Article

Wavelet Modeling of Control Stochastic Systems at Complex Shock Disturbances

Igor Sinitsyn ^{1,2}, Vladimir Sinitsyn ^{1,2}, Eduard Korepanov ¹ and Tatyana Konashenkova ^{1,*}

¹ Federal Research Center “Computer Science and Control”, Russian Academy of Sciences (FRC CSC RAS), 119333 Moscow, Russia; sinitsin@dol.ru (I.S.); vsinitsin@ipiran.ru (V.S.); ekorepanov@ipiran.ru (E.K.)

² Moscow Aviation Institute, National Research University, 125993 Moscow, Russia

* Correspondence: tkonashenkova64@mail.ru

Abstract: This article is devoted to the development of methodological supports and experimental software tools for accuracy analysis and information processing in control stochastic systems (CStS) with complex shock disturbances (ShD) by means of wavelet Haar–Galerkin technologies. Basic new results include methods and algorithms of stochastic covariance analysis and modeling on the basis of the Galerkin method and wavelet expansion for linear, linear with parametric noises, and quasilinear CStS with ShD. Results are illustrated by an information-control system at ShD. New stochastic effects accumulation for systematic and random errors are detected and investigated.

Keywords: complex shock disturbances; covariance modeling; Haar wavelet; Haar–Galerkin method; multi-echelon shock; control stochastic systems

Citation: Sinitsyn, I.; Sinitsyn, V.; Korepanov, E.; Konashenkova, T. Wavelet Modeling of Control Stochastic Systems at Complex Shock Disturbances. *Mathematics* **2021**, *9*, 2544. <https://doi.org/10.3390/math9202544>

Academic Editors:

José A. Tenreiro Machado and
Mehdi Salimi

Received: 17 August 2021

Accepted: 6 October 2021

Published: 10 October 2021

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1. Introduction

Control stochastic systems (CStS) described by nonstationary linear stochastic differential equations (SDE) in the Ito sense with parametric noises are the most adequate models at single shock disturbance (ShD) and Gaussian and non-Gaussian multi-ShD. The exact accuracy theory of stochastic processes for such SDE is developed in [1,2]. For StS with impulse, ShD asymptotic theory is given in [3,4]. Nowadays wavelet methods are intensively applied to the problem of deterministic and stochastic numerical analysis and modeling. In the past few decades, a broad range of numerical methods based on Haar wavelet methods achieved great success [5]. These methods are simple in sense versatility and flexibility and possess less computational cost for accuracy problems. A wavelet is a numerical mathematical function used to divide a given function into different scale components. The usage and study of wavelets has attained its modern growth due to mathematical analysis of the wavelet in [6–8]. The concept of multiresolution analysis (MRA) was developed in [9]. In [10], z method to construct wavelets with compact support and scale function was presented. A review of basic properties of wavelets and MRA is given in [11]. Among the wavelet families, which are described by an analytical expression, special attention deserves the Haar wavelets. Haar wavelets are very effective and popular for solving ordinary differential equations [11–15]. The wavelet solution of integral and evolutionary equation is discussed in [16–21].

The application of a wavelet for canonical expansions of random functions and SDE has been suggested in [22] and developed [23,24]. In [25,26], a combination of Haar wavelets and the Galerkin method [22] numerical solution of linear equations were presented. Development of the Haar–Galerkin method and experimental software tools for accuracy analysis of CStS at single ShD is given in [19,20].

Let us consider modeling the wavelet Haar–Galerkin methodology and experimental software tools for covariance accuracy analysis of CStS at complex multi-ShD.

2. Stochastic Systems at Complex Shock Disturbances

Let us consider the differential stochastic system described by the following Ito vector equation:

$$\dot{X} = a^{sh}(X, U, t) + b^{sh}(X, U, t)V^{sh}, \quad X(t_0) = X_0. \quad (1)$$

Here, X ($\dim X = p$) is the Markovian state vector, V^{sh} is the shock white noise (non-Gaussian in the general case), and $a^{sh} = a^{sh}(X, U, t)$ and $b^{sh} = b^{sh}(X, U, t)$ are known functions defined by ShD.

For single ShD in cases when Equation (1) satisfies the “filter hypotheses”, nonlinear effects are not able and it is enough to use linear or linear with parametric noises models:

$$\dot{X} = a^{sh}X + a_0^{sh} + b^{sh}V^{sh}, \quad X(t_0) = X_0, \quad (2)$$

$$\dot{X} = a^{sh}X + a_0^{sh} + (b_0^{sh} + \sum_{n=1}^p b_n X_n)V^{sh}, \quad X(t_0) = X_0. \quad (3)$$

In case of independent multi-echelon ShD at time moments, it is necessary to input additional items:

$$a^{sh}(X, U, t) = a_0^{sh}(t, U) + a_1^{sh}(t, U)X + \mu \bar{F}_{0t}, \quad (4)$$

$$b^{sh}(X, U, t) = b_0^{sh}(t, U) + \sum_{n=1}^p b_n^{sh}(t, U)X_n + \sqrt{\mu} \bar{F}_{1t}. \quad (5)$$

Here, $\mu \bar{F}_{0t}$ and $\sqrt{\mu} \bar{F}_{1t}$ are small functions depending on t and U . At practice functions $a_0^{sh}, a_1^{sh}, b_0^{sh}, b_n^{sh}$ are approximated by formulae:

$$a_0^{sh}(t, U) = \sum_{T_k} A_{0k}^a \varphi_k^a(t, U, T_k), \quad a_1^{sh}(t, U) = \sum_{T_k} A_{1k}^a \varphi_k^a(t, U, T_k),$$

$$b_0^{sh}(t, U) = \sum_{T_k} A_{0k}^b \varphi_k^b(t, U, T_k), \quad b_n^{sh}(t, U) = \sum_{T_k} A_{nk}^b \varphi_k^b(t, U, T_k).$$

The shock white noise V^{sh} in Equation (2) may be non-Gaussian in general cases, whereas V^{sh} in Equation (3) is strictly Gaussian.

The analytical off-line modeling Equations (2)–(5) at given ShD and system parameters U are based on known methods [1,2].

