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On a Coupled System of Stochastic Itô-Differential and the Arbitrary (Fractional) Order Differential Equations with Nonlocal Random and Stochastic Integral Conditions

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Abstract: The fractional stochastic differential equations had many applications in interpreting many events and phenomena of life, and the nonlocal conditions describe numerous problems in physics and finance. Here, we are concerned with the combination between the three senses of derivatives, the stochastic Itô-differential and the fractional and integer orders derivative for the second order stochastic process in two nonlocal problems of a coupled system of two random and stochastic differential equations with two nonlocal stochastic and random integral conditions and a coupled system of two stochastic and random integral conditions. We study the existence of mean square continuous solutions of these two nonlocal problems by using the Schauder fixed point theorem. We discuss the sufficient conditions and the continuous dependence for the unique solution.

Keywords: stochastic processes; Itô-differential equations; random differential equations; stochastic differential equation; coupled system; fractional order derivative; nonlocal stochastic integral conditions

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1. Introduction

The existence and uniqueness of solutions to stochastic differential equations driven by Brownian motion have been studied by many authors (see [1–3]).

Also the non-local coupled system was studied by some authors (see for example [4–8] and references therein).

The results are important since they cover non-local generalizations of fractional stochastic differential equations (FSDE), more applications are arising in fields such as heat conduction, electromagnetic theory and dynamic system (see for example [9,10]).

Many authors have been interested to study the fractional stochastic differential equations see [11–14] and investigate their results all the time.

Let (Ω, G, μ) be a probability space see [15].

The motive of this work is to generalize the scope results of A.M.A. El-Sayed [16,17] on the stochastic fractional operators and the solution of non-local coupled systems of stochastic differential equations see [7,16].

Also, we study the existence of solutions of a coupled system of Itô-differential equation and arbitrary (fractional) orders random differential equation subject to two coupled systems of non-local random and stochastic integral conditions. The effect of random functions and data which ensures the continuous dependence of the solution has been proved.

Let $I = [0, T]$ and $X(t; \omega) = X(t)$, $t \in I$, $\omega \in \Omega$ be such that $E(X^2(t)) < \infty$, $t \in I$.

Let $C = C(I, L_2(\Omega))$ be the class of all mean square (m.s) continuous stochastic processes on I with norm

$$\|X\|_C = \sup_{t \in I} \|X(t)\|_2, \quad \|X(t)\|_2 = (EX^2(t))^{1/2}.$$

Definition 1. Let $X \in C(I, L_2(\Omega))$ and $\mu, \nu \in (0, 1]$. The stochastic integral operator of order ν is defined by

$$I^\nu X(t) = \int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} X(s) ds$$

and the stochastic fractional order derivative is defined by

$$D^\mu X(t) = I^{1-\mu} \frac{dX}{dt}.$$

For the properties of stochastic fractional calculus see [16].

Let $\alpha, \beta \in (0, 1]$ and $T \geq 1$. Here we prove the existence of solutions $X, Y \in C([0, T], L_2(\Omega))$ of the two nonlocal fractional coupled system of the two Itô-deferential and arbitrary orders, differential equations

$$\frac{dX(t)}{dt} = g_1(t, Y(\phi(t))), \quad t \in (0, T] \quad (1)$$

and

$$dY(t) = g_2(t, D^\alpha X(t)) dW(t), \quad t \in (0, T] \quad (2)$$

with the stochastic and random integral conditions

$$X(0) + \int_0^\tau h_1(s, D^\beta X(s)) dW(s) = X_0, \quad Y(0) + \int_0^\eta h_2(s, Y(s)) ds = Y_0 \quad (3)$$

and the coupled system of the two stochastic and random non-local integral conditions

$$X(0) + \int_0^\tau h_1(s, Y(s)) dW(s) = X_0, \quad Y(0) + \int_0^\eta h_2(s, X(s)) ds = Y_0 \quad (4)$$

where X_0 and Y_0 are two second order random variables.

The existence of solutions $X, Y \in C(I, L_2(\Omega))$ of the problems (1)–(3) and (1)–(2) and (4) are proved. The continuous dependence of the unique solutions $X, Y \in C(I, L_2(\Omega))$ on X_0 and Y_0 , h_1 and h_2 and the solution $Y \in C(I, L_2(\Omega))$ on $D^\alpha X(t)$ will be studied.

2. Integral Representations of the Solution

Consider the following assumptions:

- (a1) $\phi : I \rightarrow I$, $\phi(t) \leq t$ is a continuous on I .
- (a2) $g_i : I \times L_2(\Omega) \rightarrow L_2(\Omega)$ are (Caratheodory) measurable in $t \in I$, $\forall X \in L_2(\Omega)$ and continuous in $X \in L_2(\Omega) \forall t \in I$ and there exist $b_i > 0$ and two bounded measurable functions $\xi_i : I \rightarrow R$ such that

$$\|g_i(t, X)\|_2 \leq |\xi_i(t)| + b_i \|X(t)\|_2, \quad i = 1, 2. \quad (5)$$

- (a3) $h_i : [0, T] \times L_2(\Omega) \rightarrow L_2(\Omega)$ are Caratheodory. There exist $c_i > 0$ and two bounded measurable functions $\sigma_i : I \rightarrow R$ such that

$$\|h_i(t, X)\|_2 \leq |\sigma_i(t)| + c_i \|X(t)\|_2, \quad i = 1, 2. \quad (6)$$

- (a4) $M = \max\{\sup_{t \in I} |\xi_1(t)|, \sup_{t \in I} |\xi_2(t)|\}$, $b = \max\{b_1, b_2\}$.
- (a5) $K = \max\{\sup_{t \in I} |\sigma_1(t)|, \sup_{t \in I} |\sigma_2(t)|\}$, $c = \max\{c_1, c_2\}$.
- (a6) $(b + c)T + b^2 T^2 (2 - \alpha) < 1$.

