_{i+1}}) < \varepsilon/2\,,\; \alpha \in]\alpha_i, \alpha_{i+1}]\,,\; i = 0, 1, 2..., m-1\,. \tag{5}$$

By the hypothesis, the set $Per(\overline{f})$ is dense on $\mathcal{K}(X)$, then there exist m compact sets K_1, K_2,...,K_m in $Per(\overline{f})$ such that:

$$d_H(u_{\alpha_i}, K_i) < \varepsilon/2\,,\; i = 1, 2, ..., m\,. \tag{6}$$

There exist $n_1, n_2, ..., n_m$ in $\mathbb{N}_+$ satisfying $\overline{f}^{n_i}(K_i) = K_i$, $i = 1, 2, ..., m$. Let n be the least common multiple of $n_1, n_2, ..., n_m$, then $\overline{f}^n(K_i) = K_i$, for every $i = 1, 2, ..., m$.

We define the compact sets:

$$\omega_{\alpha_i} := \bigcup_{j \geq i} K_j\,,\; i = 1, 2, ..., m\,.$$

They satisfy that $\omega_{\alpha_{i+1}} \subseteq \omega_{\alpha_i}$, $i = 1, 2, ..., m-1$, and:

$$\overline{f}^n(\omega_{\alpha_i}) = \overline{f}^n(\bigcup_{j \geq i} K_j) = \bigcup_{j \geq i} \overline{f}^n(K_j) = \bigcup_{j \geq i} K_j = \omega_{\alpha_i}\,.$$

Therefore, $\omega_{\alpha_i} \in Per(\overline{f})$ for every $i = 1, 2, ..., m$.

Notice that $u_{\alpha_{i+1}} \subseteq u_{\alpha_i}$, $i = 1, 2, ..., m-1$, implies that $u_{\alpha_i} = \bigcup_{j \geq i} u_{\alpha_j}$. Then, Equation (6) and the property (1) of Hausdorff's metric imply:

$$d_H(u_{\alpha_i}, \omega_{\alpha_i}) = d_H(\bigcup_{j \geq i} u_{\alpha_j}, \bigcup_{j \geq i} K_j) \leq \max_{j \geq i}\{d_H(u_{\alpha_j}, K_j)\} < \varepsilon/2 \, , \, i = 1, 2, ..., m \, . \tag{7}$$

We define the family ω_α for each $\alpha \in [0,1]$ as follows:

$$\omega_\alpha = \begin{cases} \omega_{\alpha_1} \, , & 0 \leq \alpha \leq \alpha_1 \\ \omega_{\alpha_i} \, , & \alpha_{i-1} < \alpha \leq \alpha_i \, , \; 2 \leq i \leq m \, . \end{cases}$$

The family $\{\omega_\alpha \, : \, \alpha \in [0,1]\} \subset \mathcal{K}(X)$ is a decreasing family satisfying the conditions of Proposition 4.9 in [18]; therefore, there exists a unique $\overline{\omega} \in \mathcal{F}_\infty(X)$ such that $\overline{\omega}_\alpha = \omega_\alpha$ for each $\alpha \in [0,1]$. Notice that $\overline{f^n}(\overline{\omega}_\alpha) = \overline{\omega}_\alpha$, for each $\alpha \in [0,1]$.

Let us show that this $\overline{\omega} \in \mathcal{F}_\infty(X)$ is periodic and the distance between u and $\overline{\omega}$ is less than ε:

We recall that $\omega_{\alpha_i} \in Per(\overline{f})$ for each $i = 1, 2, ..., m$ and the definition of the family ω_α, $\alpha \in [0,1]$, yield:

$$\left[\hat{f}^n(\overline{\omega})\right]_\alpha = \overline{f}^n(\overline{\omega}_\alpha) = \overline{f}^n(\omega_\alpha) = \omega_\alpha = \overline{\omega}_\alpha \, , \, \alpha \in [0,1] \, .$$