3. Wavelet Covariance Modeling at Additive Complex ShD

Let us consider linear nonstationary differential StS:

$$\dot{Y}_1 = a^1 Y_1 + a_0^1 + b^1 X^{sh}, \quad Y_1(t_0) = Y_{10}, \quad (6)$$

$$X^{sh} = \sum_k [A_k \varphi_k(t, T_k) + B_k Y_2], \quad (7)$$

$$\dot{Y}_2 = a^2 Y_2 + a_0^2 + b^2 V, \quad Y_2(t_0) = Y_{20}, \quad (8)$$

Here, Y_1 is the state vector; $\dim Y_1 = n_1$, Y_2 is the vector of the forming filter $\dim Y_2 = n_2$; X^{sh} is the additive complex ShD; a^1, a^2, a_0^1, a_0^2 are coefficients of corresponding dimensions; A_k and B_k, T_k are random variables; $\varphi_k(t, T_k)$ is the impulse forming functions; V is non-Gaussian in the general-case white noise, $\dim V = n^V$, with the matrix intensity v . Using:

$$Y = \begin{bmatrix} Y_1^T & Y_2^T \end{bmatrix}^T, \quad a_0 = \begin{bmatrix} a_0^1 + \sum_k A_k \varphi_k(t, T_k) \\ a_0^2 \end{bmatrix}, \quad a = \begin{bmatrix} a^1 & \sum_k B_k \\ 0 & a^2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ b^2 \end{bmatrix}$$

we come to the following linear vector stochastic equation:

$$\dot{Y} = a^{sh}Y + a_0^{sh} + b^{sh}V, \quad Y(t_0) = Y_0. \quad (9)$$

Using Equation (9), $a = a^{sh}$, $a_0 = a_0^{sh}$, $b = b^{sh}$, we have the following set of equations for the mathematical expectation m_y , covariance matrix K_y , and matrix of covariance functions $K_y(t_1, t_2)$ [1,2]:

$$\dot{m}_y = am_y + a_0, \quad m_y(t_0) = m_0; \quad (10)$$

$$\dot{K}_y = aK_y + K_y a^T + bvb^T, \quad K(t_0) = K_0; \quad (11)$$

$$\frac{\partial K_y(t_1, t_2)}{\partial t_2} = K_y(t_1, t_2)a(t_2)^T, \quad K_y(t_1, t_1) = K_y(t_1), \quad t_1 < t_2; \quad (12)$$

$$K_y(t_1, t_2) = K_y(t_2, t_1)^T, \quad t_1 > t_2; \quad (13)$$

Let us consider Equation (9) on the time interval $[t_0, T]$. After a change of variables at

$$\bar{t} = \frac{(t - t_0)}{(T - t_0)} \quad (14)$$

we transform Equation (14) to the following:

$$\bar{Y}' = \bar{a}(\bar{t})\bar{Y} + \bar{a}_0(\bar{t}) + \bar{b}(\bar{t})\bar{V}(\bar{t}) \quad (15)$$

at $\bar{t} \in [0, 1]$, $\bar{Y}(0) = Y_0$ and white noise $\bar{V}(\bar{t})$ with intensity $\bar{v}(\bar{t}) = v((T - t_0)\bar{t} + t_0)$, where:

$$\bar{Y}(\bar{t}) = Y((T - t_0)\bar{t} + t_0), \quad \bar{V}(\bar{t}) = V((T - t_0)\bar{t} + t_0),$$

$$\bar{a}(\bar{t}) = (T - t_0)a((T - t_0)\bar{t} + t_0), \quad \bar{a}_0(\bar{t}) = (T - t_0)a_0((T - t_0)\bar{t} + t_0),$$

$$\bar{b}(\bar{t}) = (T - t_0)b((T - t_0)\bar{t} + t_0).$$

Here, prime is differential by $\bar{t}$. Furthermore, briefly, we use $t = \bar{t}$.

For solving equation:

$$\bar{m}' = \bar{a}\bar{m} + \bar{a}_0, \quad \bar{m}(0) = m_0 \quad (16)$$

defining $\bar{m} = E\bar{Y}$, we apply [15,16] at condition for functions $\bar{m}'_h, \bar{m}_h, \bar{a}_{hk}, \bar{a}_{0h}$ ($h, k = 1, 2, \dots, p$) belonging to $L^2[0, 1]$; as shown in [13], these functions can be expanded into convergent wavelet series. Following [15], we define the orthonormal basis of the Haar wavelet:

$$w_1(t) = \varphi(t) = \varphi_{00}(t) = \begin{cases} 1 & \text{at } t \in [0, 1], \\ 0 & \text{at } t \notin [0, 1], \end{cases} \quad (17)$$

$$w_2(t) = \Psi(t) = \Psi_{00}(t) = \begin{cases} 1 & \text{at } t \in \left[0, \frac{1}{2}\right), \\ -1 & \text{at } t \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{at } t \notin [0, 1], \end{cases} \quad (18)$$

$$w_i(t) = \Psi_{jk}(t) = \begin{cases} \sqrt{2^j} & \text{at } t \in \left[\frac{k}{l}, \frac{k+0.5}{l}\right), \\ -\sqrt{2^j} & \text{at } t \in \left[\frac{k+0.5}{l}, \frac{k+1}{l}\right), \\ 0 & \text{at } t \notin \left[\frac{k}{l}, \frac{k+1}{l}\right). \end{cases} \quad (19)$$

Here, $\varphi = \varphi(t)$ is the scaling function, $\psi = \psi(t)$ is the maternal wavelet

$$\Psi_{jk} = \Psi_{jk}(t) = \sqrt{2^j}\Psi(2^j t - k), \quad (20)$$

$K = 0, 1, \dots, l - 1; l = 2^j; j = 1, 2, \dots, J; L = 2 \cdot 2^J; i = l + k + 1; l = 3, 4, \dots, L; J$ is the maximal resolution level.

Then, we introduce integrals:

$$p_i(t) = \int_0^t w_i(\tau) d\tau \quad (i = 1, 2, \dots, L), \quad (21)$$

where:

$$p_1(t) = \begin{cases} t & \text{at } t \in [0, 1), \\ 0 & \text{at } t \notin [0, 1), \end{cases} \quad p_i(t) = \begin{cases} \sqrt{2^j} \left(t - \frac{k}{l} \right) & \text{at } t \in \left[\frac{k}{l}, \frac{k+0.5}{l} \right), \\ \sqrt{2^j} \left(\frac{k+1}{l} - t \right) & \text{at } t \in \left[\frac{k+0.5}{l}, \frac{k+1}{l} \right), \\ 0 & \text{at } t \notin \left[\frac{k}{l}, \frac{k+1}{l} \right), \end{cases} \quad (i = 2, 3, \dots, L).$$