Now, operating by $I^{1-\alpha}$ on Equation (1), we obtain

$$D^\alpha X(t) = I^{1-\alpha} \frac{dX}{dt} = I^{1-\alpha} g_1(t, Y(\phi(t))).$$

Let

$$u(t) = D^\alpha X(t), \quad (7)$$

then from (1)–(2) we obtain

$$u(t) = I^{1-\alpha} g_1(t, Y(\phi(t))) \quad (8)$$

and

$$dY(t) = g_2(t, u(t))dW(t). \quad (9)$$

Then we have the following lemma

Lemma 1. *The solutions of the problems (1)–(3) and (1), (2) and (4) can be given by*

$$X(t) = X_0 - \int_0^t h_1(s, I^{1-\beta} g_1(s, Y(\phi(s))))dW(s) + \int_0^t g_1(s, Y(\phi(s)))ds, \quad (10)$$

$$Y(t) = Y_0 - \int_0^t h_2(s, Y(s))ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s))))dW(s) \quad (11)$$

and

$$X(t) = X_0 - \int_0^t h_1(s, Y(s))dW(s) + \int_0^t g_1(s, Y(\phi(s)))ds, \quad (12)$$

$$Y(t) = Y_0 - \int_0^t h_2(s, X(s))ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s))))dW(s), \quad (13)$$

respectively.

Proof. Integrating the Equations (7) and (9) (see [12–18]) with substitution by (8) and using the non-local conditions (3) and (4) the equivalent between the problem (1)–(3) and the integral representation (10)–(11) and the problem (1), (2) and (4) and the integral representation (12)–(13) can be proved. $\square$

3. Solutions of the Problem (1)–(3)

Theorem 1. *Let the assumptions (a1)–(a6) be satisfied, then the problem (1)–(3) has at least one solution $X, Y \in C$.*

Proof. Consider the set Q_1 such that

$$Q_1 = \{Y(t) \in L_2(\Omega) : \|Y(t)\|_2 \leq r_1\} \subset C(I, L_2(\Omega))$$

and define the mapping F_1 where

$$F_1 Y(t) = Y_0 - \int_0^t h_2(s, Y(s))ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s))))dW(s). \quad (14)$$

Let $Y \in Q_1$, then

$$\begin{aligned}
 \|F_1 Y(t)\|_2 &\leq \|Y_0\|_2 + \left\| \int_0^\eta h_2(s, Y(s)) ds \right\|_2 + \left\| \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right\|_2 \\
 &\leq \|Y_0\|_2 + \int_0^\eta (\sigma_2(s) + c_2 \|Y\|_C) ds \\
 &\quad + \sqrt{\int_0^t (\xi_2(s) + b_2 \|I^{1-\alpha} g_1(s, Y(\phi(s))))\|_2^2 ds} \\
 &\leq \|Y_0\|_2 + (K + cr_1)T + \left(M + \frac{bT^{1-\alpha}(M + br_1)}{(1-\alpha)\Gamma(1-\alpha)}\right)\sqrt{T} \\
 &\leq \|Y_0\|_2 + KT + M\sqrt{T} + \frac{MbT^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)} + \left(cT + \frac{b^2T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)}\right)r_1 \\
 &\leq \|Y_0\|_2 + (K + M)T + MbT^2(2-\alpha) + (cT + b^2T^2(2-\alpha))r_1 = r_1
 \end{aligned}$$

where

$$r_1 = \frac{(\|Y_0\|_2 + (K + M))T + MbT^2(2-\alpha)}{1 - cT - b^2T^2(2-\alpha)},$$

then $F_1 : Q_1 \rightarrow Q_1$ and the class $\{F_1 Y(t)\}$, $t \in [0, T]$ is uniformly bounded on Q_1 . $\square$

Let $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
 \|F_1 Y(t_1) - F_1 Y(t_2)\|_2 &= \left\| \int_0^{t_2} g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right. \\
 &\quad \left. - \int_0^{t_1} g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right\|_2 \\
 &= \left\| \int_{t_1}^{t_2} g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right\|_2 \\
 &\leq \sqrt{\int_{t_1}^{t_2} g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s))))\|_2^2 ds} \\
 &\leq \left(M + b\left(\frac{T^{1-\alpha}(M + b\|Y\|_C)}{\Gamma(2-\alpha)}\right)\right)\sqrt{t_2 - t_1}.
 \end{aligned}$$

This proves the equicontinuity of the class $\{F_1 Y(t)\}$, $t \in [0, T]$ on Q_1 .

Let $Y_n \in Q_1$, $Y_n \rightarrow Y$ w.p.1, we get (see [15])

$$\begin{aligned}
 L.i.m_{n \rightarrow \infty} F_1 Y_n &= L.i.m_{n \rightarrow \infty} \left(Y_0 - \int_0^\eta h_2(s, Y_n(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y_n(\phi(s)))) dW(s) \right) \\
 &= Y_0 - \int_0^\eta h_2(s, L.i.m_{n \rightarrow \infty} Y_n(s)) ds \\
 &\quad + \int_0^t g_2(s, I^{1-\alpha} g_1(s, L.i.m_{n \rightarrow \infty} Y_n(\phi(s)))) dW(s) \\
 &= Y_0 - \int_0^\eta h_2(s, Y(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) = F_1 Y.
 \end{aligned}$$

This proves that the operator $F_1 : Q_1 \rightarrow Q_1$ is continuous. Consequently, the closure of $\{F_1 Q_1\}$ is compact and (see [15]) and integral Equation (11) has a solution $Y \in C([0, T], L_2(\Omega))$.

