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New Criteria for Oscillation of Half-Linear Differential Equations with p -Laplacian-like Operators

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Abstract: In this paper, new oscillatory properties for fourth-order delay differential equations with p -Laplacian-like operators are established, using the Riccati transformation and comparison method. Moreover, our results are an extension and complement to previous results in the literature. We provide some examples to examine the applicability of our results.

Keywords: fourth-order; delay differential equations; oscillation; p -Laplacian-like operator

MSC: 34C10; 34K11

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1. Introduction

Delay differential equations arise in a variety of phenomena, including mixing liquids, economics problems, biology, medicine, physics, engineering and automatic control problems, as well as vibrational motion in flight and to explain human self-balancing; see [1,2].

The aim of this article is to study the oscillation conditions of differential equations with p -Laplacian-like operators:

$$\left(\eta(t) |w'''(t)|^{p_1-2} w'''(t) \right)' + \sum_{i=1}^j \vartheta_i(t) |w(\phi_i(t))|^{p_2-2} w(\phi_i(t)) = 0 \quad (1)$$

and

$$\left(\eta(t) |w'''(t)|^{p_1-2} w'''(t) \right)' + \beta(t) |w'''(t)|^{p_1-2} w'''(t) + \sum_{i=1}^j \vartheta_i(t) |w(\phi_i(t))|^{p_2-2} w(\phi_i(t)) = 0. \quad (2)$$

Throughout this work, we suppose the following hypotheses:

- (H₁) $p_i > 1, i = 1, 2$ are real numbers and $j \geq 1$,
(H₂) $\eta, \beta, \vartheta_i \in C([t_0, \infty), [0, \infty)), \eta(t) > 0, \vartheta_i(t) > 0, \eta'(t) + \beta(t) \geq 0, \phi_i(t) \in C([t_0, \infty), \mathbb{R}), \phi_i(t) \leq t, \lim_{t \rightarrow \infty} \phi_i(t) = \infty, i = 1, 2, \dots, j$.

Moreover, we study (1) under the condition

$$\int_{t_0}^{\infty} \frac{1}{\eta^{1/p_1-1}(s)} ds = \infty \quad (3)$$

and (2) under the condition

$$\int_{t_0}^{\infty} \left[\frac{1}{\eta(s)} \exp \left(- \int_{t_0}^s \frac{\beta(x)}{\eta(x)} dx \right) \right]^{1/p_1-1} ds = \infty. \quad (4)$$

Definition 1. A solution w of (1) and (2) is said to be non-oscillatory if it is ultimately positive or negative; otherwise, it is said to be oscillatory.

Definition 2. Equations (1) and (2) are called oscillatory if all of their solutions are oscillatory.

2. Literature Review

During recent decades, there is an ongoing interest in obtaining several sufficient conditions for the oscillatory behavior of the solutions of different kinds of differential equations, especially their oscillation and asymptote. Džurina and Jadlovská [3], Bohner et al. [4] and Baculikova [5] developed approaches and techniques for studying oscillatory properties in order to improve the oscillation criteria of second-order differential equations with delay/middle terms. Baculikova et al. [6] and Grace et al. [7] also extended this evolution to delay differential equations. Therefore, there are many studies on the oscillation criteria of different orders of some differential equations with p -Laplacian-like operators; see [8,9]. With regard to their practical importance, the oscillation and asymptote of delay differential equations have been studied extensively in recent decades; see [10–28].