For every component, $\bar{m}_h = \bar{m}_h(t)$, $h = 1, \dots, p$; Equation (16) gives expressions:

$$\bar{m}'_h = \sum_{k=1}^p \bar{a}_{hk}(t) \bar{m}_k(t) + \bar{a}_{0h}(t), \quad \bar{m}_h(0) = m_{0h}. \quad (22)$$

Expanding $\bar{m}'_h$ in the form of Haar wavelet expansion (WLE),

$$\bar{m}'_h = \sum_{i=1}^L c_{hi} w_i(t), \quad \text{where } c_{hi} = \int_0^1 \bar{m}'_h(\tau) w_i(\tau) d\tau, \quad (23)$$

we obtain from (22):

$$\bar{m}_h(t) = \int_0^t \sum_{i=1}^L c_{hi} w_i(\tau) d\tau + m_{0h} = \sum_{i=1}^L c_{hi} p_i(t) + m_{0h} \quad (h = 1, \dots, p). \quad (24)$$

After substituting Equations (23) and (24) into (22), we have expression:

$$\sum_{i=1}^L c_{hi} w_i(t) = \sum_{k=1}^p \bar{a}_{hk}(t) \left[\sum_{i=1}^L c_{ki} p_i(t) + m_{0k} \right] + \bar{a}_{0h}(t). \quad (25)$$

Projecting (25) on $w_s(t)$, we receive $(L \times p)$ linear algebraic Equations

$$\sum_{i=1}^L c_{hi} (w_i(t), w_s(t)) = \sum_{k=1}^p \sum_{i=1}^L c_{ki} (\bar{a}_{hk}(t) p_i(t), w_s(t)) + \sum_{k=1}^p m_{0k} (\bar{a}_{hk}(t), w_s(t)) + (\bar{a}_{0h}(t), w_s(t)) \quad (s = 1, 2, \dots, L; h = 1, 2, \dots, p). \quad (26)$$

Due to orthonormality $w_s(t)$, we obtain from (26) the following Equations:

$$c_{hs} = \sum_{k=1}^p \sum_{i=1}^L c_{ki} (\bar{a}_{hk}(t) p_i(t), w_s(t)) + \sum_{k=1}^p m_{0k} (\bar{a}_{hk}(t), w_s(t)) + (\bar{a}_{0h}(t), w_s(t)) \quad (s = 1, 2, \dots, L; h = 1, 2, \dots, p). \quad (27)$$

Then, we expand functions $\bar{a}_{hk}(t) p_i(t)$, $\bar{a}_{hk}(t)$, $\bar{a}_{0h}(t)$ in the form of WLE:

$$\bar{a}_{hk}(t) p_i(t) = \sum_{j=1}^L g_j^{hki} w_j(t) \quad (i = 1, 2, \dots, L; h, k = 1, 2, \dots, p), \quad (28)$$

$$\bar{a}_{hk}(t) = \sum_{j=1}^L q_j^{hk} w_j(t) \quad (h, k = 1, 2, \dots, p), \quad (29)$$

$$\bar{a}_{0h}(t) = \sum_{j=1}^L \rho_j^h w_j(t) \quad (h = 1, 2, \dots, p), \quad (30)$$

where:

$$g_j^{hki} = (\bar{a}_{hk}(t) p_i(t), w_j(t)) = \int_0^1 \bar{a}_{hk}(\tau) p_i(\tau) w_j(\tau) d\tau, \quad (31)$$

$$q_j^{hk} = (\bar{a}_{hk}(t), w_j(t)) = \int_0^1 \bar{a}_{hk}(\tau) w_j(\tau) d\tau, \quad (32)$$

$$\rho_j^h = (\bar{a}_{0h}(t), w_j(t)) = \int_0^1 \bar{a}_{0h}(\tau) w_j(\tau) d\tau. \quad (33)$$

Equation (27) may be presented in resultant form:

$$c_{hs} = \sum_{k=1}^p \sum_{i=1}^L c_{ki} g_s^{hki} + \sum_{k=1}^p m_{0k} q_s^{hk} + \rho_j^h \quad (s = 1, 2, \dots, L; h = 1, 2, \dots, p). \quad (34)$$

So we obtain the following algorithm.

Theorem 1. Let the following conditions be satisfied:

- Equations (6)–(8) are reducible to Equation (9) and Equation (15) for composed vector Y ;
- Scalar functions $\bar{m}'_h, \bar{m}_h, \bar{a}_{hk}, \bar{a}_{0h}$ ($h, k = 1, 2, \dots, p$) belong to space $L^2[0, 1]$;
- In space $L^2[0, 1]$ the wavelet basic Haar is defined by Formulaes (17)–(20).

Then, the solution of Equation (22) for component $\bar{m}_h$ of composed vector $\bar{Y}(t)$ is as follows:

$$\bar{m}_h(t) = \sum_{i=1}^L c_{hi} p_i(t) + m_{0h} \quad (h = 1, \dots, p), \quad (35)$$

where functions $p_i(t)$ are defined by (21), coefficients c_{hi} are the solutions of linear algebraic Equation (34).

From Equation (11), for every element $\bar{K}_{r_1 r_2} = \bar{K}_{r_1 r_2}(t)$ covariance matrix $\bar{K} = E[(\bar{Y} - \bar{m})(\bar{Y} - \bar{m})^T]$, we have the following ordinary differential equation:

$$\begin{aligned} \bar{K}'_{r_1 r_2}(t) &= \sum_{h=1}^p \bar{a}_{r_1 h}(t) \bar{K}_{h r_2}(t) + \sum_{h=1}^p \bar{K}_{r_1 h}(t) \bar{a}_{r_2 h}(t) + \sum_{h=1}^p \sum_{s=1}^n \bar{b}_{r_1 h}(t) \bar{v}_{hs}(t) \bar{b}_{r_2 h}(t), \\ \bar{K}_{r_1 r_2}(0) &= K_{0r_1 r_2}. \end{aligned} \quad (36)$$

Due to the $\bar{K}$ symmetry, it is sufficient to compose Equations for $r_1 = \overline{1, p}, r_2 = \overline{r_1, p}$. In this case, elements $\bar{K}_{h r_2}$ at $h > r_2$ are replaced by $\bar{K}_{r_2 h}$ and $\bar{K}_{r_1 h}$ at $r_1 > h$ by $\bar{K}_{h r_1}$. As a result, we have only $p(p+1)/2$ equations. Let us introduce notation:

$$B_{r_1 r_2}(t) = \sum_{h=1}^p \sum_{s=1}^n \bar{b}_{r_1 h}(t) \bar{v}_{hs}(t) \bar{b}_{r_2 h}(t) \quad (37)$$