Let the set Q_2 be such that

$$Q_2 = \{X(t) \in L_2(\Omega) : \|X(t)\|_2 \leq r_2\} \subset C([0, T], L_2(\Omega))$$

and define the mapping F_2X such that

$$F_2X(t) = X_0 - \int_0^t h_1(s, I^{1-\beta} g_1(s, Y(\phi(s)))) dW(s) + \int_0^t g_1(s, Y(\phi(s))) ds. \quad (15)$$

Let $X \in Q_2$, then

$$\begin{aligned} \|F_2X(t)\|_2 &\leq \|X_0\|_2 + \sqrt{\int_0^t \|h_1(s, I^{1-\beta} g_1(s, Y(\phi(s)))) dW(s)\|_2^2 ds} + \int_0^t \|g_1(s, Y(\phi(s)))\|_2 ds \\ &\leq \|X_0\|_2 + \sqrt{\int_0^t (|\sigma_1(s)| + c_1 \|I^{1-\beta} g_1(s, Y(\phi(s)))\|_2)^2 ds} + \int_0^t (|\xi_1(s)| + b_1 \|Y\|_C) ds \\ &\leq \|X_0\|_2 + (K + \frac{cT^{1-\beta}(M + br_1)}{\Gamma(2-\alpha)})\sqrt{T} + (M + br_1)T \\ &\leq \|X_0\|_2 + (K + M)T + \frac{cMT^{\frac{3}{2}-\beta}}{\Gamma(2-\alpha)} + (\frac{cbT^{\frac{3}{2}-\beta}}{\Gamma(2-\alpha)} + bT)r_1 \\ &\leq \|X_0\|_2 + (K + M)T + McT^2(2-\beta) + (bT + bcT^2(2-\beta))r_1 = r_2 \end{aligned}$$

where

$$r_2 = \|X_0\|_2 + (K + M)T + McT^2(2-\beta) + (bT + bcT^2(2-\beta))r_1,$$

then the class $\{F_2X(t)\}$, $t \in [0, T]$ is uniformly bounded.

Let $X \in Q_2$; $t_1, t_2 \in [0, T]$, then

$$\begin{aligned} \|F_2X(t_2) - F_2X(t_1)\|_2 &= \left\| \int_{t_1}^{t_2} g_1(s, Y(\phi(s))) ds \right\|_2 \\ &\leq (M + br_1)(t_2 - t_1), \end{aligned}$$

then the class $\{F_2X(t)\}$, $t \in [0, T]$ is equicontinuous.

Let $X_n \in Q_2$ be such that $X_n \rightarrow Y$ w.p.1.

Using Theorem 1, we get

$$\begin{aligned} L.i.m_{n \rightarrow \infty} F_2X_n &= L.i.m_{n \rightarrow \infty} (X_0 - \int_0^t h_1(s, I^{1-\beta} g_1(s, Y_n(\phi(s)))) dW(s) + \int_0^t g_1(s, Y_n(\phi(s))) ds) \\ &= X_0 - \int_0^t h_1(s, L.i.m_{n \rightarrow \infty} I^{1-\beta} g_1(s, Y_n(\phi(s)))) dW(s) + \int_0^t g_1(s, L.i.m_{n \rightarrow \infty} Y_n(\phi(s))) ds \\ &= X_0 - \int_0^t h_1(s, I^{1-\beta} g_1(s, Y(\phi(s)))) dW(s) + \int_0^t g_1(s, Y(\phi(s))) ds = F_2X. \end{aligned}$$

Applying Schauder Fixed Point Theorem [15], (10) has a solution $X \in C(I, L_2(\Omega))$.

3.1. Uniqueness Theorem

To discuss the uniqueness of the solution Y of (11) consider the assumptions $(a^*1) - (a^*2)$ instead of $(a1) - (a2)$ such that

$(a^*1) g_i : I \times L_2(\Omega) \rightarrow L_2(\Omega)$; $i = 1, 2$ are Caratheodory and satisfy second argument Lipschitz condition

$$\|g_i(t, u(t)) - g_i(t, v(t))\|_2 \leq b \|u(t) - v(t)\|_2.$$

$(a^*2) h_i : I \times L_2(\Omega) \rightarrow L_2(\Omega)$; $i = 1, 2$ Caratheodory and satisfy second argument Lipschitz condition

$$\|h_i(t, u(t)) - h_i(t, v(t))\|_2 \leq c \|u(t) - v(t)\|_2.$$

It is clear that the assumptions $(a^*1) - (a^*2)$ imply the assumptions $(a1) - (a2)$.

Theorem 2. Let $(a^*1) - (a^*2)$ and $(a3) - (a6)$ be satisfied, then the solution of (1)–(3) is unique.

Proof. Let Y_1 and Y_2 be two solutions of (11), then

$$\begin{aligned} \|Y_1(t) - Y_2(t)\|_2 &\leq \int_0^\eta \|h_2(s, Y_2(s)) - h_2(s, Y_1(s))\|_2 ds \\ &\quad + \sqrt{\int_0^t \|g_2(s, I^{1-\alpha}g_1(s, Y_1(\phi(s)))) - g_2(s, I^{1-\alpha}g_1(s, Y_2(\phi(s))))\|_2^2 ds} \\ &\leq cT\|Y_1 - Y_2\|_C + \frac{b^2T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)}\|Y_1 - Y_2\|_C \\ &\leq (cT + b^2T^2(2-\alpha))\|Y_1 - Y_2\|_C. \end{aligned}$$

and

$$(1 - cT - b^2T^2(2-\alpha))\|Y_1 - Y_2\|_C \leq 0.$$

Using (a6) we can get

$$\|Y_1 - Y_2\|_C = 0,$$

then the solution of (11) is unique. Consequently, the solution of (10) is unique. $\square$

Combining the results, then we deduce that the solution $X, Y \in C$ of the problem the problem (1)–(3) is unique.