Li et al. [8] considered the oscillation for the delay equation

$$\left(\eta(t) |z'''(t)|^{p-2} z'''(t) \right)' + \sum_{i=1}^j \vartheta_i(t) w(\phi_i(t)) = 0,$$

where $\phi_i(t) \leq t$, and they used the Riccati technique to find oscillation conditions for this equation. Park et al. [26] studied the asymptotic properties of the solutions of the delay equation

$$\left(\eta(t) |w^{(\kappa-1)}(t)|^{p-2} w^{(\kappa-1)}(t) \right)' + \vartheta(t) g(w(\phi(t))) = 0,$$

where $\phi(t) \leq t$, κ is even, and they used the integral average technique to obtain some oscillation results for this equation under the condition

$$\int_{v_0}^{\infty} \frac{1}{\eta^{1/(p-1)}(s)} ds = \infty.$$

Zhang et al. [9] discussed the equation

$$\left(\eta(t) \left(w^{(\kappa-1)}(t) \right)^{p-1} \right)' + \beta(t) |w'''(t)|^{p_1-2} w'''(t) + \vartheta(t) f(w(\phi(t))) = 0.$$

The purpose of this paper is to continue the authors' work [13,14].

Many researchers have used the comparison method to find oscillation conditions for this equation.

The authors in [8,9,26] used the integral average and comparison techniques that differ from the approach used in this article. Their approach is based on using the comparison technique to reduce Equations (1) and (2) into a first-order equation, while our article is based on using the Riccati technique to reduce Equations (1) and (2) into a first-order inequality to find more effective oscillation conditions for Equations (1) and (2).

Motivated by the reasons mentioned above, in this paper, we extend the results using Riccati and comparison techniques under (3) and (4). These results contribute to adding some important conditions that were previously studied in the subject of oscillation of differential equations with neutral terms. To prove our main results, we give some examples.

We shall establish asymptotic properties for (2) by converting into the form (1). It is not difficult

to see that

$$\begin{aligned} \frac{1}{\zeta_{i_0}(t)} \frac{d}{dt} \left(\mu(t) \eta(t) (w'''(t))^{p_1-1} \right) &= \frac{1}{\zeta_{i_0}(t)} \left(\zeta_{i_0}(t) \left(\eta(t) (w'''(t))^{p_1-1} \right)' \right) \\ &+ \frac{1}{\zeta_{i_0}(t)} \left(\zeta_{i_0}'(t) \eta(t) (w'''(t))^{p_1-1} \right), \\ &= \left(\eta(t) (w'''(t))^{p_1-1} \right)' + \frac{\zeta_{i_0}'(t)}{\zeta_{i_0}(t)} \eta(t) (w'''(t))^{p_1-1}, \\ &= \left(\eta(t) (w'''(t))^{p_1-1} \right)' + \beta(t) (w'''(t))^{p_1-1}, \end{aligned}$$

which with (2) gives

$$\left(\zeta_{i_0}(t) \eta(t) (w'''(t))^{p_1-1} \right)' + \zeta_{i_0}(t) \sum_{i=1}^j \vartheta_i(t) w^{p_2-1}(\phi_i(t)) = 0.$$

To prove the main results, we present some lemmas:

Lemma 1. In [15], if the function w satisfies $w^{(i)}(t) > 0$, $i = 0, 1, \dots, n$, and $w^{(n+1)}(t) < 0$, then

$$\frac{w(t)}{t^n/n!} \geq \frac{w'(t)}{t^{n-1}/(n-1)!}.$$

Lemma 2. In [16], let $h \in C^n([t_0, \infty), (0, \infty))$. Suppose that $h^{(n)}(t)$ is of a fixed sign, on $[t_0, \infty)$, $h^{(n)}(t)$, not identically zero and that there exists an $t_1 \geq t_0$ such that, for all $t \geq t_1$,

$$h^{(n-1)}(t)h^{(n)}(t) \leq 0.$$

If we have $\lim_{t \rightarrow \infty} h(t) \neq 0$, then there exists an $t_\lambda \geq t_0$ such that

$$h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} |h^{(n-1)}(t)|,$$

for all $\lambda \in (0, 1)$ and $t \geq t_\lambda$.