As a result of applying the Haar–Galerkin method to the solution of the Equation (36), we obtain the following formulae:

$$\bar{K}'_{r_1 r_2} = \sum_{i=1}^L c_i^{r_1 r_2} w_i(t), \quad (38)$$

$$c_i^{r_1 r_2} = \int_0^1 \bar{K}'_{r_1 r_2}(\tau) w_i(\tau) d\tau, \quad (39)$$

$$\bar{K}_{r_1 r_2} = \sum_{i=1}^L c_i^{r_1 r_2} p_i(t) + K_{0r_1 r_2}. \quad (40)$$

After substituting (38) and (40) into (37), we have:

$$\begin{aligned} \sum_{i=1}^L c_i^{r_1 r_2} w_i(t) &= \sum_{h=1}^p \sum_{i=1}^L c_i^{h r_2} \bar{a}_{r_1 h}(t) p_i(t) + \sum_{h=1}^p K_{0r_2 h} \bar{a}_{r_1 h}(t) + \\ &+ \sum_{i=1}^p \sum_{i=1}^L c_i^{r_1 h} \bar{a}_{r_2 h}(t) p_i(t) + \sum_{h=1}^p K_{0r_1 h} \bar{a}_{r_2 h}(t) + B_{r_1 r_2}(t). \end{aligned} \quad (41)$$

Protecting (41) on basis $w_s(t)$, we receive $p(p+1)L/2$ equations for $c_i^{r_1 r_2}$:

$$c_s^{r_1 r_2} = \sum_{i=1}^L \sum_{h=1}^p c_i^{h r_2} g_s^{r_1 h i} + \sum_{i=1}^L \sum_{h=1}^p c_i^{r_1 h} g_s^{r_2 h i} + \sum_{h=1}^p K_{0h r_2} q_s^{r_1 h} + \sum_{h=1}^p K_{0r_1 h} q_s^{r_2 h} + \rho_s^{r_1 r_2}, \quad (42)$$

where:

$$\rho_s^{r_1 r_2} = \int_0^1 B_{r_1 r_2}(\tau) w_i(\tau) d\tau.$$

So, we come to the following algorithm:

Theorem 2. Let the conditions of Theorem 1 and additional conditions be considered:

- Equations for elements of covariance matrix Equation (11) are reduced to Equation (36);
- The scalar functions $\bar{K}_{r_1 r_2}'(t)$, $B_{r_1 r_2}(t)$ ($r_1, r_2 = \overline{1, p}$) belonging to space $L^2[0, 1]$ are fulfilled.

Then, the solution of Equation (36) for elements $\bar{K}_{r_1 r_2}$ of the covariance matrix of the composed vector $\bar{Y}$ is expressed by Equation (40), where $p_i(t)$ is defined by (21), coefficients $c_i^{r_1 r_2}$ being the solution of linear algebraic Equation (42).

From Equation (12) for every element $\bar{K}_{r_1 r_2}(t_1, t_2)$ of matrix $\bar{K}(t_1, t_2) = E[(\bar{Y}(t_1) - \bar{m}(t_1))(\bar{Y}(t_1) - \bar{m}(t_1))^T]$ at $t = t_2$ we have the following ordinary equations with corresponding initial condition:

$$\frac{\partial \bar{K}_{r_1 r_2}(t_1, t_2)}{\partial t_2} = \sum_{h=1}^p \bar{K}_{r_1 h}(t_1, t_2) \bar{a}_{r_2 h}(t_2), \quad (43)$$

$$\bar{K}_{r_1 r_2}(t_1, t_1) = \bar{K}_{r_1 r_2}(t_1) \quad (r_1, r_2 = \overline{1, p}; t_1, t_2 \in [0, 1]; t_1 < t_2). \quad (44)$$

As a result of applying the Haar–Galerkin method to the solution of the Equation (43) with initial conditions (44), we obtain the following expressions:

$$\frac{\partial \bar{K}_{r_1 r_2}(t_1, t_2)}{\partial t_2} = \sum_{i=1}^p d_i^{r_1 r_2} w_i(t_2), \quad (45)$$

$$d_i^{r_1 r_2} = \int_0^1 \frac{\partial \bar{K}_{r_1 r_2}(t_1, \tau_2)}{\partial \tau_2} w_i(\tau_2) d\tau_2, \quad (46)$$

$$\bar{K}_{r_1 r_2}(t_1, t_2) = \sum_{i=1}^L d_i^{r_1 r_2} p_i(t_2) + \bar{K}_{r_1 r_2}(t_1), \quad (47)$$

$$\sum_{i=1}^p d_i^{r_1 r_2} w_i(t_2) = \sum_{h=1}^p \bar{a}_{r_2 h}(t_2) \left[\sum_{i=1}^L d_i^{r_1 h} p_i(t_2) + \bar{K}_{r_1 h}(t_1) \right] \quad (48)$$

After projecting (47) on basis $w_s(t_2)$ and tauting into consideration on Expressions (31), (32), we have a system of $p^2 \times L$ linear equations for coefficients $d_i^{r_1 r_2}$:

$$d_s^{r_1 r_2} = \sum_{i=1}^L \sum_{h=1}^p d_i^{r_1 h} g_s^{r_2 h i} + \sum_{h=1}^p \bar{K}_{r_1 h}(t_1) q_s^{r_2 h} \quad (r_1, r_2 = \overline{1, p}; s = \overline{1, L}). \quad (49)$$

Theorem 3. Let the conditions of Theorems 1, 2, and additional conditions be considered:

- Equation (12) may be reduced to Equations (43) and (44);
- functions $\frac{\partial \bar{K}_{r_1 r_2}(t_1, t_2)}{\partial t_2}$, $\bar{K}_{r_1 r_2}(t_1, t_2)$ ($r_1, r_2 = \overline{1, p}$) at $t = t_2$ belong to space $L^2[0, 1]$ at fixed $t_1 \in [0, 1]$ all valid.

Then, the solution of Equations (43) and (44) for elements $\bar{K}_{r_1 r_2}(t_1, t_2)$ ($r_1, r_2 = \overline{1, p}$) of the covariance matrix of the composed vector $\bar{Y}$ for $t_1 < t_2$ is as follows: (47). Here, $p_i(t)$ is defined by (21), coefficients $d_i^{r_1 r_2}$ are defined by (49), and the values of $\bar{K}_{r_1 r_2}(t_1)$ are taken from Theorem 2.