3.2. Continuous Dependence

Theorem 3. The unique solution of the problem (1)–(3) depends continuously on X_o, Y_o .

Proof. Let $\hat{X}, \hat{Y}$ be the solution of

$$\begin{aligned} \hat{X}(t) &= \hat{X}_o - \int_0^\tau h_1(s, I^{1-\beta}g_1(s, \hat{Y}(\phi(s))))dW(s) + \int_0^t g_1(s, \hat{Y}(\phi(s)))ds, \\ \hat{Y}(t) &= \hat{Y}_o - \int_0^\eta h_2(s, \hat{Y}(s))ds + \int_0^t g_2(s, I^{1-\alpha}g_1(s, \hat{Y}(\phi(s))))dW(s). \end{aligned}$$

Let

$$\|X_o - \hat{X}_o\|_2 \leq \delta \quad \text{and} \quad \|Y_o - \hat{Y}_o\|_2 \leq \delta,$$

then

$$\begin{aligned} \|Y - \hat{Y}\|_C &\leq \|Y_o - \hat{Y}_o\|_2 + c\|Y - \hat{Y}\|_C T + \frac{b^2T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)}\|Y - \hat{Y}\|_C \\ &\leq \|Y_o - \hat{Y}_o\|_2 + (cT + b^2T^2(2-\alpha))\|Y - \hat{Y}\|_C, \\ (1 - cT - b^2T^2(2-\alpha))\|Y - \hat{Y}\|_C &\leq \|Y_o - \hat{Y}_o\|_2 \end{aligned}$$

and

$$\|Y - \hat{Y}\|_C \leq \frac{\delta}{1 - cT - b^2T^2(2-\alpha)} = \epsilon. \quad (16)$$

In the same way, we have

$$\begin{aligned} \|X - \hat{X}\|_C &\leq \|X_o - \hat{X}_o\|_2 + \left[\frac{cT^{1-\beta}b\|Y - \hat{Y}\|_C}{\Gamma(2-\beta)} \right] \sqrt{T} + bT\|Y - \hat{Y}\|_C, \\ \|X - \hat{X}\|_C &\leq \|X_o - \hat{X}_o\|_2 + [cbT^2(2-\beta) + bT]\|Y - \hat{Y}\|_2 \end{aligned}$$

and

$$\|X - \hat{X}\|_C \leq \delta \left\{ 1 + \frac{cbT^2(2-\beta) + bT}{1 - cT - b^2T^2(2-\alpha)} \right\} = \epsilon. \quad (17)$$

□

Theorem 4. The solution of (1)–(3) depends continuously on h_1 and h_2 .

Proof. Let $\hat{X}, \hat{Y}$ be the solution of

$$\begin{aligned}\hat{X}(t) &= X_0 - \int_0^t h_1^*(s, I^{1-\beta} g_1(s, \hat{Y}(\phi(s)))) dW(s) + \int_0^t g_1(s, \hat{Y}(\phi(s))) ds, \\ \hat{Y}(t) &= Y_0 - \int_0^t h_2^*(s, \hat{Y}(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, \hat{Y}(\phi(s)))) dW(s),\end{aligned}$$

If $\|h_j^*(s, \cdot) - h(s, \cdot)\|_2 \leq \delta_1, \quad j = 1, 2$ then

$$\begin{aligned}\|Y(t) - \hat{Y}(t)\|_2 &= \left\| \int_0^t [h_2^*(s, \hat{Y}(s)) - h_2(s, Y(s))] ds \right. \\ &\quad \left. + \int_0^t [g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) - g_2(s, I^{1-\alpha} g_1(s, \hat{Y}(\phi(s))))] dW(s) \right\|_2 \\ &\leq (\delta_1 + c\|Y - \hat{Y}\|_C)T + \frac{b^2 T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)} \|Y - \hat{Y}\|_C \\ &\leq \delta_1 T + [cT + b^2 T^2(2-\alpha)] \|Y - \hat{Y}\|_C\end{aligned}$$

and

$$\|Y - \hat{Y}\|_C \leq \frac{\delta_1 T}{1 - cT - b^2 T^2(2-\alpha)} \quad (18)$$

and similarly we have

$$\begin{aligned}\|X(t) - \hat{X}(t)\|_2 &\leq \left\| \int_0^t [h_1^*(s, I^{1-\beta} g_1(s, \hat{Y}(\phi(s)))) - h_1(s, I^{1-\beta} g_1(s, Y(\phi(s))))] dW(s) \right\|_2 \\ &\quad + \left\| \int_0^t [g_1(s, Y(\phi(s))) - g_1(s, \hat{Y}(\phi(s)))] ds \right\|_2 \\ &\leq \sqrt{\int_0^t (\delta_1 + \frac{bcT^{1-\beta}\|Y - \hat{Y}\|_C}{\Gamma(2-\beta)})^2 ds} + bT\|Y - \hat{Y}\|_C \\ &\leq (\delta_1 + \frac{bcT^{1-\beta}\|Y - \hat{Y}\|_C}{\Gamma(2-\beta)})\sqrt{T} + bTc \\ &\leq \delta_1\sqrt{T} + [bcT^2(2-\beta) + bT]\|Y - \hat{Y}\|_C\end{aligned}$$

and

$$\|X - \hat{X}\|_C \leq \delta_1 T \left[1 + \frac{bcT^2(2-\beta) + bT}{1 - cT - b^2 T^2(2-\alpha)} \right] = \epsilon. \quad (19)$$

From (18) and (19) the solution X, Y depends continuously on h_1, h_2 . □

Theorem 5. The solution of (1)–(3) depends continuously on $D^\alpha X(t) = u(t)$.