Since $u = v$ if and only if $u_\alpha = v_\alpha$ for each $\alpha \in [0,1]$, we conclude that:

$$\hat{f}^n(\overline{\omega}) = \overline{\omega} \, . \quad \rightarrow \quad \overline{\omega} \in Per(\hat{f}) \, .$$

Finally, by using the triangular inequality for d_H, Relation (5), and the definition of $\omega_\alpha = \overline{\omega}_\alpha$ in each subinterval, it holds that:

$$d_H(u_\alpha, \overline{\omega}_\alpha) < d_H(u_\alpha, u_{\alpha_i}) + d_H(u_{\alpha_i}, \omega_{\alpha_i}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, , \text{ for } \alpha \in]\alpha_i, \alpha_{i+1}] \, , \, 0 \leq i \leq m-1 \, .$$

Since $\alpha_0 = 0$ and $u_{0^+} = u_0$, the last expression is also fulfilled for $\alpha = 0$. Hence, we conclude that:

$$d_H(u_\alpha, \overline{\omega}_\alpha) < \varepsilon \, , \text{ for each } \alpha \in [0,1] \, .$$

which implies that:

$$d_\infty(u, \overline{\omega}) < \varepsilon \, ,$$

and therefore, the set $Per(\hat{f})$ is dense in $\mathcal{F}_\infty(X)$.

(ii) $\Rightarrow$ (iii): This is obvious from the fact that, in general, $d_0(u,v) \leq d_\infty(u,v)$ for every $u, v \in \mathcal{F}(X)$, i.e., the topology induced in $\mathcal{F}(X)$ by d_0 is weaker than the topology induced by d_∞.

(iii) $\Rightarrow$ (i): Let K be a nonempty compact set in $\mathcal{K}(X)$ and $\varepsilon > 0$. Let us consider the characteristic function of K:

$$u := \chi_K \in \mathcal{F}_0(X)$$

Notice that,

$$u_\alpha = \{x \in X : u(x) \geq \alpha\} = K \, , \alpha \in]0,1] \text{ and } \; u_0 = \overline{\bigcup_{\alpha \in]0,1]} u_\alpha} = K \, .$$

By the hypothesis, there exists $v \in Per(\hat{f})$ such that:

$$\hat{f}^n(v) = v \, , \quad \text{for some } \; n \in \mathbb{N}_+ \text{ and } \; d_0(u,v) < \varepsilon \, .$$

Consider the family of compact sets $\{v_\alpha \, , \, \alpha \in [0,1]\} \subset \mathcal{K}(X)$ defined as usual. Since by definition $u = \chi_K$, Proposition 1 tells us that $d_\infty(u,v) = d_0(u,v) < \varepsilon$ for each $v \in \mathcal{F}(X)$, which implies that $d_H(u_\alpha, v_\alpha) < \varepsilon$ for each $\alpha \in [0,1]$.

Take a fixed $\tilde{\alpha} \in [0,1]$, and define $M := v_{\tilde{\alpha}} \in \mathcal{K}(X)$. It is easy to check that M is a periodic point of the map $\overline{f}$,

$$\overline{f}^n(M) = \overline{f}^n(v_{\tilde{\alpha}}) = \left[\hat{f}^n(v)\right]_{\tilde{\alpha}} = v_{\tilde{\alpha}} = M,$$

and the distance between K and M is less than ε,

$$d_H(K,M) = d_H(K,v_{\tilde{\alpha}}) = d_H(u_{\tilde{\alpha}},v_{\tilde{\alpha}}) \leq d_\infty(u,v) = d_0(u,v) < \varepsilon\,.$$

Therefore, the set of periodic points $\mathrm{Per}(\overline{f})$ is dense in $\mathcal{K}(X)$. □

A direct consequence of the previous result and Theorem 3 in [4] is the following:

Corollary 1. *Let f be a continuous map on a metric space X. The following assertions are equivalent:*