Lemma 3. In [17], let κ be a ratio of two odd numbers, $V > 0$ and U , that are constants. Then,

$$Uu - Vu^{(\kappa+1)/\kappa} \leq \frac{\kappa^\kappa}{(\kappa+1)^{\kappa+1}} U^{\kappa+1} V^{-\kappa}. \quad (5)$$

3. Oscillation Criteria

For convenience, we denote

$$\begin{aligned} R(t) &:= \int_t^\infty \left(\frac{1}{\eta(x)} \int_x^\infty \sum_{i=1}^j \vartheta_i(s) ds \right)^{1/(p_1-1)} dx, \\ \tilde{R}(t) &:= \mu_2^{(p_2-1)/(p_1-1)} \int_t^\infty \left(\frac{1}{\eta(x)} \int_x^\infty \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s} \right)^{p_2-1} ds \right)^{1/(p_1-1)} dx, \\ \zeta_{i_0}(t) &:= \exp \left(\int_{t_0}^t \frac{\beta(x)}{\eta(x)} dx \right) \end{aligned}$$

and

$$\hat{R}(t) := \mu_2^{(p_2-1)/(p_1-1)} \int_t^\infty \left(\frac{1}{\eta(x) \zeta_{i_0}(t)} \int_x^\infty \zeta_{i_0}(t) \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s} \right)^{p_2-1} ds \right)^{1/(p_1-1)} dx,$$

where $\mu_2 \in (0, 1)$.

Lemma 4. Let (3) hold. If w is an eventually positive solution of (1), then $w' > 0$ and $w''' > 0$.

Proof. The proof is obvious and therefore is omitted. $\square$

Theorem 1. If the equation

$$x'(t) + \frac{\lambda^{p_2-1} \sum_{i=1}^j \vartheta_i(t) \phi_i^{3(p_2-1)}(t)}{6^{p_2-1} \eta^{(p_2-1)/(p_1-1)}(\phi_i(t))} x^{(p_2-1)/(p_1-1)}(\phi_i(t)) = 0 \quad (6)$$

is oscillatory, then (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution in $[t_0, \infty)$. Then, there exists a $t_1 \geq t_0$ such that $w(t) > 0$ and $w(\phi_i(t)) > 0$ for $t \geq t_1$. Let

$$x(t) := \eta(t)(w'''(t))^{p_1-1} > 0 \quad [\text{from Lemma 4}]. \quad (7)$$

It is known that

$$|w'''(t)|^{p_1-2} w'''(t) = (w'''(t))^{p_1-1} \text{ and } |w(\phi_i(t))|^{p_2-2} w(\phi_i(t)) = w^{p_2-1}(\phi_i(t)).$$

From (1) and (7), we obtain

$$x'(t) + \sum_{i=1}^j \vartheta_i(t) w^{p_2-1}(\phi_i(t)) = 0. \quad (8)$$

Since w is positive and increasing, we see $\lim_{t \rightarrow \infty} w(t) \neq 0$. So, using Lemma 2, we find

$$w^{p_2-1}(\phi_i(t)) \geq \frac{\lambda^{p_2-1}}{6^{p_2-1}} \phi_i^{3(p_2-1)}(t) (w'''(\phi_i(t)))^{p_2-1}, \quad (9)$$

for all $\lambda \in (0, 1)$. By (8) and (9), we see that

$$x'(t) + \frac{\lambda^{p_2-1}}{6^{p_2-1}} \sum_{i=1}^j \vartheta_i(t) \phi_i^{3(p_2-1)}(t) (w'''(\phi_i(t)))^{p_2-1} \leq 0.$$

Thus, x is a positive solution of the inequality

$$x'(t) + \frac{\lambda^{p_2-1} \sum_{i=1}^j \vartheta_i(t) \phi_i^{3(p_2-1)}(t)}{6^{p_2-1} \eta^{(p_2-1)/(p_1-1)}(\phi_i(t))} x^{(p_2-1)/(p_1-1)}(\phi_i(t)) \leq 0.$$

By using Theorem 1 [22], we find that (6) also has a positive solution, which is a contradiction. The proof is complete. $\square$

Corollary 1. Assume that $p_2 = p_1$ and (3) holds. If

$$\liminf_{t \rightarrow \infty} \int_{\phi(t)}^t \frac{\lambda^{p_2-1} \sum_{i=1}^j \vartheta_i(s) \phi_i^{3(p_2-1)}(s)}{6^{p_2-1} \eta^{(p_2-1)/(p_1-1)}(\phi_i(s))} ds > \frac{1}{e}, \quad (10)$$

then (1) is oscillatory.