Proof. Let $\hat{X}, \hat{Y}$ be the solution of

$$\begin{aligned}\hat{X}(t) &= X_0 - \int_0^t h_1(s, I^{\alpha-\beta} \hat{u}(s)) dW(s) + I^\alpha \hat{u}(t), \\ \hat{Y}(t) &= Y_0 - \int_0^t h_2(s, \hat{Y}(s)) ds + \int_0^t g_2(s, \hat{u}(s)) dW(s)\end{aligned}$$

such that $\|\hat{u}(t) - u(t)\|_2 \leq \delta_2$, then

$$\begin{aligned}\|X(t) - \hat{X}(t)\|_2 &\leq \left\| \int_0^\eta [h_1(s, I^{\alpha-\beta} \hat{u}(s)) - h_1(s, I^{\alpha-\beta} u(s))] dW(s) \right\|_2 + \|I^\alpha u(t) - I^\alpha \hat{u}(t)\|_2 \\ &\leq \frac{cT^{\alpha-\beta} \delta_2 \sqrt{T}}{\Gamma(\alpha - \beta + 1)} + \frac{T^\alpha \delta_2}{\Gamma(\alpha + 1)}\end{aligned}$$

and

$$\|X - \hat{X}\|_C \leq T^\alpha \delta_2 \left[\frac{cT^{\frac{1}{2}-\beta}}{\Gamma(\alpha - \beta + 1)} + \frac{1}{\Gamma(\alpha + 1)} \right] = \epsilon. \quad (20)$$

By the same way

$$\begin{aligned}\|Y(t) - \hat{Y}(t)\|_2 &\leq \left\| \int_0^\eta [h_2(s, \hat{Y}(s)) - h_2(s, Y(s))] ds \right\|_2 + \left\| \int_0^t [g_2(s, u(s)) - g_2(s, \hat{u}(s))] dW(s) \right\|_2 \\ &\leq cT \|Y - \hat{Y}\|_C + b\sqrt{T} \delta_2 = \epsilon.\end{aligned}$$

Now

$$\|Y - \hat{Y}\|_C \leq \frac{bT \delta_2}{(1 - cT)} = \epsilon \quad (21)$$

which is complete the proof. $\square$

4. Solutions of the Problem (1)–(2) and (4)

Let $\Lambda = C(I, L_2(\Omega)) \times C(I, L_2(\Omega))$ be the set of ordered pairs (X, Y) , $X, Y \in C$ and

$$\|(X, Y)\|_\Lambda = \|X\|_C + \|Y\|_C. \quad (22)$$

Define the mapping $P(X, Y) = (P_1 Y, P_2 X)$ where $P_1 Y, P_2 X$ are given by

$$P_1 Y(t) = X_0 - \int_0^\tau h_1(s, Y(s)) dW(s) + \int_0^t g_1(s, Y(\phi(s))) ds, \quad (23)$$

$$P_2 X(t) = Y_0 - \int_0^\eta h_2(s, X(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s). \quad (24)$$

Consider the set Q ,

$$Q = \{(X, Y) \in L_2(\Omega), (X, Y) \in \Lambda : \|(X, Y)\|_\Lambda = \|X(t)\|_2 + \|Y(t)\|_2 \leq r\}.$$

4.1. Existence Theorem

Theorem 6. Let $T \geq 1$ and (a1) – (a6) be satisfied, then (1)–(2) and (4) has a solution $(X, Y) \in \Lambda$.

Proof. Let $(X, Y) \in Q$, then we have

$$\begin{aligned}\|P_1 Y(t)\|_2 &\leq \|X_0\|_2 + \left\| \int_0^\eta h_1(s, Y(s)) dW(s) \right\|_2 + \left\| \int_0^t g_1(s, Y(\phi(s))) ds \right\|_2 \\ &\leq \|X_0\|_2 + (K + c\|Y\|_C) \sqrt{T} + (M + b\|Y\|_C) T \\ &\leq \|X_0\|_2 + MT + K\sqrt{T} + (c\sqrt{T} + bT) \|Y\|_C \\ &\leq \|X_0\|_2 + (M + K)T + (c + b)T \|Y\|_C,\end{aligned}$$

$$\begin{aligned}
\|P_2 X(t)\|_2 &\leq \|Y_0\|_2 + \left\| \int_0^t h_2(s, X(s)) ds \right\|_2 + \left\| \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right\|_2 \\
&\leq \|Y_0\|_2 + \int_0^t \|h_2(s, X(s))\|_2 ds + \sqrt{\int_0^t \|g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s))))\|_2^2 ds} \\
&\leq \|Y_0\|_2 + (K + c\|X\|_C)T + (M + \frac{bT^{1-\alpha}}{\Gamma(2-\alpha)})(M + b\|Y\|_C)\sqrt{T} \\
&\leq \|Y_0\|_2 + KT + M\sqrt{T} + \frac{bMT^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)} + \frac{b^2T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)}\|Y\|_C + Tc\|X\|_C \\
&\leq \|Y_0\|_2 + (K + M)T + MbT^2(2-\alpha) + b^2T^2(2-\alpha)\|Y\|_C + Tc\|X\|_C.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|P(X, Y)\|_X &= \|(P_1 Y, P_2 X)\|_\Lambda \\
&= \|P_1 Y\|_C + \|P_2 X\|_C \\
&\leq \|X_0\|_2 + \|Y_0\|_2 + 2(K + M)T + MbT^2(2-\alpha) \\
&\quad + [(c + b)T + b^2T^2(2-\alpha)]\{\|X\|_C + \|Y\|_C\} \\
&\leq \|(X_0, Y_0)\|_\Lambda + 2(K + M)T + MbT^2(2-\alpha) + [(c + b)T + b^2T^2(2-\alpha)]\|(X, Y)\|_\Lambda \\
&\leq \|(X_0, Y_0)\|_\Lambda + 2(K + M)T + MbT^2(2-\alpha) + [(c + b)T + b^2T^2(2-\alpha)]r = r
\end{aligned}$$

where

$$r = \frac{\|(X_0, Y_0)\|_\Lambda + 2(K + M)T + MbT^2(2-\alpha)}{1 - cT - bT - b^2T^2(2-\alpha)},$$

then the class $\{P(X, Y)(t), t = [0, T]\}$ is uniformly bounded and $P(x, y) : Q \rightarrow Q$. $\square$