(i) $\overline{f}$ *is Devaney chaotic on* $\mathcal{K}(X)$*;*
(ii) $\hat{f}$ *is Devaney chaotic on* $\mathcal{F}_\infty(X)$*;*
(iii) $\hat{f}$ *is Devaney chaotic on* $\mathcal{F}_0(X)$*.*

The equivalence of Devaney's chaos for a continuous map f on a metric space and the induced map $\overline{f}$ on the hyperspace $\mathcal{K}(X)$ does not hold in general in the context of the dynamics of continuous maps on compact metric spaces. It is a well-known fact that both implications:

$$f \text{ Devaney chaotic } \rightleftarrows \overline{f} \text{ Devaney chaotic}$$

are false (in general) for the nonlinear setting. See Remark 13 and Theorem 14 in [3]. However, this equivalence holds (see Theorem 2.2 in [2]) for continuous linear operators on a complete locally convex space X. Although, traditionally, the concept of chaos was associated with the behavior of certain nonlinear dynamical systems, it has been well known since more than 100 y ago that chaos can also occur in linear systems, provided they are infinite-dimensional. The dynamics of linear operators in infinite-dimensional spaces has become a very active research area and has been extensively studied for more than twenty years, especially for operators on Fréchet spaces (i.e., metrizable and complete locally convex spaces). An overview of the state-of-the-art in the area of linear chaos can be found in the monographs [25,26].

In the linear framework, the Devaney chaotic behavior of a continuous linear operator T on a space X and that of the associated dynamical systems $(\mathcal{K}(X), \overline{T})$ and $(\mathcal{F}(X), \hat{T})$ are equivalent.

Theorem 1. *Let T be a continuous linear operator on Fréchet space X. The following assertions are equivalent:*

(i) T *is Devaney chaotic on* X*;*
(ii) $\overline{T}$ *is Devaney chaotic on* $\mathcal{K}(X)$*;*
(iii) $\hat{T}$ *is Devaney chaotic on* $\mathcal{F}_\infty(X)$*;*
(iv) $\hat{T}$ *is Devaney chaotic on* $\mathcal{F}_0(X)$*.*

Proof. The equivalence between (i) and (ii) was proven in [2] (see Theorem 2.2) and the equivalences among (ii), (iii), and (iv) are given in the previous Corollary 1. □

3. Other Dynamical Properties Related to Chaos

The purpose of this section is to deal with the concepts of $\mathcal{A}$-transitivity, Li–Yorke chaos, and distributional chaos. Since the weak mixing property is required to have at least topological transitivity on the hyperspace or on the space of fuzzy sets, we concentrate on $\mathcal{A}$-transitivity for a proper filter $\mathcal{A}$. Typical examples of proper filters are the family $\mathcal{A}_{cf}$ of cofinite subsets of $\mathbb{Z}_+$, so that $\mathcal{A}_{cf}$-transitivity is exactly the *mixing* property, and the family

$\mathcal{A}_{ts}$ of thickly syndetic sets. We recall that a strictly increasing sequence $(n_j)_{j\in\mathbb{N}} \in \mathbb{Z}_+^{\mathbb{N}}$ is *syndetic* if:

$$\sup_{j\in\mathbb{N}}(n_{j+1} - n_j) < \infty.$$

A subset $A \subset \mathbb{Z}_+$ is *thickly syndetic* if, for each $N \in \mathbb{N}$, the set $\{j \in \mathbb{Z}_+ : \{j, j+1, \ldots, j+N\} \subset A\}$ is syndetic.

We are now in conditions to establish the equivalence of $\mathcal{A}$-transitivity in our different frameworks (the original system and its associated hyperspace and space of fuzzy sets). We recall that the equivalence of Properties (i) and (ii) in the following theorem was given in [27]. For the equivalence with (iii) and (iv), we essentially followed the arguments taken from [4].