4. Wavelet Covariance Modeling at Parametric Complex Shd

Let us consider the Ito linear with parametric noises nonstationary differential StS:

$$\dot{Y} = aY + a_0 + X^{sh}, \quad (50)$$

$$\begin{aligned} X^{sh} &= X_1^{sh} + X_2^{sh}, X_1^{sh} = \sum_{T_k} (A_k \varphi_k + \psi_k V^{sh}), \\ X_2^{sh} &= \sum_{h=1}^p \sum_{T_k} \alpha_h Y_h (A_k \varphi_k + \psi_k V^{sh}). \end{aligned} \quad (51)$$

Here, V^{sh} is Gaussian white noise with intensity matrix $v^{sh} = v(t)$. Equations (50) and (51) may be transformed into the following form:

$$\dot{Y} = a^{sh} Y + a_0^{sh} + \left(b_0^{sh} + \sum_{h=1}^p b_h^{sh} Y_h \right) V^{sh}, Y(t_0) = Y_0, \quad (52)$$

where:

$$\begin{aligned} a_0^{sh} &= a_0 + \sum_{T_k} A_k \varphi_k, \quad a^{sh} Y = aY + \sum_{h=1}^p \sum_{T_k} A_k \mathfrak{a}_h \varphi_k Y_h, \\ b_0^{sh} &= \sum_{T_k} \psi_k, \quad \sum_{h=1}^p b_h^{sh} Y_h V^{sh} = \sum_{h=1}^p \sum_{T_k} (\psi_k \mathfrak{a}_h Y_h) V^{sh}. \end{aligned} \quad (53)$$

For Equations (52) and (53), the following covariance equations are valid [1,2]:

$$\dot{m} = am + a_0, \quad m(t_0) = m_0; \quad (54)$$

$$\dot{K} = aK + Ka^T + b_0 v b_0^T + \sum_{h=1}^p (b_h v b_h^T + b_0 v b_h^T) m_h + \sum_{h=1}^p \sum_{l=1}^p b_h v b_l^T (m_h m_l + K_{hl}), \quad K(t_0) = K_0; \quad (55)$$

$$\frac{\partial K(t_1, t_2)}{\partial t_2} = K(t_1, t_2) a(t_2)^T, \quad K(t_1, t_1) = K(t_1), \quad t_1 < t_2; \quad (56)$$

Here, $m = EY = [m_1 \ m_2 \ \dots \ m_p]^T$, $K = K(t) = [K_{ij}(t)]_{i,j=1}^p$, $K = K(t_1, t_2) = [K_{ij}(t_1, t_2)]_{i,j=1}^p$.

Equation (52) for $\bar{t}$ (Formula (14)) may be transformed to the following one:

$$\bar{Y}' = \bar{a}(\bar{t}) \bar{Y} + \bar{a}_0(\bar{t}) + (\bar{b}_0(\bar{t}) + \sum_{n=1}^p \bar{b}_n(\bar{t}) \bar{Y}_n) \bar{V}(\bar{t}), \quad (57)$$

$$\bar{v}(\bar{t}) = v((T - t_0)\bar{t} + t_0), \quad \bar{Y}(\bar{t}) = Y((T - t_0)\bar{t} + t_0), \quad \bar{V}(\bar{t}) = V((T - t_0)\bar{t} + t_0),$$

$$\bar{a}(\bar{t}) = (T - t_0)a((T - t_0)\bar{t} + t_0), \quad \bar{a}_0(\bar{t}) = (T - t_0)a_0((T - t_0)\bar{t} + t_0),$$

$$\bar{b}_0(\bar{t}) = (T - t_0)b_0((T - t_0)\bar{t} + t_0), \quad \bar{b}_n(\bar{t}) = (T - t_0)b_n((T - t_0)\bar{t} + t_0).$$

The equation for mathematical expectation $\bar{m} = E\bar{Y}$ is separated from equations for covariance characteristics and is defined by Equation (16) (see Theorem 1).

The equations for elements $\bar{K}_{r_1 r_2} = \bar{K}_{r_1 r_2}(t)$, $t \in [0, 1]$, due to their symmetry ($r_1 = \overline{1, p}$; $r_2 = \overline{r_1, p}$), have the form:

$$\begin{aligned} \bar{K}'_{r_1 r_2}(t) &= \sum_{l=1}^p \bar{a}_{r_1 l}(t) \bar{K}_{l r_2}(t) + \sum_{l=1}^p \bar{a}_{r_2 l}(t) \bar{K}_{r_1 l}(t) + \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^0(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^0(t) + \\ &+ \sum_{k=1}^p \bar{m}_k(t) \left[\sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^k(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^0(t) + \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^0(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^k(t) \right] + \\ &+ \sum_{k=1}^p \sum_{l=1}^p ((\bar{m}_k(t) \bar{m}_l(t) + \bar{K}_{kl}(t)) \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^k(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^l(t)), \quad \bar{K}_{r_1 r_2}(t_0) = K_{r_1 r_2}^0. \end{aligned} \quad (58)$$

Using notations:

$$B_{r_1 r_2}^{00}(t) = \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^0(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^0(t),$$

$$B_{r_1 r_2}^{k0}(t) = \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^k(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^0(t),$$

$$B_{r_1 r_2}^{0k}(t) = \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^0(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^k(t),$$

$$B_{r_1 r_2}^{kl}(t) = \sum_{s_1=1}^n \sum_{s_2=1}^n \bar{b}_{r_1 s_1}^k(t) \bar{v}_{s_1 s_2}(t) \bar{b}_{r_2 s_2}^l(t),$$

we transform Equation (58) into:

$$\begin{aligned} \bar{K}'_{r_1 r_2}(t) &= \sum_{l=1}^p \bar{a}_{r_1 l}(t) \bar{K}_{l r_2}(t) + \sum_{l=1}^p \bar{a}_{r_2 l}(t) \bar{K}_{r_1 l}(t) + B_{r_1 r_2}^{00}(t) + \\ &+ \sum_{k=1}^p \bar{m}_k(t) [B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k}(t)] + \sum_{k=1}^p \sum_{l=1}^p ((\bar{m}_k(t) \bar{m}_l(t) + \bar{K}_{kl}(t)) B_{r_1 r_2}^{kl}(t)). \end{aligned} \quad (59)$$

