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Oscillation and Asymptotic Properties of Second Order Half-Linear Differential Equations with Mixed Deviating Arguments

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Abstract: In this paper, we study oscillation and asymptotic properties for half-linear second order differential equations with mixed argument of the form $(r(t)(y'(t))^\alpha)' = p(t)y^\alpha(\tau(t))$. Such differential equation may possess two types of nonoscillatory solutions either from the class $\mathcal{N}_0$ (positive decreasing solutions) or $\mathcal{N}_2$ (positive increasing solutions). We establish new criteria for $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_2 = \emptyset$ provided that delayed and advanced parts of deviating argument are large enough. As a consequence of these results, we provide new oscillatory criteria. The presented results essentially improve existing ones even for a linear case of considered equations.

Keywords: second order differential equations; delay; advanced; mixed argument; monotonic properties; oscillation

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1. Introduction

We consider the half-linear second order differential equations with mixed deviating argument of the following form.

$$(r(t)(y'(t))^\alpha)' = p(t)y^\alpha(\tau(t)). \quad (1)$$

Throughout this paper, it is assumed that the following is the case:

- (H_1) $p, r \in C([t_0, \infty))$, $p(t) > 0$, $r(t) > 0$, α is the ratio of two positive odd integers;
 (H_2) $\tau(t) \in C([t_0, \infty))$, $\tau'(t) > 0$; $\lim_{t \rightarrow \infty} \tau(t) = \infty$.

By a proper solution of Equation (1), we mean a function $y : [T_y, \infty) \rightarrow \mathbb{R}$ that satisfies Equation (1) for all sufficiently large t and $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_y$. We make the standing hypothesis that (1) does possess proper solutions. A proper solution is called oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. An equation itself is said to be oscillatory if all its proper solutions are oscillatory. There are numerous papers devoted to oscillation theory of differential equations, see, e.g., [1–12].

It is known (see, e.g., [7]) that if $y(t)$ is a nonoscillatory solution of (1), then eventually either:

$$y(t)r(t)(y'(t))^\alpha < 0 \quad \text{and} \quad y(t)(r(t)(y'(t))^\alpha)' > 0$$

and we say that $y(t)$ is of degree 0, and we denote the set of such solutions by $\mathcal{N}_0$ or

$$y(t)r(t)(y'(t))^\alpha > 0 \quad \text{and} \quad y(t)(r(t)(y'(t))^\alpha)' > 0,$$

and we say that $y(t)$ is of degree two, and we denote the corresponding set by $\mathcal{N}_2$.

Consequently, the set $\mathcal{N}$ of all nonoscillatory solutions of (1) has the following decomposition.

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

The first aim of this paper is to establish criteria for $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_2 = \emptyset$. This problem has been solved by many authors, and we mention here the pioneering works of Ladas et al. [12] and Koplatadze and Chanturija [9]; in general, authors discuss the condition for $\mathcal{N}_0 = \emptyset$ only when the deviating argument is delayed ($\tau(t) < t$) and criteria for $\mathcal{N}_2 = \emptyset$ only for advanced arguments ($\tau(t) > t$), (see [1–12]).

In this paper, we establish the desired criteria when deviating argument $\tau(t)$ is of a mixed type, which means that its delayed part:

$$\mathcal{D}_\tau = \{t \in (t_0, \infty) : \tau(t) < t\}$$

and its advanced part

$$\mathcal{A}_\tau = \{t \in (t_0, \infty) : \tau(t) > t\}$$

are both unbounded subsets of (t_0, ∞) .

The second aim of this paper is to join the criteria obtained for $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_2 = \emptyset$ to establish the oscillation of (1).

Our basic results will be formulated for general Equation (1), i.e., without additional conditions imposed on function $r(t)$. Then, we provide significant improvements for two partial cases, namely when (1) is in either canonical form, that is, when it has the following form.

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} ds = \infty \quad (2)$$

When this situation occurs we employ the following function:

$$R(t) = \int_{t_0}^t \frac{1}{r^{1/\alpha}(s)} ds$$

or in noncanonical form (opposite case) when the following is the case.

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} ds < \infty \quad (3)$$

In this case, we shall use the auxiliary function of the following form.

$$\rho(t) = \int_t^{\infty} \frac{1}{r^{1/\alpha}(s)} ds.$$

Thus, our results are of high generality and, what is more, they hold for all $\alpha > 0$, and our technique does not require discussing cases $\alpha \in (0, 1)$ and $\alpha > 1$ separately as it is common, see [1–10].

2. Materials and Methods

We have used the methods of mathematical analysis.