Theorem 2. *If $\mathcal{A}$ is a proper filter and (X, f) is a dynamical system on a metric space X, then the following assertions are equivalent:*

(i) (X, f) *is $\mathcal{A}$-transitive;*
(ii) $(\mathcal{K}(X), \overline{f})$ *is $\mathcal{A}$-transitive;*
(iii) $(\mathcal{F}_\infty(X), \hat{f})$ *is $\mathcal{A}$-transitive;*
(iv) $(\mathcal{F}_0(X), \hat{f})$ *is $\mathcal{A}$-transitive.*

Proof. (ii) $\Rightarrow$ (iii): Given arbitrary $u, v \in \mathcal{F}(X)$ and $\varepsilon > 0$, we have to show that $N_{\hat{f}}(U, V) \in \mathcal{A}$, where $U = B_\infty(u, \varepsilon)$ and $V = B_\infty(v, \varepsilon)$. By Lemma 2, there exist numbers $0 = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_m = 1$ such that:

$$d_H(u_\alpha, u_{\alpha_{i+1}}) < \varepsilon/3\,, \quad \text{for each } \alpha \in]\alpha_i, \alpha_{i+1}] \text{ and } i = 0, 1, 2..., m-1;$$
$$d_H(v_\alpha, v_{\alpha_{i+1}}) < \varepsilon/3\,, \quad \text{for each } \alpha \in]\alpha_i, \alpha_{i+1}] \text{ and } i = 0, 1, 2..., m-1.$$

Since $(\mathcal{K}(X), \overline{f})$ is $\mathcal{A}$-transitive and $\mathcal{A}$ is a proper filter, we have that:

$$A := \bigcap_{i=1}^{m} N_{\overline{f}}(U_i, V_i) \in \mathcal{A},$$

where $U_i = B_H(u_{\alpha_i}, \varepsilon/3)$ and $V_i = B_H(v_{\alpha_i}, \varepsilon/3)$, $i = 1, \ldots, m$. Given $n \in A$, we find $K_i \in U_i$ such that $L_i := f^n(K_i) \in V_i$, $i = 1, \ldots, m$.

As before, we consider the increasing family of compact sets:

$$\omega_{\alpha_i} := \bigcup_{j\geq i} K_j\,,\ i = 1, 2, ..., m\,,$$

and we have $d_H(u_{\alpha_i}, \omega_{\alpha_i}) < \varepsilon/3$, $i = 1, \ldots, m$. We also set:

$$\omega_\alpha = \begin{cases} \omega_{\alpha_1}\,, & 0 \leq \alpha \leq \alpha_1 \\ \omega_{\alpha_i}\,, & \alpha_{i-1} < \alpha \leq \alpha_i\,,\ 2 \leq i \leq m\,. \end{cases}$$

for each $\alpha \in [0, 1]$, which determines $\omega \in \mathcal{F}(X)$ with $d_\infty(\omega, u) < 2\varepsilon/3$, thus $\omega \in U$. Analogously, by setting:

$$\eta_{\alpha_i} := \bigcup_{j\geq i} L_j\,,\ i = 1, 2, ..., m\,,$$

we have $d_H(v_{\alpha_i}, \eta_{\alpha_i}) < \varepsilon/3$, $i = 1, \ldots, m$, and:

$$\eta_\alpha = \begin{cases} \eta_{\alpha_1}\,, & 0 \leq \alpha \leq \alpha_1 \\ \eta_{\alpha_i}\,, & \alpha_{i-1} < \alpha \leq \alpha_i\,,\ 2 \leq i \leq m\,. \end{cases}$$

for each $\alpha \in [0, 1]$, determines $\eta \in \mathcal{F}(X)$ with $d_\infty(\eta, v) < 2\varepsilon/3$, and we obtain $\eta \in V$.

By construction, we have that $f^n(\omega_{\alpha_i}) = \eta_{\alpha_i}$, $i = 1, \ldots, m$, so $\hat{f}^n(\omega) = \eta$. That is, $n \in N_{\hat{f}}(U, V)$. Since $n \in A$ was arbitrary, we obtain that $\mathcal{A} \ni A \subset N_{\hat{f}}(U, V)$, which yields $N_{\hat{f}}(U, V) \in \mathcal{A}$, as desired.