After, the Haar expanding left hand of Equation (58) is equal to:

$$\bar{K}'_{r_1 r_2}(t) = \sum_{i=1}^L c_i^{r_1 r_2} w_i(t), \quad (60)$$

$$c_i^{r_1 r_2} = \int_0^1 \bar{K}'_{r_1 r_2}(\tau) w_i(\tau).$$

We obtain the solution (59):

$$\bar{K}_{r_1 r_2}(t) = \sum_{i=1}^L c_i^{r_1 r_2} p_i(t) + K_{r_1 r_2}^0. \quad (61)$$

Substituting Equations (60) and (61) into (58), we have the following results:

$$\begin{aligned} \sum_{i=1}^L c_i^{r_1 r_2} w_i(t) &= \sum_{i=1}^L \sum_{l=1}^p c_i^{l r_2} \bar{a}_{r_1 l}(t) p_i(t) + \sum_{l=1}^p \bar{a}_{r_1 l}(t) K_{l r_2}^0(t) + \\ &+ \sum_{i=1}^L \sum_{l=1}^p c_i^{r_1 l} \bar{a}_{r_2 l}(t) p_i(t) + \sum_{l=1}^p \bar{a}_{r_2 l}(t) K_{r_1 l}^0(t) + B_{r_1 r_2}^{00}(t) + \\ &+ \sum_{k=1}^p \sum_{i=1}^L c_{ki} \left[B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k}(t) \right] p_i(t) + \sum_{k=1}^p m_k^0 \left[B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k}(t) \right] + \\ &+ \sum_{k=1}^p \sum_{l=1}^p \sum_{i_1=1}^L \sum_{i_2=1}^L c_{ki_1} c_{li_2} B_{r_1 r_2}^{kl}(t) p_{i_1}(t) p_{i_2}(t) + \sum_{k=1}^p m_k^0 \sum_{l=1}^p \sum_{i_2=1}^L c_{li_2} B_{r_1 r_2}^{kl}(t) p_{i_2}(t) + \\ &+ \sum_{l=1}^p m_l^0 \sum_{k=1}^p \sum_{i_1=1}^L c_{ki_1} B_{r_1 r_2}^{kl}(t) p_{i_1}(t) + \sum_{k=1}^p \sum_{l=1}^p B_{r_1 r_2}^{kl}(t) \left(m_k^0 m_l^0 + K_{kl}^0 \right) + \\ &+ \sum_{i=1}^L \sum_{k=1}^p \sum_{l=1}^p c_i^{kl} B_{r_1 r_2}^{kl}(t) p_i(t). \end{aligned} \quad (62)$$

At last taking into account the notations:

$$\begin{aligned} \bar{a}_{lk}(t) p_i(t) &= \sum_{j=1}^L g_j^{lki} w_j(t), \\ \bar{a}_{lk}(t) &= \sum_{j=1}^L q_j^{lk} w_j(t), \\ B_{r_1 r_2}^{00}(t) &= \sum_{j=1}^L h_{r_1 r_2 j}^{00} w_j(t), \\ B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k} &= \sum_{j=1}^L h_{r_1 r_2 j}^{0k} w_j(t), \\ B_{r_1 r_2}^{kl}(t) &= \sum_{j=1}^L h_{r_1 r_2 j}^{kl} w_j(t), \\ \left[B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k} \right] p_i(t) &= \sum_{j=1}^L u_{r_1 r_2 j}^{0ki} w_j(t), \\ B_{r_1 r_2}^{kl}(t) p_i(t) &= \sum_{j=1}^L u_{r_1 r_2 j}^{kli} w_j(t), \\ B_{r_1 r_2}^{kl}(t) p_{i_1}(t) p_{i_2}(t) &= \sum_{j=1}^L u_{r_1 r_2 j}^{kli_1 i_2} w_j(t) \end{aligned}$$

and projecting (62) on wavelet basis $w_s(t)$, we obtain the $p(p+1)L/2$ linear algebraic Equation relatively on coefficients $c_s^{r_1 r_2}$ ($r_1 = \overline{1, p}$; $r_2 = \overline{1, r_1}$; $s = \overline{1, L}$):

$$\begin{aligned} c_s^{r_1 r_2} &= \sum_{i=1}^L \sum_{l=1}^p c_i^{l r_2} g_s^{r_1 li} + \sum_{l=1}^p q_s^{r_1 l} K_{l r_2}^0 + \sum_{i=1}^L \sum_{l=1}^p c_i^{r_1 l} g_s^{r_2 li} + \sum_{l=1}^p q_s^{r_2 l} K_{r_1 l}^0 + h_{r_1 r_2 s}^{00} + \\ &+ \sum_{k=1}^p \sum_{i=1}^L c_{ki} u_{r_1 r_2 s}^{0ki} + \sum_{k=1}^p m_k^0 h_{r_1 r_2 s}^{0k} + \sum_{k=1}^p \sum_{l=1}^p \sum_{i_1=1}^L \sum_{i_2=1}^L c_{ki_1} c_{li_2} u_{r_1 r_2 s}^{kli_1 i_2} + \\ &+ \sum_{k=1}^p m_k^0 \sum_{l=1}^p \sum_{i_2=1}^L c_{li_2} u_{r_1 r_2 s}^{kli} + \sum_{l=1}^p m_l^0 \sum_{k=1}^p \sum_{i_1=1}^L c_{ki_1} u_{r_1 r_2 s}^{kli_1} + \\ &+ \sum_{k=1}^p \sum_{l=1}^p h_{r_1 r_2 s}^{kl} \left(m_k^0 m_l^0 + K_{kl}^0 \right) + \sum_{i=1}^L \sum_{k=1}^p \sum_{l=1}^p c_i^{kl} u_{r_1 r_2 s}^{kli}. \end{aligned} \quad (63)$$

Theorem 4. Let the conditions of Theorem 1 and:

- Equations (55) are reduced to Equations (58);
- scalar functions $\bar{a}_{lk}(t) p_i(t)$, $\bar{a}_{lk}(t)$, $B_{r_1 r_2}^{00}(t)$, $\left[B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k} \right]$, $B_{r_1 r_2}^{kl}(t)$, $\left[B_{r_1 r_2}^{k0}(t) + B_{r_1 r_2}^{0k} \right] p_i(t)$, $B_{r_1 r_2}^{kl}(t) p_i(t)$, $B_{r_1 r_2}^{kl}(t) p_{i_1}(t) p_{i_2}(t)$ ($l, k, r_1, r_2 = \overline{1, p}$; $i, i_1, i_2 = \overline{1, L}$) belong to space $L^2[0, 1]$.

Then, the solutions of Equation (58) is expressed by Equations (61) and (63).