3. Basic Results

Our first result is applicable to both canonical and noncanonical equations. In all our results, we employ two sequences $\{t_k\}$ and $\{s_k\}$ such that the following is the case:

$$t_k \in \mathcal{D}_\tau, \quad t_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty \quad (4)$$

and we have the following.

$$s_k \in \mathcal{A}_\tau, \quad s_k \rightarrow \infty \quad \text{as} \quad k \rightarrow \infty. \quad (5)$$

Theorem 1. Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ satisfying (4) and (5), respectively. If the following is the case:

$$\limsup_{k \rightarrow \infty} \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} p(s) ds \right)^{1/\alpha} du > 1, \quad (6)$$

and

$$\limsup_{k \rightarrow \infty} \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left(\int_{s_k}^u p(s) ds \right)^{1/\alpha} du > 1, \quad (7)$$

then, (1) is oscillatory.

Proof. Assume on the contrary, that (1) possesses an eventually positive solution $y(t)$. Then, either $y(t) \in \mathcal{N}_0$ or $y(t) \in \mathcal{N}_2$. We shall show that (6) and (7) imply $\mathcal{N}_0 = \emptyset$ and $\mathcal{N}_2 = \emptyset$, respectively.

At first, we admit that $y(t) \in \mathcal{N}_0$. We remark that since $\tau(t)$ is increasing, then $t_k \in \mathcal{D}_\tau$ implies $(\tau(t_k), t_k) \subset \mathcal{D}_\tau$. Let $u \in (\tau(t_k), t_k)$ for some $k \in \{1, 2, \dots\}$. Using the fact that $y^\alpha(t)$ is decreasing, an integration of (1) from u to t_k yields the following.

$$-r(u)(y'(u))^\alpha \geq \int_u^{t_k} p(s)y^\alpha(\tau(s)) ds \geq y^\alpha(\tau(t_k)) \int_u^{t_k} p(s) ds.$$

Extracting the α root and integrating from $\tau(t_k)$ to t_k , we are led to the following:

$$y(\tau(t_k)) \geq y(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} p(s) ds \right)^{1/\alpha} du$$

which contradicts the condition (6), and we conclude that class $\mathcal{N}_0 = \emptyset$.

Now, we assume that $y(t) \in \mathcal{N}_2$. Then, $y(t)$ is increasing. It is useful to notice that since $\tau(t)$ is increasing, it follows from $s_k \in \mathcal{A}_\tau$ that $(s_k, \tau(s_k)) \subset \mathcal{A}_\tau$. Let $u \in (s_k, \tau(s_k))$ for some $k \in \{1, 2, \dots\}$. By integrating (1) from s_k to u , one obtains the following.

$$r(u)(y'(u))^\alpha \geq \int_{s_k}^u p(t)y^\alpha(\tau(t)) dt \geq y^\alpha(\tau(s_k)) \int_{s_k}^u p(t) dt.$$

An integration of the last inequality from s_k to $\tau(s_k)$ provides the following:

$$y(\tau(s_k)) \geq y(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left(\int_{s_k}^u p(s) ds \right)^{1/\alpha} du$$

which contradicts (7) and so $\mathcal{N}_2 = \emptyset$, and the proof is complete. $\square$

Theorem 1 extends the corresponding result of Kusano [11] formulated for $r(t) \equiv 1$. For the linear case of (1), namely when $\alpha = 1$, we can change the order of integration in (6) and (7), which essentially simplifies evaluation of these criteria.

Corollary 1. Let $\alpha = 1$ and (2) hold. Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ satisfying (4) and (5). If the following is the case:

$$\limsup_{k \rightarrow \infty} \int_{\tau(t_k)}^{t_k} p(s)(R(s) - R(\tau(t_k))) ds > 1 \quad (8)$$

and

$$\limsup_{k \rightarrow \infty} \int_{s_k}^{\tau(s_k)} p(t)(R(\tau(s_k)) - R(t)) dt > 1, \quad (9)$$

then, (1) is oscillatory.

Corollary 2. Let $\alpha = 1$ and (3) hold. Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ satisfying (4) and (5). If the following is the case:

$$\limsup_{k \rightarrow \infty} \int_{\tau(t_k)}^{t_k} p(s)(\rho(\tau(t_k)) - \rho(s)) \, ds > 1 \quad (10)$$

and

$$\limsup_{k \rightarrow \infty} \int_{s_k}^{\tau(s_k)} p(t)(\rho(t) - \rho(\tau(s_k))) \, dt > 1, \quad (11)$$

then, (1) is oscillatory.

To illustrate the above mentioned criteria, we provide the following couple of examples.

Example 1. We consider the second order linear functional differential equation in the canonical form.

$$\left(t^{1/2}y'(t)\right)' = \frac{a}{t^{3/2}}y(t(1 + 0.5\sin(\ln t))), \quad a > 0. \quad (12)$$

Clearly, the deviating argument $\tau(t) = t(1 + 0.5\sin(\ln t))$ is of mixed type such that $t/2 \leq \tau(t) \leq 3t/2$ and $R(t) = 2\sqrt{t}$.