(iii) $\Rightarrow$ (iv) is trivial, since $\tau_0 \subset \tau_\infty$.

(iv) $\Rightarrow$ (i): We suppose that $(\mathcal{F}_0(X), \hat{f})$ is $\mathcal{A}$-transitive, and we pick arbitrary $x, y \in X$ and $\varepsilon > 0$. We need to show that $N_f(U, V) \in \mathcal{A}$, where $U = B(x, \varepsilon)$ and $V = B(y, \varepsilon)$. To do this, we set $u = \chi_{\{x\}}$, $v = \chi_{\{y\}}$, $\hat{U} = B_0(u, \varepsilon)$, and $\hat{V} = B_0(v, \varepsilon)$. By the hypothesis, $A := N_{\hat{f}}(\hat{U}, \hat{V}) \in \mathcal{A}$. If $n \in A$, we find $u' \in \hat{U}$ and $v' \in \hat{V}$ such that $\hat{f}^n(u') = v'$. This implies that, by selecting any $x' \in u'_0$, then $y' := f^n(x') \in v'_0$ and:

$$d(x', x) \leq d_H(u'_0, \{x\}) = d_H(u'_0, u_0) \leq d_0(u', u) < \varepsilon.$$

Analogously, $d(y', y) \leq d_0(v', v) < \varepsilon$, and we obtain that $n \in N_f(U, V)$. We conclude that $A \subset N_f(U, V)$ and (X, f) is $\mathcal{A}$-transitive. □

As mentioned before, since $\mathcal{A}_{cf}$ is a proper filter and $\mathcal{A}_{cf}$-transitivity is the topological mixing property, one immediately has the equivalence of the four properties above in the case of topological mixing.

The previous theorem has also some consequences for linear dynamics. We recall that a dynamical system (X, f) is said to be *topologically ergodic* if for any pair $U, V \subset X$ of nonempty open sets, there is a syndetic sequence (n_j) in $\mathbb{N}$ such that $f^{n_j}(U) \cap V \neq \emptyset$ for all $j \in \mathbb{N}$. Actually, topologically ergodic operators are $\mathcal{A}_{ts}$-transitive (see the exercises in [26] (Chapter 2)), and $\mathcal{A}_{ts}$ is a proper filter. The following is then an easy consequence of the previous theorem.

Corollary 2. *If T is a continuous linear operator on a metrizable topological vector space X, then the following assertions are equivalent:*

(i) *T is topologically ergodic;*
(ii) *$\overline{T}$ is topologically ergodic;*
(iii) *$\hat{T}$ is topologically ergodic.*

We recall that irrational rotations of the circle are not weakly mixing, but they are topologically ergodic, so the above corollary cannot be extended to the nonlinear setting.

We finally turn our attention to Li–Yorke chaos. The first result is essentially easy, but there are still some natural questions that remain open.

Proposition 3. *Let f be a continuous map on a metric space X. Then:*

(i) *If there exists a (ε-distributionally) scrambled set S for f, then there exist (ε-distributionally) scrambled sets $\overline{S}$ and $\hat{S}$ for $\overline{f}$ and $\hat{f}$, respectively, with the same cardinality as S;*
(ii) *If there exists a (ε-distributionally) scrambled set $\overline{S}$ for $\overline{f}$, then there exists a (ε-distributionally) scrambled set $\hat{S}$ for $\hat{f}$ with the same cardinality as $\overline{S}$;*
(iii) *If f is Li–Yorke (distributionally) chaotic on X, then $\overline{f}$ is Li–Yorke (distributionally) chaotic on $\mathcal{K}(X)$;*
(iv) *If $\overline{f}$ is Li–Yorke (distributionally) chaotic on $\mathcal{K}(X)$, then $\hat{f}$ is Li–Yorke (distributionally) chaotic on $\mathcal{F}_\infty(X)$ and in $\mathcal{F}_0(X)$.*