Lemma 5. If

$$\int_{t_0}^{\infty} \left(M^{p_2-p_1} \zeta(s) \sum_{i=1}^j \vartheta_i(s) \frac{\phi_i^{3(p_2-1)}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{p_1^{p_1}} \frac{\eta(s)(\zeta'(s))^{p_1}}{\mu^{p_1-1} s^{2(p_1-1)} \zeta^{p_1-1}(s)} \right) ds = \infty, \quad (11)$$

for some $\mu \in (0, 1)$, then $w'' < 0$.

Proof. Let $w''(t) > 0$. From Lemmas 1 and 2, we find

$$\frac{w(\phi_i(t))}{w(t)} \geq \frac{\phi_i^3(t)}{t^3} \quad (12)$$

and

$$w'(t) \geq \frac{\mu}{2} t^2 w'''(t), \quad (13)$$

Let

$$\psi(t) := \zeta(t) \frac{\eta(t)(w'''(t))^{p_1-1}}{w^{p_1-1}(t)} > 0. \quad (14)$$

From (12)–(14), we obtain

$$\begin{aligned} \psi'(t) \leq & \frac{\zeta'(t)}{\zeta(t)} \sigma(t) - \zeta(t) \sum_{i=1}^j \vartheta_i(t) \frac{\phi_i^{3(p_1-1)}(t)}{t^{3(p_1-1)}} w^{p_2-p_1}(\phi_i(t)) \\ & - \frac{(p_1-1)\mu}{2} \frac{t^2}{\zeta^{1/p_1-1}(t) \eta^{1/p_1-1}(t)} \psi^{1+(1/(p_1-1))}(t). \end{aligned} \quad (15)$$

Since $w'(t) > 0$, there exist a $t_2 \geq t_1$ and a constant $M > 0$ such that $w(t) > M$, for all $t \geq t_2$. Using the inequality (5) with $U = \zeta'/\zeta$, $V = \kappa \mu t^2 / (2\eta^{1/\kappa}(t)\zeta^{1/\kappa}(t))$ and $u = \psi$, we get

$$\psi'(t) \leq -M^{p_2-p_1} \zeta(t) \sum_{i=1}^j \vartheta_i(t) \frac{\phi_i^{3(p_1-1)}(t)}{t^{3(p_1-1)}} + \frac{2^{p_1-1}}{p_1^{p_1}} \frac{\eta(t)(\zeta'(t))^{p_1}}{\mu^{p_1-1} t^{2(p_1-1)} \zeta^{p_1-1}(t)}.$$

This implies that

$$\int_{t_1}^t \left(M^{p_2-p_1} \zeta(s) \sum_{i=1}^j \vartheta_i(s) \frac{\phi_i^{3(p_2-1)}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{p_1^{p_1}} \frac{\eta(s)(\zeta'(s))^{p_1}}{\mu^{p_1-1} s^{2(p_1-1)} \zeta^{p_1-1}(s)} \right) ds \leq \psi(t_1),$$

which contradicts (11). The proof is complete. $\square$

Theorem 2. Assume that $p_2 \geq p_1$ and (11) hold, for some $\mu \in (0, 1)$. If

$$u''(t) + M^{p_2-p_1} \tilde{R}(t)u(t) = 0 \quad (16)$$

is oscillatory, then (1) is oscillatory.