We place $t_k = e^{3\pi/2+2k\pi}$, $k = 1, 2, \dots$. Then, $t_k \in \mathcal{D}_\tau$ and $\tau(t_k) = 0.5e^{3\pi/2+2k\pi}$. Condition (8) takes the following form:

$$\begin{aligned} \lim_{k \rightarrow \infty} 2a \int_{\tau(t_k)}^{t_k} \left(s^{-1} - \sqrt{0.5e^{(3\pi/2)+2k\pi}} s^{-3/2}\right) \, ds = \\ 2a \left(\ln 2 - 2\left(1 - \frac{1}{\sqrt{2}}\right)\right) > 1 \end{aligned}$$

which means that for $a > 4.65722$, the class $\mathcal{N}_0 = \emptyset$ for (12).

On the other hand, if we set $s_k = e^{\pi/2+2k\pi}$, $k = 1, 2, \dots$, then $s_k \in \mathcal{A}_\tau$ and, moreover, $\tau(s_k) = 1.5e^{\pi/2+2k\pi}$. Condition (9) reduces to the following:

$$\begin{aligned} \lim_{k \rightarrow \infty} 2a \int_{s_k}^{\tau(s_k)} \left(\sqrt{1.5e^{\pi/2+2k\pi}} t^{-3/2} - t^{-1}\right) \, dt = \\ 2a \left(2\left(\sqrt{1.5} - 1\right) - \ln 1.5\right) > 1 \end{aligned}$$

which ensures that $\mathcal{N}_2 = \emptyset$ provided that $a > 11.3573$. Picking up both criteria, we observe that the following condition:

$$a > 11.3573$$

implies oscillation of (12).

Example 2. We consider the second order linear functional differential equation in the noncanonical form.

$$\left(t^2y'(t)\right)' = a y(t(1 + 0.5\sin(\ln t))), \quad a > 0. \quad (13)$$

It is easy to observe that $\rho(t) = 1/t$. We again substitute $t_k = e^{3\pi/2+2k\pi}$ and $k \in \{1, 2, \dots\}$. Condition (10) takes the following form:

$$\lim_{k \rightarrow \infty} a \int_{\tau(t_k)}^{t_k} \left(\frac{1}{0.5e^{(3\pi/2)+2k\pi}} - \frac{1}{s}\right) \, ds = a(1 - \ln 2) > 1$$

which means that for $a > 3.25892$, the class $\mathcal{N}_0 = \emptyset$.

To eliminate class $\mathcal{N}_2$, we set $s_k = e^{\pi/2+2k\pi}$, $k = 1, 2, \dots$. Condition (11) simplifies to the following:

$$\lim_{k \rightarrow \infty} a \int_{s_k}^{\tau(s_k)} \left(\frac{1}{s} - \frac{2}{3e^{\pi/2+2k\pi}} \right) ds = a \left(\ln \frac{3}{2} - \frac{1}{3} \right) > 1$$

which ensures that $\mathcal{N}_2 = \emptyset$ provided that $a > 13.86352$. Therefore, the following condition guarantees oscillation of (13).

$$a > 13.86352$$

In the next two sections, we essentially improve conditions (6)–(11) for eliminations of classes $\mathcal{N}_0$ and $\mathcal{N}_2$. To achieve our goals, it is necessary to study canonical and noncanonical equations separately.

4. Canonical Equations

We establish new monotonic properties of possible nonoscillatory solutions and then apply them to improve the above mentioned criteria. The progress will be presented via Equation (12). In the first part, we focus our considerations to eliminate class $\mathcal{N}_0$.

Lemma 1. Let (2) hold. Assume that there exist a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_c such that for $k \in \{1, 2, \dots\}$, we have the following.

$$R(\tau(s)) \left(\int_{\tau(s)}^s p(t) dt \right)^{1/\alpha} \geq \gamma_c \quad \text{on} \quad (\tau(t_k), t_k). \quad (14)$$

If $y(s)$ is a positive solution of (1) such that $y(s) \in \mathcal{N}_0$, then the following is the case.

$$R^{\gamma_c}(\tau(s))y(\tau(s)) \downarrow \quad \text{on} \quad (\tau(t_k), t_k), \quad k = 1, 2, \dots$$

Proof. Assume that $s \in (\tau(t_k), t_k)$ for some $k \in \{1, 2, \dots\}$. Since $y^\alpha(t)$ is decreasing, an integration of (1) from $\tau(s)$ to s yields the following.

$$\begin{aligned} -r^{1/\alpha}(\tau(s))y'(\tau(s)) &\geq \left(\int_{\tau(s)}^s p(t)y^\alpha(\tau(t)) dt \right)^{1/\alpha} \\ &\geq y(\tau(s)) \left(\int_{\tau(s)}^s p(t) dt \right)^{1/\alpha}. \end{aligned} \quad (15)$$

It is easy to see that the last inequality, in view of (14), implies the following.