For solution of Equation (56), we use Theorem 3.

5. Examples

Let us consider the information-control system (ICS) of the third order at ShD described by the following equations:

$$\begin{aligned}\dot{Y}_1 &= Y_2, \\ \dot{Y}_2 &= -\omega_c^2 Y_1 - 2\varepsilon\omega_c Y_2 + \left(S + n_t^{sh} + V_t^{sh}\right)(\cos\alpha + Y_1 \sin\alpha), \\ \dot{Y}_3 &= Y_1, \\ Y_i(t_0) &= Y_{i0} (i = 1, 2, 3).\end{aligned}$$

Here, Y_i represents the components of the state vector $Y = [Y_1 \ Y_2 \ Y_3]^T$, α , ε , ω_c , S are the system parameters; n_t^{sh} is the rectangular impulse function at time moment t^{sh} ; α is the angle between acceleration $(S + n_t^{sh} + V_t^{sh})$ and longitudinal ICS axis; V_t^{sh} is the white noise with intensity t^{sh} . The equation may be presented in the vector form:

$$\dot{Y} = a_0^{sh} + a^{sh}Y + (b_0^{sh} + b_1^{sh}Y)V_t^{sh},$$

where:

$$\begin{aligned}a_0^{sh} &= \begin{bmatrix} 0 \\ (S + n_t^{sh})\cos\alpha \\ 0 \end{bmatrix}, \quad a^{sh} = \begin{bmatrix} 0 & 1 & 0 \\ -\Omega_t^{sh} & -2\varepsilon\omega_c & 0 \\ 1 & 0 & 0 \end{bmatrix}, \\ b_0^{sh} &= \begin{bmatrix} 0 \\ \cos\alpha \\ 0 \end{bmatrix}, \quad b_1^{sh} = \begin{bmatrix} 0 \\ \sin\alpha \\ 0 \end{bmatrix}, \quad \Omega_t^{sh} = \omega_c^2 - (S + n_t^{sh})\sin\alpha.\end{aligned}$$

Equations for mathematical expectations $m_i = EY_i$ and elements of covariance matrix $k_{ij} = E[(Y_i - m_i)(Y_j - m_j)]$ are as follows:

$$\begin{aligned}\dot{m}_1 &= m_2, \\ \dot{m}_2 &= a_{21}^{sh}m_1 + a_{22}^{sh}m_2 + a_{02}^{sh}, \\ \dot{m}_3 &= m_1, \\ m_i(t_0) &= m_{i0} (i = 1, 2, 3); \\ \dot{k}_{11} &= 2k_{12}, \\ \dot{k}_{12} &= a_{21}^{sh}k_{11} + a_{22}^{sh}k_{12} + k_{22}, \\ \dot{k}_{13} &= k_{11} + k_{23}, \\ \dot{k}_{22} &= 2a_{21}^{sh}k_{12} + 2a_{22}^{sh}k_{22} + v_t^{sh}\cos^2\alpha + v_t^{sh}m_1\sin 2\alpha + v_t^{sh}(m_1^2 + k_{11})\sin^2\alpha, \\ \dot{k}_{23} &= k_{12} + a_{21}^{sh}k_{13} + a_{22}^{sh}k_{23}, \\ \dot{k}_{33} &= 2k_{13}, \\ k_{ij}(t_0) &= k_{ij0} (i, j = 1, 2, 3).\end{aligned}$$

Typical Figures 1 and 2 for ICS with parameters $J = 5$; $L = 64$; $\varepsilon = 0,7$; $\omega_c = 1$; $S = 1$; $t^{sh} = 0,135$;

$$n_t^{sh} = \begin{cases} 10, & t \in [t_{sh}; t_{sh} + 1], \\ 0, & t \notin [t_{sh}; t_{sh} + 1]. \end{cases} \quad v_t^{sh} = \begin{cases} 6, & t \in [t_{sh}; t_{sh} + 1], \\ 0, & t \notin [t_{sh}; t_{sh} + 1]. \end{cases}$$

show the accumulation effect of systematic and random errors for output variable Y_3 .

Typical Figure 1a,b for ICS with parameter $\alpha = 45^\circ$. Typical Figure 2a,b—with parameter $\alpha = 225^\circ$.

So, for single ShD and $0^\circ \leq \alpha \leq 180^\circ$, the following conclusions are valid:

- Systematic restricted error takes place for variables m_1, m_2 , whereas for variable m_3 , we have an accumulation growing effect (linear systematic drift);
- A random restricted effect takes place for variables k_{ij} ($i, j = 1, 2$), whereas for k_{33} , we have only the accumulation effect.

Figure 1. (a) Graphs m_1, m_2, m_3 at $\alpha = 45^\circ$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 45^\circ$.

Figure 2. (a) Graphs m_1, m_2, m_3 at $\alpha = 225^\circ$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 225^\circ$.

Let us consider the information-control system (ICS) of the third order at ShD described by the following equations:

$$\begin{aligned} \dot{Y}_1 &= Y_2, \\ \dot{Y}_2 &= -\omega_c^2 Y_1 - 2\varepsilon\omega_c Y_2 + \left(S + \sum_{k=1}^N \varphi_k(t_k) \left(n_t^{sh} + V_t^{sh} \right) \right) (\cos\alpha + Y_1 \sin\alpha), \\ \dot{Y}_3 &= Y_1, \\ Y_i(t_0) &= Y_{i0} (i = 1, 2, 3). \end{aligned}$$

Here, Y_i represents the components of the state vector $Y = [Y_1 \ Y_2 \ Y_3]^T$, $\alpha, \varepsilon, \omega_c, S, \varphi_k(t_k), N$ is the system parameters; n_t^{sh} is rectangular impulse function at time moment t_k ; α is the angle between acceleration $\left(S + \sum_{k=1}^N \varphi_k(t_k) \left(n_t^{sh} + V_t^{sh} \right) \right)$ and longitudinal ICS axis; V_t^{sh} is the white noise with intensity v_t^{sh} . The equation may be presented in the vector form:

$$\dot{Y} = a_0^{sh} + a^{sh} Y + \left(b_0^{sh} + b_1^{sh} Y \right) V_t^{sh},$$

where:

$$a_0^{sh} = \begin{bmatrix} 0 \\ (S + \sum_{k=1}^N \varphi_k(t_k) n_t^{sh}) \cos \alpha \\ 0 \end{bmatrix}, \quad a^{sh} = \begin{bmatrix} 0 & 1 & 0 \\ -\Omega_t^{sh} & -2\varepsilon\omega_c & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$b_0^{sh} = \begin{bmatrix} 0 \\ \cos \alpha \sum_{k=1}^N \varphi_k(t_k) \\ 0 \end{bmatrix}, \quad b_1^{sh} = \begin{bmatrix} 0 \\ \sin \alpha \sum_{k=1}^N \varphi_k(t_k) \\ 0 \end{bmatrix},$$

$$\Omega_t^{sh} = \omega_c^2 - \left(S + \sum_{k=1}^N \varphi_k(t_k) n_t^{sh} \right) \sin \alpha.$$