Let $(X, Y) \in Q$, $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
\|P_1 Y(t_2) - P_1 Y(t_1)\|_2 &= \left\| \int_{t_1}^{t_2} g_1(s, Y(\phi(s))) ds \right\|_2 \\
&\leq (M + b\|Y\|_C)(t_2 - t_1),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
\|P_2 X(t_2) - P_2 X(t_1)\|_2 &= \left\| \int_{t_1}^{t_2} g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s) \right\|_2 \\
&\leq (M + b\|I^{1-\alpha} g_1(s, Y(\phi(s)))\|_2) \sqrt{(t_2 - t_1)}.
\end{aligned} \tag{26}$$

But

$$\begin{aligned}
P(X(t_2), Y(t_2)) - P(X(t_1), Y(t_1)) &= (P_1 Y(t_2), P_2 X(t_2)) - (P_1 Y(t_1), P_2 X(t_1)) \\
&= ((P_1 Y(t_2) - P_1 Y(t_1)), (P_2 X(t_2) - P_2 X(t_1))),
\end{aligned}$$

then from (25) and (26), $\{P(X, Y)(t), t \in [0, T]\}$ is equicontinuous on Q .

Let $(X_n, Y_n) \rightarrow (X, Y)$ w.p.1, $\{(X_n, Y_n)\} \in Q$ then applying stochastic Lebesgue dominated convergence Theorem [15], we can obtain

$$\begin{aligned}
L.i.m_{n \rightarrow \infty} P(X_n, Y_n) &= (L.i.m_{n \rightarrow \infty} P_1 Y_n, L.i.m_{n \rightarrow \infty} P_2 X_n) \\
&= (L.i.m_{n \rightarrow \infty} \{X_0 - \int_0^\tau h_1(s, Y_n(s)) dW(s) + \int_0^t g_1(s, Y_n(\phi(s))) ds\}, \\
&\quad L.i.m_{n \rightarrow \infty} \{Y_0 - \int_0^\eta h_2(s, X_n(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y_n(\phi(s)))) dW(s)\}) \\
&= (X_0 - \int_0^\tau h_1(s, L.i.m_{n \rightarrow \infty} Y_n(s)) dW(s) + \int_0^t g_1(s, L.i.m_{n \rightarrow \infty} Y_n(\phi(s))) ds, \\
&\quad Y_0 - \int_0^\eta h_2(s, L.i.m_{n \rightarrow \infty} X_n(s)) ds \\
&\quad + \int_0^t g_2(s, I^{1-\alpha} g_1(s, L.i.m_{n \rightarrow \infty} Y_n(\phi(s)))) dW(s)) \\
&= (X_0 - \int_0^\tau h_1(s, Y(s)) dW(s) + \int_0^t g_1(s, Y(\phi(s))) ds, \\
&\quad Y_0 - \int_0^\eta h_2(s, X(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s),) \\
&= (P_1 Y, P_2 X) = P(x, y).
\end{aligned}$$

This proves that $P : Q \rightarrow Q$ is continuous.

Then the closure of PQ is compact and (see [15]) the problem (1)–(2) and (4) has a solution $(X, Y) \in \Lambda$, $X, Y \in C(I, L_2(\Omega))$.

4.2. Uniqueness of the Solution

Theorem 7. Let $(a^*1) - (a^*2)$ and $(a3) - (a6)$ are satisfied. Then the solution of (1)–(2) is unique.

Proof. Let (X_1, Y_1) and (X_2, Y_2) be the solution of

$$\begin{aligned}
(x(t), y(t)) &= (X_0 - \int_0^\tau h_1(s, Y(s)) dW(s) + \int_0^t g_1(s, Y(\phi(s))) ds, \\
&\quad Y_0 - \int_0^\eta h_2(s, X(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, Y(\phi(s)))) dW(s)), \quad (27)
\end{aligned}$$

then we can get

$$\begin{aligned}
\|X_1(t) - X_2(t)\|_2 &\leq c\sqrt{T}\|Y_1 - Y_2\|_C + bT\|Y_1 - Y_2\|_C \\
&\leq (c\sqrt{T} + bT)\|Y_1 - Y_2\|_C \\
&\leq (c + b)T\{\|X_1 - X_2\|_C + \|Y_1 - Y_2\|_C\}. \quad (28)
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\|Y_1(t) - Y_2(t)\|_2 &\leq cT\|X_1 - X_2\|_C + \frac{b^2 T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)} \|y_1 - y_2\|_C \\
&\leq cT\|X_1 - X_2\|_C + b^2 T^2 (2-\alpha) \|y_1 - y_2\|_C \\
&\leq (cT + b^2 T^2 (2-\alpha)) \{\|X_1 - X_2\|_C + \|Y_1 - Y_2\|_C\}. \quad (29)
\end{aligned}$$

Hence from (28) and (29)

$$\begin{aligned}
\|(X_1, Y_1) - (X_2, Y_2)\|_\Lambda &= \|(X_1 - X_2)\|_C + \|(Y_1 - Y_2)\|_C \\
&\leq ((b + c)T + b^2 T^2 (2-\alpha)) \{\|X_1 - X_2\|_C + \|Y_1 - Y_2\|_C\} \\
&\leq ((b + c)T + b^2 T^2 (2-\alpha)) \|(X_1, Y_1) - (X_2, Y_2)\|_\Lambda.
\end{aligned}$$

This implies that

$$(1 - (b + c)T - b^2T^2(2 - \alpha))\|(X_1, Y_1) - (X_2, Y_2)\|_\Lambda \leq 0.$$

then

$$\|(X_1, Y_1) - (X_2, Y_2)\|_\Lambda = 0$$

and $(X_1, Y_1) = (X_2, Y_2)$. $\square$

4.3. Continuous Dependence

Theorem 8. The solution of the problem (1)–(2) and (4) is continuously dependent on (X_0, Y_0) .