$$-r^{1/\alpha}(\tau(s))R(\tau(s))y'(\tau(s)) \geq \gamma_c y(\tau(s)).$$

Consequently, we have the following:

$$\begin{aligned} (R^{\gamma_c}(\tau(s))y(\tau(s)))' &= \\ \frac{\tau'(s)R^{\gamma_c-1}(\tau(s))}{r^{1/\alpha}(\tau(s))} \left(\gamma_c y(\tau(s)) + R(\tau(s))r^{1/\alpha}(\tau(s))y'(\tau(s)) \right) &\leq 0 \end{aligned}$$

on $(\tau(t_k), t_k)$ and $k = 1, 2, \dots$, and the proof is complete. $\square$

Theorem 2. Let (2) hold. Assume that there exists a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_c such that (14) holds. If the following is the case:

$$\limsup_{k \rightarrow \infty} R^{\gamma_c}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} \frac{p(s)}{R^{\alpha\gamma_c}(\tau(s))} ds \right)^{1/\alpha} du > 1, \quad (16)$$

then, the class $\mathcal{N}_0 = \emptyset$ for (1) is the case.

Proof. Assume on the contrary that (1) possesses an eventually positive solution $y(t) \in \mathcal{N}_0$. Let $u \in (\tau(t_k), t_k)$ for some $k \in \{1, 2, \dots\}$. Using the fact that $R^{\alpha\gamma_c}(\tau(s))y^\alpha(\tau(s))$ is decreasing on $(\tau(t_k), t_k)$, an integration of (1) from u to t_k yields the following.

$$\begin{aligned} -(y'(u))^\alpha &\geq \frac{1}{r(u)} \int_u^{t_k} p(s)y^\alpha(\tau(s)) \, ds \\ &\geq \frac{R^{\alpha\gamma_c}(\tau(t_k))y^\alpha(\tau(t_k))}{r(u)} \int_u^{t_k} \frac{p(s)}{R^{\alpha\gamma_c}(\tau(s))} \, ds. \end{aligned}$$

Extracting the α root and integrating from $\tau(t_k)$ to t_k , one obtains the following.

$$y(\tau(t_k)) \geq y(\tau(t_k))R^{\gamma_c}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} \frac{p(s)}{R^{\alpha\gamma_c}(\tau(s))} \, ds \right)^{1/\alpha} du.$$

This is a contradiction, and the proof is now complete. $\square$

For $\alpha = 1$, condition (16) can be significantly simplified.

Corollary 3. Let $\alpha = 1$ and (2) hold. Assume that there exists a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_c such that (14) holds. If the following is the case:

$$\limsup_{k \rightarrow \infty} R^{\gamma_c}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{p(s)}{R^{\gamma_c}(\tau(s))} (R(s) - R(\tau(t_k))) \, ds > 1, \quad (17)$$

then the class $\mathcal{N}_0 = \emptyset$ for (1).

Now, we turn our attention to the class $\mathcal{N}_2$.

Lemma 2. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_c such that for $k \in \{1, 2, \dots\}$, the following is the case.

$$R(\tau(u)) \left(\int_u^{\tau(u)} p(t) \, dt \right)^{1/\alpha} \geq \delta_c \quad \text{on } (s_k, \tau(s_k)). \quad (18)$$

If $y(u)$ is a positive solution of (1) such that $y(u) \in \mathcal{N}_2$, then the following is the case.

$$R^{-\delta_c}(\tau(u))y(\tau(u)) \uparrow \quad \text{on } (s_k, \tau(s_k)), \quad k = 1, 2, \dots$$

Proof. Assume that $u \in (s_k, \tau(s_k))$ for some $k \in \{1, 2, \dots\}$. Taking into account that $y^\alpha(t)$ is increasing, an integration of (1) from u to $\tau(u)$ produces the following.

$$r^{1/\alpha}(\tau(u))y'(\tau(u)) \geq \left(\int_u^{\tau(u)} p(t)y^\alpha(\tau(t)) \, dt \right)^{1/\alpha} \geq y(\tau(u)) \left(\int_{\tau(s)}^s p(t) \, dt \right)^{1/\alpha}.$$

By (18), the last inequality implies the following.

$$r^{1/\alpha}(\tau(u))R(\tau(u))y'(\tau(u)) \geq \delta_c y(\tau(u)).$$

Therefore, we have the following:

$$\begin{aligned} \left(R^{-\delta_c}(\tau(u))y(\tau(u)) \right)' &= \\ \frac{\tau'(u)R^{-\delta_c-1}(\tau(u))}{r^{1/\alpha}(\tau(u))} \left(R(\tau(u))r^{1/\alpha}(\tau(u))y'(\tau(u)) - \delta_c y(\tau(u)) \right) &\geq 0 \end{aligned}$$

on $(s_k, \tau(s_k))$ and $k = 1, 2, \dots$, and the proof is complete. $\square$

Theorem 3. Let (2) hold. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_c satisfying (18). If the following is the case:

$$\limsup_{k \rightarrow \infty} R^{-\delta_c}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left(\int_{s_k}^u p(t) R^{\alpha\delta_c}(\tau(t)) dt \right)^{1/\alpha} du > 1, \quad (19)$$

then the class $\mathcal{N}_2 = \emptyset$ for (1).