Proof. Assume the contrary, that (1) has a nonoscillatory solution in $[t_0, \infty)$. Without loss of generality, we only need to be concerned with positive solutions of Equation (1). Then, there exists a $t_1 \geq t_0$ such that $w(t) > 0$ and $w(\phi_i(t)) > 0$ for $t \geq t_1$. From Lemmas 1 and 4, we find that

$$w'(t) > 0, \quad w''(t) < 0 \text{ and } w'''(t) > 0, \quad (17)$$

for $t \geq t_2$, where t_2 is sufficiently large. Now, integrating (1) from t to l , we have

$$\eta(l)(w'''(l))^{p_1-1} = \eta(t)(w'''(t))^{p_1-1} - \int_t^l \sum_{i=1}^j \vartheta_i(s) w^{p_2-1}(\phi_i(s)) ds. \quad (18)$$

Using Lemma 3 in [17] with (17), we obtain

$$\frac{w(\phi_i(t))}{w(t)} \geq \lambda \frac{\phi_i(t)}{t},$$

which with (18) gives

$$\eta(l)(w'''(l))^{p_1-1} - \eta(t)(w'''(t))^{p_1-1} + \lambda^{p_2-1} \int_t^l \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s} \right)^{p_2-1} w^{p_1-1}(s) ds \leq 0.$$

It follows, by $w' > 0$, that

$$\eta(l)(w'''(l))^{p_1-1} - \eta(t)(w'''(t))^{p_1-1} + \lambda^{p_2-1} w^{p_1-1}(t) \int_t^l \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s} \right)^{p_2-1} ds \leq 0. \quad (19)$$

Taking $l \rightarrow \infty$, we have

$$-\eta(t)(w'''(t))^{p_1-1} + \lambda^{p_2-1}w^{p_1-1}(t) \int_t^\infty \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s}\right)^{p_2-1} ds \leq 0,$$

that is,

$$w'''(t) \geq \frac{\lambda^{(p_2-1)/(p_1-1)}}{\eta^{1/(p_1-1)}(t)} w^{(p_2-1)/(p_1-1)}(t) \left(\int_t^\infty \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s}\right)^{p_2-1} ds \right)^{1/(p_1-1)}.$$

Integrating the above inequality from t to ∞ , we obtain

$$-w''(t) \geq \lambda^{(p_2-1)/(p_1-1)} w^{(p_2-1)/(p_1-1)}(t) \int_t^\infty \left(\frac{1}{\eta(x)} \int_x^\infty \sum_{i=1}^j \vartheta_i(s) \left(\frac{\phi_i(s)}{s}\right)^{p_2-1} ds \right)^{1/(p_1-1)} dx,$$

hence

$$w''(t) \leq -\tilde{R}(t)w^{(p_2-1)/(p_1-1)}(t). \quad (20)$$

Let

$$\sigma(t) = \frac{w'(t)}{w(t)},$$

then $\sigma(t) > 0$ for $t \geq t_1$, and

$$\sigma'(t) = \frac{w''(t)}{w(t)} - \left(\frac{w'(t)}{w(t)} \right)^2.$$

By using (20) and the definition of $\sigma(t)$, we see that

$$\sigma'(t) \leq -\tilde{R}(t) \frac{w^{(p_2-1)/(p_1-1)}(t)}{w(t)} - \sigma^2(t). \quad (21)$$

Since $w'(t) > 0$, there exists a constant $M > 0$ such that $w(t) \geq M$, for all $t \geq t_2$. Then, (21) becomes

$$\sigma'(t) + \sigma^2(t) + M^{p_2-p_1}\tilde{R}(t) \leq 0, \quad (22)$$

From [21], we obtain (16) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (22) holds, which is a contradiction. The theorem is proved. $\square$

Theorem 3. Let $p_2 \geq p_1$, $\phi_i'(t) > 1$ and (11) hold, for some $\mu \in (0, 1)$. If

$$\left(\frac{1}{\phi_i'(t)} u'(t) \right)' + M^{(p_2-1)/(p_1-2)} R(t) u(t) = 0 \quad (23)$$

is oscillatory, then (1) is oscillatory.