Proof. Let $(\hat{X}, \hat{Y})$ be the solution of (1)–(2) and (4)

$$\begin{aligned}\hat{X}(t) &= \hat{X}_0 - \int_0^t h_1(s, \hat{Y}(s))dW(s) + \int_0^t g_1(s, \hat{Y}(\phi(s)))ds, \\ \hat{Y}(t) &= \hat{Y}_0 - \int_0^t h_2(s, \hat{X}(s))ds + \int_0^t g_2(s, I^{1-\alpha}g_1(s, \hat{Y}(\phi(s))))dW(s)\end{aligned}$$

such that $\|(X_0, Y_0) - (\hat{X}_0, \hat{Y}_0)\|_\Lambda < \delta_3$. Then we have

$$\begin{aligned}X(t) - \hat{X}(t) &= X_0 - \hat{X}_0 - \int_0^t [h_1(s, \hat{Y}(s)) - h_1(s, Y(s))]dW(s) \\ &\quad + \int_0^t [g_1(s, Y(\phi(s))) - g_1(s, \hat{Y}(\phi(s)))]ds\end{aligned}$$

and

$$\begin{aligned}\|X(t) - \hat{X}(t)\|_2 &\leq \|X_0 - \hat{X}_0\|_C + c\sqrt{T}\|Y - \hat{Y}\|_C + bT\|Y - \hat{Y}\|_C \\ &\leq \|X_0 - \hat{X}_0\|_2 + (b + c)T\{\|X - \hat{X}\|_C + \|Y - \hat{Y}\|_C\}.\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\|Y(t) - \hat{Y}(t)\|_2 &\leq \|Y_0 - \hat{Y}_0\|_2 + cT\|X - \hat{X}\|_C + b^2T^2(2 - \alpha)\|Y - \hat{Y}\|_C \\ &\leq \|Y_0 - \hat{Y}_0\|_2 + (cT + b^2T^2(2 - \alpha))\{\|X - \hat{X}\|_C + \|Y - \hat{Y}\|_C\}\end{aligned}$$

and

$$\begin{aligned}\|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda &= \|X - \hat{X}\|_C + \|Y - \hat{Y}\|_C \\ &\leq \|X_0 - \hat{X}_0\|_2 + \|Y_0 - \hat{Y}_0\|_2 \\ &\quad + ((b + c)T + b^2T^2(2 - \alpha))\{\|X - \hat{X}\|_C + \|Y - \hat{Y}\|_C\} \\ &\leq \|(X_0, Y_0) - (\hat{X}_0, \hat{Y}_0)\|_\Lambda \\ &\quad + ((b + c)T + b^2T^2(2 - \alpha))\|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda.\end{aligned}$$

Hence,

$$\|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda \leq \frac{\delta_3}{1 - T(b + c) - b^2T^2(2 - \alpha)} = \epsilon.$$

$\square$

Theorem 9. The solution of (1)–(2) and (4) depends continuously on h_1 and h_2 .

Proof. Let $(\hat{X}, \hat{Y})$ be the solution of (1)–(2) and (4) such that

$$\begin{aligned}\hat{X}(t) &= X_0 - \int_0^t h_1^*(s, \hat{Y}(s)) dW(s) + \int_0^t g_1(s, \hat{Y}(\phi(s))) ds, \\ \hat{Y}(t) &= Y_0 - \int_0^t h_2^*(s, \hat{X}(s)) ds + \int_0^t g_2(s, I^{1-\alpha} g_1(s, \hat{Y}(\phi(s)))) dW(s).\end{aligned}$$

Let $\|h_i^*(t, u(t)) - h(t, u(t))\|_2 \leq \delta_4$, $i = 1, 2$ then

$$\begin{aligned}\|X(t) - \hat{X}(t)\|_2 &= \left\| \int_0^t [h_1^*(s, \hat{Y}(s)) - h_1(s, Y(s))] dW(s) + \int_0^t [g_1(s, Y(\phi(s))) - g_1(s, \hat{Y}(\phi(s)))] ds \right\|_2 \\ &\leq \sqrt{\int_0^t \|h_1^*(s, \hat{Y}(s)) - h_1(s, Y(s))\|_2^2 ds} + \int_0^t \|g_1(s, Y(\phi(s))) - g_1(s, \hat{Y}(\phi(s)))\|_2 ds \\ &\leq \sqrt{\int_0^t [\|h_1^*(s, \hat{Y}(s)) - h_1^*(s, Y(s))\|_2 + \|h_1^*(s, Y(s)) - h_1(s, Y(s))\|_2]^2 ds} \\ &\quad + \int_0^t \|g_1(s, Y(\phi(s))) - g_1(s, \hat{Y}(\phi(s)))\|_2 ds \\ &\leq \sqrt{\int_0^t (c\|Y(s) - \hat{Y}(s)\|_2 + \delta_4)^2 ds} + \int_0^t b\|Y(s) - \hat{Y}(s)\|_2 ds \\ &\leq (c + b)T\|Y - \hat{Y}\|_C + \delta_4 T.\end{aligned}$$