Proof. Let us admit that (1) possesses an eventually positive solution $y(t) \in \mathcal{N}_2$. Assume that $u \in (s_k, \tau(s_k))$ for some $k \in \{1, 2, \dots\}$. Employing the fact that $R^{-\alpha\delta_c}(\tau(t))y^\alpha(\tau(t))$ is increasing on $(s_k, \tau(s_k))$ and integrating (1) from s_k to u , we obtain the following.

$$\begin{aligned} (y'(u))^\alpha &\geq \frac{1}{r(u)} \int_{s_k}^u p(t) y^\alpha(\tau(t)) dt \\ &\geq \frac{R^{-\alpha\delta_c}(\tau(s_k)) y^\alpha(\tau(s_k))}{r(u)} \int_{s_k}^u p(t) R^{\alpha\delta_c}(\tau(t)) dt. \end{aligned}$$

By extracting the α root and integrating from s_k to $\tau(s_k)$, we observe that the following is the case.

$$y(\tau(s_k)) \geq y(\tau(s_k)) R^{-\delta_c}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{\tau(s_k)} p(t) R^{\alpha\delta_c}(\tau(t)) dt \right)^{1/\alpha} du.$$

This is a contradiction, and the proof is complete now. $\square$

Corollary 4. Let $\alpha = 1$ and (2) hold. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_c satisfying (18). If the following is the case:

$$\limsup_{k \rightarrow \infty} R^{-\delta_c}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} p(t) R^{\delta_c}(\tau(t)) (R(\tau(st_k)) - R(t)) dt > 1, \quad (20)$$

then the class $\mathcal{N}_2 = \emptyset$ for (1).

By picking up the above results, we are prepared to formulate the improvement of Theorem 1 provided that (1) is a canonical form.

Theorem 4. Let (2) hold. Assume that there exist two sequences $\{t_k\}$, $\{s_k\}$ satisfying (4) and (5) and there exist positive constants γ_c and δ_c for which (14) and (18) hold. If (16) and (19) are satisfied, then (1) is oscillatory.

Note that if $\gamma_c = \delta_c = 0$, Theorem 4 reduces to Theorem 1. In the opposite case, the progress that Theorem 4 yields will be demonstrated by means of Equation (12).

Example 3. We consider again the following differential equation.

$$\left(t^{1/2} y'(t) \right)' = \frac{a}{t^{3/2}} y(t(1 + 0.5 \sin(\ln t))), \quad a > 0.$$

At first, we shall show that $\mathcal{N}_0 = \emptyset$ for $a \geq 3.419701$. Thus, we set $a = 3.419701$. Substituting again $t_k = e^{3\pi/2 + 2k\pi}$ and $k = 1, 2, \dots$, we observe that $\tau(t_k) = 0.5e^{3\pi/2 + 2k\pi}$. Consequently, Condition (14) reduces to the following.

$$4a \left(1 - \sqrt{1 + 0.5 \sin(\ln t)} \right) \geq \gamma_c \quad \text{on} \quad (\tau(t_k), t_k), \quad k = 1, 2, \dots$$

Since $4a\left(1 - \sqrt{1 + 0.5 \sin(\ln t)}\right)$ is increasing on $(\tau(t_k), t_k)$, we have the following.

$$\gamma_c = 4a\left(1 - \sqrt{1 + 0.5 \sin(\ln \tau(t_k))}\right) = 4a\left(1 - \sqrt{1 - 0.5 \cos(\ln 2)}\right) = 2.9483028,$$

On the other hand, Criterion (17) takes the following form.

$$\limsup_{k \rightarrow \infty} 2^{\gamma_c} \left(0.5 e^{3\pi/2 + 2k\pi}\right)^{\frac{\gamma_c}{2}} \int_{\tau(t_k)}^{t_k} a s^{-\frac{3}{2}} 2^{-\gamma_c} (s(1 + 0.5 \sin \ln s))^{\frac{-\gamma_c}{2}} \left(2s^{\frac{1}{2}} - 2\left(0.5 e^{3\pi/2 + 2k\pi}\right)^{\frac{1}{2}}\right) ds > 1.$$

Substituting $s = t e^{(3\pi/2) + 2k\pi}$ simplifies the above term into the following.

$$2a \left(\frac{1}{2}\right)^{\gamma_c/2} \int_{0.5}^1 t^{-(3+\gamma_c)/2} (1 - 0.5 \cos \ln t)^{-\gamma_c/2} (\sqrt{t} - \sqrt{0.5}) dt > 1$$

We used Matlab for evaluating (with $\gamma_c = 2.9483028$) the following.

$$\int_{0.5}^1 t^{-(3+\gamma_c)/2} (1 - 0.5 \cos \ln t)^{-\gamma_c/2} (\sqrt{t} - \sqrt{0.5}) dt = 0.406205505$$

Finally, we conclude that the following is the case:

$$\limsup_{k \rightarrow \infty} R^{\gamma_c}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} \frac{p(s)}{R^{\alpha\gamma_c}(\tau(s))} ds \right)^{1/\alpha} du = 1.00000035 > 1$$

which by Corollary 3 guarantees that $\mathcal{N}_0 = \emptyset$. We obtain essentially better results for value of a than it has been presented in Example 1.