Proof. From the proof of Theorem 2, we find that (18) holds. Thus, it follows from $\phi_i'(t) \geq 0$ and $w'(t) \geq 0$ that

$$\eta(l)(w'''(l))^{p_1-1} - \eta(t)(w'''(t))^{p_1-1} + w^{p_2-1}(\phi_i(t)) \int_t^l \sum_{i=1}^j \vartheta_i(s) ds \leq 0. \quad (24)$$

Thus, (17) becomes

$$w''(t) \leq -R(t)w^{(p_2-1)/(p_1-1)}(\phi_i(t)). \quad (25)$$

Let

$$\omega(t) = \frac{w'(t)}{w(\phi_i(t))}, \quad (26)$$

then $\omega(t) > 0$ for $t \geq t_1$, and

$$\begin{aligned}\omega'(t) &= \frac{w''(t)}{w(\phi_i(t))} - \frac{w'(t)}{w^2(\phi_i(t))} w'(\phi_i(t)) \phi_i'(t) \\ &\leq \frac{w''(t)}{w(\phi_i(t))} - \phi_i'(t) \left(\frac{w'(t)}{w(\phi_i(t))} \right)^2.\end{aligned}$$

From (25) and (26), we find

$$\omega'(t) + M^{(p_2-1)/(p_1-2)} R(t) + \phi_i'(t) \omega^2(t) \leq 0. \quad (27)$$

From [21], we find (23) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that (27) holds, which is a contradiction. The theorem is proved. $\square$

Corollary 2. Let $p_2 = p_1$ and (11) hold. If

$$\lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s) \tilde{R}(s) - \frac{1}{4} h^2(t, s) \right) ds = \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^\infty \tilde{R}(s) ds > \frac{1}{4}, \quad (28)$$

then (1) is oscillatory.

Corollary 3. Let $p_2 = p_1$ and (11) hold. If $\varepsilon \in (0, 1/4]$ such that

$$t^2 \tilde{R}(s) \geq \varepsilon$$

and

$$\limsup_{t \rightarrow \infty} \left(t^{\varepsilon-1} \int_{t_0}^t s^{2-\varepsilon} \tilde{R}(s) ds + t^{1-\tilde{\varepsilon}} \int_t^\infty s^{\tilde{\varepsilon}} \tilde{R}(s) ds \right) > 1,$$

where $\tilde{\varepsilon} = \frac{1}{2}(1 - \sqrt{1 - 4\varepsilon})$, then (1) is oscillatory.

Corollary 4. Let $p_1 = p_2$ and (4) holds. If

$$\liminf_{t \rightarrow \infty} \int_{\phi(t)}^t \frac{\lambda^{p_2-1}}{6^{p_2-1}} \frac{\zeta_{t_0}(s) \sum_{i=1}^j \vartheta_i(s) \phi_i^{3(p_2-1)}(s)}{\zeta_{t_0}^{(p_2-1)/(p_1-1)}(\phi_i(s)) \eta^{(p_2-1)/(p_1-1)}(\phi_i(s))} ds > \frac{1}{e},$$

then (2) is oscillatory.

Corollary 5. Let $p_1 = p_2$, (4) and

$$\int_{t_0}^\infty \left(M^{p_2-p_1} \zeta(s) \zeta_{t_0}(s) \sum_{i=1}^j \vartheta_i(s) \frac{\phi_i^{3\kappa}(s)}{s^{3\kappa}} - \frac{2^{p_1-1}}{p_1^{p_1}} \frac{\eta(s) (\zeta'(s))^{p_1}}{\mu^{p_1-1} s^{2(p_1-1)} \zeta^{p_1-1}(s)} \right) ds = \infty, \quad (29)$$

hold, for some $\mu \in (0, 1)$. If

$$\lim_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \left(H(t, s) \hat{R}(s) - \frac{1}{4} h^2(t, s) \right) ds = \infty$$

or

$$\liminf_{t \rightarrow \infty} \int_t^\infty \hat{R}(s) ds > \frac{1}{4},$$

then (2) is oscillatory.