Proof. Everything is a consequence of the fact that the dynamical system (X, f) can be regarded as a subsystem of the dynamical system $(\mathcal{K}(X), \overline{f})$, and in turn, $(\mathcal{K}(X), \overline{f})$ is a subsystem of $(\mathcal{F}(X), \hat{f})$ by means of the isometric embeddings:

$$x \in X \mapsto \{x\} \in \mathcal{K}(X)$$
$$K \in \mathcal{K} \mapsto \chi_K \in \mathcal{F}(X)$$

both for $\mathcal{F}_\infty(X)$ and $\mathcal{F}_0(X)$. □

Remark 1. *In Theorem 10 of [3], an example was provided of a dynamical system (X, f) that admits no Li–Yorke pairs, but $(\mathcal{K}(X), \overline{f})$ (and therefore, $(\mathcal{F}_\infty(X), \hat{f})$ or $(\mathcal{F}_0(X), \hat{f})$) is distributionally chaotic. However, we do not know if there are examples of dynamical systems (X, f) for which $(\mathcal{F}_\infty(X), \hat{f})$ or $(\mathcal{F}_0(X), \hat{f})$ is (distributionally) Li–Yorke chaotic and $(\mathcal{K}(X), \overline{f})$ is not.*

Within the framework of linear dynamics, we can obtain a characterization under very general conditions, in the line of Theorem 3.2 in [2].

Theorem 3. *Let T be a continuous linear operator on a Fréchet space X, and define:*

$$NS(T) := \{x \in X : (T^n x)_{n \in \mathbb{Z}_+} \text{ has a subsequence converging to } 0\}.$$

If $\mathrm{span}(NS(T))$ is dense in X, then the following assertions are equivalent:

(i) T is Li–Yorke chaotic;

(ii) $\overline{T}$ is Li–Yorke chaotic;

(iii) $(\mathcal{F}_\infty(X), \hat{T})$ or $(\mathcal{F}_0(X), \hat{T})$ is Li–Yorke chaotic.

Proof. The equivalence of (i) and (ii) was shown in [2] (Theorem 3.2), and we already to know the implication of (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii): Again, by [2] (Theorem 3.2), we just need to show that $\overline{T}$ admits a Li–Yorke pair. Since the fuzzy system admits a Li–Yorke pair, say (u, v), by compactness and by the fact that (u, v) is a Li–Yorke pair, we obtain $K, L \in \mathcal{K}(X)$ with $K \subset u_0$, $L \subset v_0$ such that:

$$\liminf_{n\to\infty} d_H(T^n(K), T^n(L)) = 0 \text{ and } \limsup_{n\to\infty} d_H(T^n(K), T^n(L)) > 0.$$

That is, (K, L) is a Li–Yorke pair for $(\mathcal{K}(X), \overline{T})$, and we conclude the result. □

4. Conclusions

For a discrete dynamical system (X, f) on a metric space (X, d), we studied the interplay with the dynamics of its induced maps $\overline{f} : \mathcal{K}(X) \to \mathcal{K}(X)$ on the hyperspace of nonempty compact subspaces of X and $\hat{f} : \mathcal{F}(X) \to \mathcal{F}(X)$ on the space of normal fuzzy sets. We concentrated on dynamical properties related to chaotic notions. Devaney chaos was shown to be equivalent in the hyperspace or in the spaces of fuzzy sets and also equivalent to Devaney chaos in the original system when X is a complete and metrizable locally convex space and f is a continuous and linear operator. We also studied $\mathcal{A}$-transitivity for Furstenberg families $\mathcal{A}$, which are proper filters, and showed the equivalence of this property, in general, in all the systems (original, hyperspace, and spaces of fuzzy sets) considered here. Finally, with Li–Yorke and distributional chaos, we observed that from the smaller to the bigger space, the implications are fine, left open the intriguing question of whether we can go from the space of fuzzy sets to the hyperspace, and obtained a characterization, under very general assumptions, of Li–Yorke chaos in the four systems considered here within the framework of linear dynamics.

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