We claim that $\mathcal{N}_2 = \emptyset$ for $a \geq 7.364929976$. To verify this, we let $a = 7.364929976$ and $s_k = e^{\pi/2 + 2k\pi}$ and $k = 1, 2, \dots$. Then, $\tau(s_k) = 1.5 e^{\pi/2 + 2k\pi}$. Equation (18) implies the following.

$$4a \left(\sqrt{1 + 0.5 \sin(\ln t)} - 1 \right) \geq \delta_c \quad \text{on each } (s_k, \tau(s_k)).$$

Consequently, we have the following.

$$\delta_c = 4a \left(\sqrt{1 + 0.5 \sin(\ln \tau(s_k))} - 1 \right) = 4a \left(\sqrt{1 + 0.5 \cos(\ln 1.5)} - 1 \right) = 6.1300057.$$

Condition (20) reduces to the following.

$$\limsup_{k \rightarrow \infty} 2^{-\delta_c} \left(1.5 e^{\pi/2 + 2k\pi}\right)^{\frac{-\delta_c}{2}} \int_{s_k}^{\tau(s_k)} a t^{-\frac{3}{2}} 2^{\delta_c} (t(1 + 0.5 \sin \ln t))^{\frac{\delta_c}{2}} \left(2\left(1.5 e^{\pi/2 + 2k\pi}\right)^{\frac{1}{2}} - 2t^{\frac{1}{2}}\right) dt > 1.$$

To simplify the last integral, we use the substitution $t = e^{\pi/2 + 2k\pi} x$, and we obtain the following.

$$2a \left(\frac{2}{3}\right)^{\delta_c/2} \int_1^{1.5} x^{(-3+\delta_c)/2} (1 + 0.5 \cos \ln x)^{\delta_c/2} (\sqrt{1.5} - \sqrt{x}) dx > 1$$

By employing Matlab, we find out that for $\delta_c = 6.1300057$, the following is the case.

$$\int_1^{1.5} x^{(-3+\delta_c)/2} (1 + 0.5 \cos \ln x)^{\delta_c/2} (\sqrt{1.5} - \sqrt{x}) dx = 0.23524565$$

Therefore, the following is the case:

$$\limsup_{k \rightarrow \infty} R^{-\delta_c}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} p(t) R^{\delta_c}(\tau(t)) (R(t) - R(\tau(t_k))) dt = 1.000000006 > 1$$

which by Corollary 4 implies that $\mathcal{N}_2 = \emptyset$. Again we obtain better results than in Example 1.

By combining both criteria, we observe that condition $a \geq 7.364929976$ implies oscillation of (12), while Theorem 1 requires $a > 13.8635$.

5. Noncanonical Equations

Now, we turn our attention to noncanonical equation. Similarly as in the previous section, we introduce new monotonic properties of nonoscillatory solutions and then apply them to improve criteria concerning noncanonical equations. The progress will be demonstrated via Equation (13).

Lemma 3. Let (3) hold. Assume that there exists a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_n such that for $k \in \{1, 2, \dots\}$, we have the following.

$$\rho(\tau(s)) \left(\int_{\tau(s)}^s p(t) dt \right)^{1/\alpha} \geq \gamma_n \quad \text{on} \quad (\tau(t_k), t_k). \quad (21)$$

If $y(s)$ is a positive solution of (1) such that $y(s) \in \mathcal{N}_0$, then the following is the case.

$$\rho^{-\gamma_c}(\tau(s)) y(\tau(s)) \downarrow \quad \text{on} \quad (\tau(t_k), t_k), \quad k = 1, 2, \dots$$

Proof. Assume that $s \in (\tau(t_k), t_k)$ for some $k \in \{1, 2, \dots\}$ and $y(t) \in \mathcal{N}_0$ is a solution of (1). Then, (15) implies the following.

$$-r^{1/\alpha}(\tau(s)) \rho(\tau(s)) y'(\tau(s)) \geq \gamma_n y(\tau(s)).$$

Consequently, we have the following:

$$\begin{aligned} (\rho^{-\gamma_c}(\tau(s)) y(\tau(s)))' &= \\ \frac{\tau'(s) \rho^{-\gamma_c-1}(\tau(s))}{r^{1/\alpha}(\tau(s))} \left(\gamma_c y(\tau(s)) + \rho(\tau(s)) r^{1/\alpha}(\tau(s)) y'(\tau(s)) \right) &\leq 0 \end{aligned}$$

on $(\tau(t_k), t_k)$ and $k = 1, 2, \dots$, and the proof is complete. $\square$

Theorem 5. Let (3) hold. Assume that there exists a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_n such that (21) holds. If the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{-\gamma_n}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} p(s) \rho^{\alpha\gamma_n}(\tau(s)) ds \right)^{1/\alpha} du > 1, \quad (22)$$

then the class $\mathcal{N}_0 = \emptyset$ for (1).