Similarly we can get

$$\begin{aligned}\|Y - \hat{Y}\|_C &\leq cT\|X - \hat{X}\|_C + \frac{b^2 T^{\frac{3}{2}-\alpha}}{\Gamma(2-\alpha)} \|Y - \hat{Y}\|_C + \delta_4 T \\ &\leq cT\|X - \hat{X}\|_C + b^2 T^2 (2-\alpha) \|Y - \hat{Y}\|_C + \delta_4 T\end{aligned}$$

and

$$\begin{aligned}\|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda &= \|(X - \hat{X})\|_C + \|(Y - \hat{Y})\|_C \\ &\leq 2\delta_4 T + cT\|X - \hat{X}\|_C + (cT + bT + b^2 T^2 (2-\alpha)) \|Y - \hat{Y}\|_C \\ &\leq 2\delta_4 T + [(b+c)T + b^2 T^2 (2-\alpha)] \{ \|Y - \hat{Y}\|_C + \|X - \hat{X}\|_C \} \\ &\leq 2\delta_4 T + [(b+c)T + b^2 T^2 (2-\alpha)] \|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda.\end{aligned}$$

This implies that

$$\|(X, Y) - (\hat{X}, \hat{Y})\|_\Lambda \leq \frac{2\delta_4 T}{1 - (b+c)T - b^2 T^2 (2-\alpha)} = \epsilon$$

which completes the proof. $\square$

Theorem 10. The solution (X, Y) of (1)–(2) and (4) depends continuously on $u(t) = D^\alpha X(t)$, $\alpha \in (0, 1]$.

Proof. Let $(\hat{X}, \hat{Y})$ be the solution of

$$\begin{aligned}\hat{X}(t) &= X_0 - \int_0^t h_1(s, \hat{Y}(s)) dW(s) + I^\alpha \hat{u}(t), \\ \hat{Y}(t) &= Y_0 - \int_0^t h_2(s, \hat{X}(s)) ds + \int_0^t g_2(s, \hat{u}(s)) dW(s)\end{aligned}$$

such that $\|\hat{u}(t) - u(t)\|_2 \leq \delta_5$, then

$$\begin{aligned}
\|(X, Y) - (\hat{X}, \hat{Y})\|_{\Lambda} &= \|(X - \hat{X})\|_C + \|(Y - \hat{Y})\|_C \\
&\leq \left\| \int_0^{\eta} [h_1(s, \hat{Y}(s)) - h_1(s, Y(s))] dW(s) \right\|_2 + \|I^{\alpha} u(t) - I^{\alpha} \hat{u}(t)\|_2 \\
&+ \left\| \int_0^{\eta} [h_2(s, \hat{X}(s)) - h_2(s, X(s))] ds \right\|_2 + \left\| \int_0^t [g_2(s, u(s)) - g_2(s, \hat{u}(s))] dW(s) \right\|_2 \\
&\leq c\sqrt{T}\|Y - \hat{Y}\|_C + \frac{T^{\alpha}\delta_5}{\Gamma(\alpha+1)} + cT\|X - \hat{X}\|_C + b\sqrt{T}\delta_5 \\
&\leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + bT \right) \delta_5 + cT\|(X, Y) - (\hat{X}, \hat{Y})\|_{\Lambda}.
\end{aligned}$$

Now

$$\|(X, Y) - (\hat{X}, \hat{Y})\|_{\Lambda} \leq \frac{\delta_5}{1 - cT} \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + bT \right) \quad (30)$$

which completes the proof. $\square$

5. Example

Consider the coupled system

$$\begin{aligned}
\frac{dX}{dt}(t) &= \frac{1 - t^2 + Y(t)}{30(1 + \|Y(t)\|_2)}, \quad t \in (0, 1], \\
dY(t) &= \frac{e^{-t} + D^{\frac{1}{2}}X(t)}{120(1 + \|X(t)\|_2)} dW(t), \quad t \in (0, 1]
\end{aligned} \quad (31)$$

subject to

$$X_0 = \int_0^{\tau} \frac{sX(s)}{80 + s^2} dW(s), \quad Y_0 = \int_0^{\eta} \frac{Y(s)}{\sqrt{s+36}} ds \quad (32)$$

where

$$\|g_1(t, Y(t))\|_2 \leq \frac{1}{30}\|Y(t)\|_2, \quad \|g_2(t, X(t))\|_2 \leq \frac{1}{120}[1 + \|X(t)\|_2]$$

and

$$\|h_1(t, X(t))\|_2 \leq \frac{\|X(t)\|_2}{80}, \quad \|h_2(t, Y(t))\|_2 \leq \frac{\|Y(t)\|_2}{6}.$$

Easily, the coupled system (31) with nonlocal integral conditions (32) satisfies all the assumptions (a1)–(a6) of Theorem 1. with $b = \frac{1}{30}$, $c = \frac{1}{6}$.

6. Conclusions

Here, we have combined between the three senses of derivatives, the stochastic Ito-differential and the fractional and integer orders derivatives for the second order stochastic process in two non-local problems of the coupled system of the two random and stochastic differential Equations (1) and (2) with the stochastic and random integral conditions (3) and the coupled system of the two stochastic and random nonlocal integral conditions (4) where X_0 and Y_0 are two second order random variables.

The existence of solutions $X, Y \in C(I, L_2(\Omega))$ of the problems (1)–(3) and (1)–(2) and (4) are proved. The unique solution and the sufficient conditions are discussed. The continuous dependence of the solution $X, Y \in C(I, L_2(\Omega))$ on the two random variables X_0 and Y_0 and on the two random functions h_1 and h_2 and the continuous dependence of the solution $Y \in C(I, L_2(\Omega))$ on the fractional order derivative $D^{\alpha}X(t)$ are studied. An example is given.

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