Corollary 6. Let $p_1 = p_2$ and (29) hold. If $\varepsilon \in (0, 1/4]$ such that

$$t^2 \hat{R}(s) \geq \varepsilon$$

and

$$\limsup_{t \rightarrow \infty} \left(t^{\varepsilon-1} \int_{t_0}^t s^{2-\varepsilon} \hat{R}(s) ds + t^{1-\tilde{\varepsilon}} \int_t^\infty s^{\tilde{\varepsilon}} \hat{R}(s) ds \right) > 1,$$

where $\tilde{\varepsilon}$ is defined as in Corollary 3, then (2) is oscillatory.

Example 1. For $\iota \geq 1$, consider the equation:

$$\left(\iota^3(w''(\iota))^3\right)' + \frac{\vartheta_0}{\iota^7}w^3(\gamma\iota) = 0, \quad (30)$$

we see that $p_1 = p_2 = 4$, $\eta(\iota) = \iota^3$, $\phi_i(\iota) = \gamma\iota$ and $\vartheta(\iota) = \vartheta_0/\iota^7$, $\gamma \in (0, 1]$ and $\vartheta_0 > 0$. Thus, we obtain

$$\tilde{R}(\iota) = \lambda \left(\frac{\vartheta_0}{6}\right)^{1/3} \gamma \frac{1}{2\iota^2}.$$

By Corollaries 1 and 2, Equation (30) is oscillatory if

$$\vartheta_0 > \frac{6^3}{e\left(\ln \frac{1}{\gamma}\right)\gamma^6},$$

$$\vartheta_0 > \left(\frac{3^4}{2}\right) \frac{1}{\gamma^9}$$

and

$$\vartheta_0 > 6\left(\frac{1}{4\gamma}\right)^3,$$

respectively. Thus, Equation (30) is oscillatory if

$$\vartheta_0 > \max\left\{\left(\frac{3^4}{2}\right) \frac{1}{\gamma^9}, 6\left(\frac{1}{4\gamma}\right)^3\right\} = \left(\frac{3^4}{2}\right) \frac{1}{\gamma^9}. \quad (31)$$

Example 2. Consider the equation

$$\left(\iota^3(w'''(\iota))^3\right)' + (w'''(\iota))^3 + \frac{\vartheta_0}{\iota^5}w^3(\iota/2) = 0, \iota \geq 1, \vartheta_0 > 0. \quad (32)$$

Let $p_1 = p_2 = 4$, $\eta(\iota) = \iota^3$, $\beta(\iota) = 1$, $\phi_i(\iota) = \iota/2$ and $\vartheta(\iota) = \vartheta_0/\iota^5$. Then, it is easy to verify that

$$\begin{aligned} & \int_{\iota_0}^{\infty} \left[\frac{1}{\eta(s)} \exp\left(-\int_{\iota_0}^s \frac{\beta(x)}{\eta(x)} dx\right) \right]^{1/p_1-1} ds \\ &= \int_{\iota_0}^{\infty} \left[\frac{1}{s^3} \exp\left(-\int_{\iota_0}^s \frac{1}{s^3} dx\right) \right]^{1/3} ds = \infty. \end{aligned}$$

By Corollary 5, Equation (32) is oscillatory.

4. Conclusions

In this work, we study the asymptotic and oscillatory properties of solutions of the fourth-order delay differential equations with p -Laplacian-like operators. Using the Riccati transformation, we obtained new criteria that guarantee the oscillation of all solutions of the studied equations. In future work, we will study oscillatory properties of Equation (1) under the condition

$$\int_{\iota_0}^{\infty} \frac{1}{\eta^{1/p_1-1}(s)} ds < \infty. \quad (33)$$

An interesting problem is to extend our results to even-order damped differential equations with p -Laplacian-like operators

$$\left(\eta(\iota)\left(w^{(\kappa-1)}(\iota)\right)^{p-1}\right)' + \beta(\iota)|w'''(\iota)|^{p_1-2}w'''(\iota) + \vartheta(\iota)f(w(\phi(\iota))) = 0.$$

under the condition

$$\int_{t_0}^{\infty} \left[\frac{1}{\eta(s)} \exp \left(- \int_{t_0}^s \frac{\beta(x)}{\eta(x)} dx \right) \right]^{1/p_1-1} ds < \infty.$$

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