Proof. Assume that (1) has an eventually positive solution $y(t) \in \mathcal{N}_0$. Let $u \in (\tau(t_k), t_k)$ for some $k \in \{1, 2, \dots\}$. Employing that $\rho^{-\alpha\gamma_n}(\tau(s)) y^\alpha(\tau(s))$ is decreasing on $(\tau(t_k), t_k)$ and integrating (1) from u to t_k , one obtains the following.

$$\begin{aligned} -(y'(u))^\alpha &\geq \frac{1}{r(u)} \int_u^{t_k} p(s) y^\alpha(\tau(s)) ds \\ &\geq \frac{\rho^{-\alpha\gamma_n}(\tau(t_k)) y^\alpha(\tau(t_k))}{r(u)} \int_u^{t_k} p(s) \rho^{\alpha\gamma_n}(\tau(s)) ds. \end{aligned}$$

By extracting the α root and integrating $\tau(t_k)$ to t_k , we obtain the following.

$$y(\tau(t_k)) \geq y(\tau(t_k))\rho^{-\gamma_n}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} \frac{1}{r^{1/\alpha}(u)} \left(\int_u^{t_k} p(s)\rho^{\alpha\gamma_n}(\tau(s)) ds \right)^{1/\alpha} du.$$

This is a contradiction, and the proof is complete now. $\square$

Corollary 5. Let $\alpha = 1$ and (3) hold. Assume that there exists a sequence $\{t_k\}$ satisfying (4) and a positive constant γ_n such that (21) holds. If the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{-\gamma_n}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} p(s)\rho^{\gamma_n}(\tau(s))(\rho(\tau(t_k)) - \rho(s)) ds > 1, \quad (23)$$

then the class $\mathcal{N}_0 = \emptyset$ for (1).

Now, we turn our attention to the class $\mathcal{N}_2$. Since the proofs of the following results are very similar to those presented for canonical equations, they will be omitted.

Lemma 4. Let (3) hold. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_n such that, for $k \in \{1, 2, \dots\}$, we have the following.

$$\rho(\tau(u)) \left(\int_u^{\tau(u)} p(t) dt \right)^{1/\alpha} \geq \delta_n \quad \text{on} \quad (s_k, \tau(s_k)). \quad (24)$$

If $y(u)$ is a positive solution of (1) such that $y(u) \in \mathcal{N}_2$, then the following is the case.

$$\rho^{\delta_n}(\tau(u))y(\tau(u)) \uparrow \quad \text{on} \quad (s_k, \tau(s_k)), \quad k = 1, 2, \dots$$

Theorem 6. Let (3) hold. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_n satisfying (24). If the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{\delta_n}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} \frac{1}{r^{1/\alpha}(u)} \left(\int_{s_k}^u p(t)\rho^{-\alpha\delta_n}(\tau(t)) dt \right)^{1/\alpha} du > 1, \quad (25)$$

then the class $\mathcal{N}_2 = \emptyset$ for (1).

Corollary 6. Let $\alpha = 1$ and (3) hold. Assume that there exists a sequence $\{s_k\}$ satisfying (5) and a positive constant δ_n such that (24) holds. If the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{\delta_n}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} p(t)\rho^{-\delta_n}(\tau(t))(\rho(t) - \rho(\tau(s_k))) dt > 1, \quad (26)$$

then the class $\mathcal{N}_2 = \emptyset$ for (1).

Picking up the above results, we immediately obtain the following improvement of Theorem 1 for noncanonical (1).

Theorem 7. Let (3) hold. Assume that there exist two sequences $\{t_k\}$ and $\{s_k\}$ satisfying (4) and (5), and there exist positive constants γ_n and δ_n such that (21) and (24) hold. If (22) are (25) are satisfied, then (1) is oscillatory.

The progress that Theorem 7 yields will be demonstrated via equation (13).