Equations for mathematical expectations $m_i = EY_i$ and elements of covariance matrix $k_{ij} = E[(Y_i - m_i)(Y_j - m_j)]$ are as follows:

$$\begin{aligned} \dot{m}_1 &= m_2, \\ \dot{m}_2 &= a_{21}^{sh} m_1 + a_{22}^{sh} m_2 + a_{02}^{sh}, \\ \dot{m}_3 &= m_1 \\ m_i(t_0) &= m_{i0} (i = 1, 2, 3); \\ \dot{k}_{11} &= 2k_{12}, \\ \dot{k}_{12} &= a_{21}^{sh} k_{11} + a_{22}^{sh} k_{12} + k_{22}, \\ \dot{k}_{13} &= k_{11} + k_{23}, \\ \dot{k}_{22} &= 2a_{21}^{sh} k_{12} + 2a_{22}^{sh} k_{22} + v_t^{sh} \left[(b_{02}^{sh})^2 + 2b_{02}^{sh} b_{12}^{sh} m_1 + (b_{02}^{sh})^2 (m_1^2 + k_{11}) \right], \\ \dot{k}_{23} &= k_{12} + a_{21}^{sh} k_{13} + a_{22}^{sh} k_{23}, \\ \dot{k}_{33} &= 2k_{13}, \\ k_{ij}(t_0) &= k_{ij0} (i, j = 1, 2, 3). \end{aligned}$$

Typical Figures 3–10 for ICS with parameters $J = 5; L = 64; \varepsilon = 0, 7; \omega_c = 1; S = 1; t_1 = 0, 135; t_k = t_1 + 2(k - 1); n_t^{sh} = 10; \dot{s}_t^{sh} = 6; k = \overline{1, N}$.

$$\varphi_k(t_k) = \begin{cases} 1, & t \in [t_k; t_k + 1], \\ 0, & t \notin [t_k; t_k + 1]. \end{cases}$$

show the accumulation effect of systematic and random errors for output variable Y_3 .

Typical Figure 3a,b for ICS with parameters $\alpha = 0^\circ, N = 1$. Typical Figure 4a,b—with parameters $\alpha = 180^\circ, N = 1$.

Typical Figure 5a,b for ICS with parameters $\alpha = 0^\circ, N = 2$. Typical Figure 6a,b—with parameters $\alpha = 180^\circ, N = 2$.

Typical Figure 7a,b for ICS with parameters $\alpha = 0^\circ, N = 4$. Typical Figure 8a,b—with parameters $\alpha = 180^\circ, N = 4$.

Typical Figure 9a,b for ICS with parameter $\alpha = 0^\circ, N = 10$. Typical Figure 10a,b—with parameters $\alpha = 180^\circ, N = 10$.

So, for multi-echelon ShD and $0^\circ \leq \alpha \leq 180^\circ$, we have the nonlinear systematic and random accumulation and drift effects for the corresponding variables.

At fixed N and various α , we have different quality graphs. So, at $N \rightarrow \infty$ mathematical expectations, m_1, m_2 are restricted, whereas m_3 is not restricted. Variances k_{11}, k_{22} are restricted and k_{33} is not restricted.

Figure 3. (a) Graphs m_1, m_2, m_3 at $\alpha = 0^\circ$, $N = 1$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 0^\circ$, $N = 1$.

Figure 4. (a) Graphs m_1, m_2, m_3 at $\alpha = 180^\circ$, $N = 1$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 180^\circ$, $N = 1$.

Figure 5. (a) Graphs m_1, m_2, m_3 at $\alpha = 0^\circ$, $N = 2$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 0^\circ$, $N = 2$.

Figure 6. (a) Graphs m_1, m_2, m_3 at $\alpha = 180^\circ$, $N = 2$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 180^\circ$, $N = 2$.

Figure 7. (a) Graphs m_1, m_2, m_3 at $\alpha = 0^\circ$, $N = 4$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 0^\circ$, $N = 4$.

Figure 8. (a) Graphs m_1, m_2, m_3 at $\alpha = 180^\circ$, $N = 4$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 180^\circ$, $N = 4$.

Figure 9. (a) Graphs m_1, m_2, m_3 at $\alpha = 0^\circ$, $N = 10$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 0^\circ$, $N = 10$.

Figure 10. (a) Graphs m_1, m_2, m_3 at $\alpha = 180^\circ$, $N = 10$; (b) Graphs k_{11}, k_{22}, k_{33} at $\alpha = 180^\circ$, $N = 10$.

6. Conclusions

For control stochastic systems at nonstationary shock disturbances described by linear stochastic differential equations with stochastic parametric noises, corresponding modeling methodological support and experimental software tools are developed. The methodology is based on the deterministic Haar–Galerkin algorithms for the solution of equations for the mathematical expectation, covariance matrix, and matrix of covariance functions.

Original new results include methods and algorithms of stochastic covariance analysis and modeling on the basis of the Galerkin method and wavelet expansion for linear, linear with parametric noises, and quasilinear control stochastic systems with complex shock disturbances. A new methodology may be called quick “analytical numerical modeling”. This methodology does not use Monte Carlo methods.

For the accuracy confirmation wavelet methodology, two special examples in the form of an information-control system at single and multi-echelon shock disturbances was presented. New nondeterministic error accumulations and drift effects are detected. Future works include nonlinear covariance analysis and probabilistic distributions problems in the field of nonlinear stochastic systems and dependent multi-shock disturbances.

The research was carried out using the infrastructure of shared research facilities CKP «Informatics» of FRC CSC RAS [27].

Author Contributions: Conceptualization, I.S.; methodology, I.S., V.S., T.K.; software, E.K., T.K. All authors have read and agreed to the published version of the manuscript.

Funding: The research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

CStS	Control Stochastic Systems
ICS	Information–Control System
MRA	MultiResolution Analysis
SDE	Stochastic Differential Equations
ShD	Shock Disturbances
WLE	Wavelet Expansion

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