Example 4. We consider again the following differential equation.

$$\left(t^2 y'(t) \right)' = a y(t(1 + 0.5 \sin(\ln t))), \quad a > 0.$$

At first, we shall show that $\mathcal{N}_0 = \emptyset$ for $a \geq 1.979968$. Thus, we set $a = 1.979968$. Substituting again $t_k = e^{3\pi/2+2k\pi}$ and $k = 1, 2, \dots$, we observe that $\tau(t_k) = 0.5e^{3\pi/2+2k\pi}$. Condition (21) takes the following form.

$$a \left(\frac{1}{1 + 0.5 \sin(\ln t)} - 1 \right) \geq \gamma_n \quad \text{on } (\tau(t_k), t_k). \quad k = 1, 2, \dots$$

Taking the monotonicity of the above function into account, we see that the following is the case.

$$\gamma_n = a \left(\frac{1}{1 + 0.5 \sin(\ln(\tau(t_k)))} - 1 \right) = a \left(\frac{1}{1 - 0.5 \cos(\ln 2)} - 1 \right) = 2.4750025,$$

Criterion (17) in terms of coefficients of (13) yields the following.

$$\limsup_{k \rightarrow \infty} \left(0.5 e^{3\pi/2+2k\pi} \right)^{\gamma_n} \int_{\tau(t_k)}^{t_k} \frac{a}{(s(1 + 0.5 \sin \ln s))^{\gamma_n}} \left(\frac{1}{0.5 e^{3\pi/2+2k\pi}} - \frac{1}{s} \right) ds > 1.$$

By substituting $s = t e^{(3\pi/2)+2k\pi}$, one can observe that the above inequality transforms into the following.

$$\frac{a}{2^{\gamma_n}} \int_{0.5}^1 t^{-\gamma_n} (1 - 0.5 \cos \ln t)^{-\gamma_n} \left(2 - \frac{1}{t} \right) dt > 1$$

We employ Matlab for evaluating the following (with $\gamma_n = 2.4750025$).

$$\int_{0.5}^1 t^{-\gamma_n} (1 - 0.5 \cos \ln t)^{-\gamma_n} \left(2 - \frac{1}{t} \right) dt = 2.80796782$$

Finally, we conclude that the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{-\gamma_n}(\tau(t_k)) \int_{\tau(t_k)}^{t_k} p(s) \rho^{\gamma_n}(\tau(s)) (\rho(\tau(t_k)) - \rho(s)) ds = 1.00000069 > 1$$

which by Corollary 5 guarantees that $\mathcal{N}_0 = \emptyset$. It is useful to notice that we obtained essentially the better parameter a than in Example 2.

We shall verify that $\mathcal{N}_2 = \emptyset$ for $a \geq 9.095512$. To show this, we let $a = 9.095512$ and $s_k = e^{\pi/2+2k\pi}$ and $k = 1, 2, \dots$. Then, $\tau(s_k) = 1.5e^{\pi/2+2k\pi}$, and it follows from (24) that the following is the case.

$$a \left(1 - \frac{1}{1 + 0.5 \sin(\ln u)} \right) \geq \delta_n \quad \text{on each } (s_k, \tau(s_k)).$$

Thus, we have the following.

$$\delta_n = a \left(1 - \frac{1}{1 + 0.5 \sin(\ln(\tau(t_k)))} \right) = a \left(1 - \frac{1}{1 + 0.5 \cos(\ln 1.5)} \right) = 2.8634021.$$

Condition (26) yields the following.

$$\limsup_{k \rightarrow \infty} \left(1.5 e^{3\pi/2+2k\pi} \right)^{-\delta_n} \int_{s_k}^{\tau(s_k)} a(t(1 + 0.5 \sin(\ln t)))^{\delta_n} \left(\frac{1}{t} - \frac{1}{1.5 e^{3\pi/2+2k\pi}} \right) dt > 1.$$

The substitution $t = e^{\pi/2+2k\pi}x$ results in the following.

$$a\left(\frac{2}{3}\right)^{\delta_n} \int_1^{1.5} x^{\delta_n} (1 + 0.5 \cos \ln x)^{\delta_n} \left(\frac{1}{x} - \frac{2}{3}\right) dx > 1$$

By employing Matlab, we find out that, for $\delta_n = 2.8634021$, the following is the case.

$$\int_1^{1.5} x^{\delta_n} (1 + 0.5 \cos \ln x)^{\delta_n} \left(\frac{1}{x} - \frac{2}{3}\right) dx = 0.35106945782$$

Therefore, the following is the case:

$$\limsup_{k \rightarrow \infty} \rho^{\delta_n}(\tau(s_k)) \int_{s_k}^{\tau(s_k)} p(t) \rho^{-\delta_n}(\tau(t)) (\rho(t) - \rho(\tau(t_k))) dt = 1.00000024 > 1$$

which by Corollary 6 implies that $\mathcal{N}_2 = \emptyset$. Again, we obtain better results than in Example 1.

By combining both criteria, we observe that condition $a \geq 9.095512$ implies oscillation of (13), while Theorem 1 requires $a > 13.8635$.

6. Discussion

In this paper, we tried to fulfill the certain gap in the oscillation theory concerning differential equations with mixed arguments. Our results are of high generality. Our basic criteria are applicable to the general equation, and the improved ones are applicable to canonical and noncanonical equations, separately. The progress is demonstrated via a set of examples.

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