

Article

# Subordination Properties of Meromorphic Kummer Function Correlated with Hurwitz–Lerch Zeta-Function

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**Abstract:** Recently, Special Function Theory (SPFT) and Operator Theory (OPT) have acquired a lot of concern due to their considerable applications in disciplines of pure and applied mathematics. The Hurwitz-Lerch Zeta type functions, as a part of Special Function Theory (SPFT), are significant in developing and providing further new studies. In complex domain, the convolution tool is a salutary technique for systematic analytical characterization of geometric functions. The analytic functions in the punctured unit disk are the so-called meromorphic functions. In this present analysis, a new convolution complex operator defined on meromorphic functions related with the Hurwitz-Lerch Zeta type functions and Kummer functions is considered. Certain sufficient stipulations are stated for several formulas of this defining operator to attain subordination. Indeed, these outcomes are an extension of known outcomes of starlikeness, convexity, and close to convexity.



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## 1. Introduction

During the 18th century, complex analysis (complex function theory) had been launched, which has become thereafter one of the major disciplines of mathematics. Prominent complex analysts include Euler, Riemann, and Cauchy. This realm has had a great influence on a variety of research subjects in, for example, engineering, physics, and mathematics, due to its efficient applications to numerous conceptions and problems. Researchers have happened to meet certain unexpected relationships among obviously different research areas. The study of the intriguing and fascinating interplay of geometry and complex analysis has been famed as Geometric Analytic Function Theory (GAFT). In other words, it deals with the structure of analytic functions in the complex domain whose specific geometries are starlike, close-to-starlike, convex, close-to-convex, spiral, and so on. In 1851, Riemann contributed to the origin of GAFT by presenting the first significant outcome, namely the Riemann mapping theorem (RMT). Koebe followed suit in 1907 and proceeded to the study of univalent function. In light of RMT, he initiated the discussion of the merits for univalent analytical functions over the open unit disk rather than in a complex domain. This modified version led to the creation of the Univalent Analytic Function Theory (UAFT). One of the gorgeous problems in UAFT is Bieberbach’s conjecture “coefficient conjecture” posed by Bieberbach in 1916. It states the upper bounds of the coefficient of the univalent function in the unit disk [1]. For many years, this conjecture posed a challenge to researchers in the field. Until 1985, De Branges [2] settled all attempts and resolved it.

The difficulty in resolving this conjecture led to several profound and significant contributions in GAFT along with the development of several gadgets. These involve Loewner’s parametric technique, Milin’s and Fitz Gerald’s techniques of exponentiating the Grunsky inequalities, Baernstein’s technique of maximal function, and variational techniques in addition to new subclasses of univalent functions imposed by geometric stipulation. Among the subclasses considered are the subclasses of convex functions, starlike functions, close-to-convex function, and quasi-convex functions, consistently. Besides, de Branges employed hypergeometric function, as a sort of the Special Function Theories (SPFT) in order to resolve the Bieberbach problem. From an application point of sight, SPFT are such significant mathematical tools for their interesting merits and remarkable role in the study of the Fractional Calculus (FRC) and Operator Theory (OPT), for instance, Ghanim and Al-Janaby [3]. Accordingly, SPFT plays a giant pivotal role in the development of research in the area of GAFT which includes a lot of new implementations and generalizations. For instance, Noor [4], El-Ashwah and Hassan [5], Xing and Jose Xing, Rassias and Yang ([6,7]), Ghanim and Al-Janaby ([8,9]) and Al-Janaby and Ghanim ([10,11]).

In this context, the term hypergeometric function, first coined by Wallis in the year 1655, also known as the hypergeometric series is in the complex plane  $\mathbb{C}$  and the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . This function was discussed by Euler first, and then systematically investigated by Gauss in 1813. It is formulated as [12]:

$${}_2F_1(q, v; \omega; z) = \sum_{\kappa=0}^{\infty} \frac{(q)_{\kappa} (v)_{\kappa} z^{\kappa}}{(\omega)_{\kappa} \kappa!}, \quad (q, v \in \mathbb{C}, \omega \in \mathbb{C} \setminus \{0, -1, \dots\}, |z| < 1).$$

Here  $(\omega)_{\kappa}$  is the Pochhammer (rising) symbol and is defined as:

$$(\omega)_{\kappa} = \begin{cases} 1 & \kappa = 0, \\ \omega(\omega + 1) \cdots (\omega + \kappa - 1) & \kappa \in \mathbb{N} = \{1, 2, \dots\}. \end{cases}$$

Subsequently, in 1837, Kummer presented the Kummer function, namely confluent hypergeometric function, as a solution of a Kummer differential equation. This function is written as [12]:

$$K(q; \omega, z) = \sum_{\kappa=0}^{\infty} \frac{(q)_{\kappa} z^{\kappa}}{(\omega)_{\kappa} \kappa!} = {}_1F_1(q; \omega; z), \tag{1}$$

$$(q \in \mathbb{C}, \omega \in \mathbb{C} \setminus \{0, -1, \dots\}, |z| < 1).$$

Furthermore, the Zeta functions constitute some phenomenal special functions that appear in the study of Analytic Number Theory (ANT). There are a number of generalizations of the Zeta function, such as Euler–Riemann Zeta function, Hurwitz Zeta function, and Lerch Zeta function. The Euler–Riemann Zeta function plays a pioneering role in ANT, due to its advantages in discussing the merits of prime numbers. It also has fruitful implementations in probability theory, applied statistics, and physics. Euler first formulated this function, as a function of a real variable, in the first half of the 18th century. Then, in 1859, Riemann utilized complex analysis to expand on Euler’s definition to a complex variable. Symbolized by  $S(z)$ , the definition was posed as the Dirichlet series:

$$S(z) = \sum_{\kappa=1}^{\infty} \frac{1}{\kappa^z}, \quad \text{for } \Re(z) > 1.$$

Later, the more general Zeta function, currently called Hurwitz Zeta function, was also propounded by Adolf Hurwitz in 1882, as a general formula of the Riemann Zeta function considered as [13]:

$$S(\mu, \varkappa) = \sum_{\kappa=0}^{\infty} \frac{1}{(\kappa + \mu)^{\varkappa}}, \quad \text{for } \Re(\varkappa) > 1, \Re(\mu) > 1.$$

More generally, the famed Hurwitz–Lerch Zeta function  $f(\mu, \varkappa, z)$  is described as [14]:

$$\phi_{\mu, \varkappa}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{(\kappa + \mu)^{\varkappa}}, \quad \text{for } \Re(\varkappa) > 1, \Re(\mu) > 1. \tag{2}$$

$$(\mu \in \mathbb{C} \setminus \mathbb{Z}_0^-, \varkappa \in \mathbb{C} \text{ when } |z| < 1; \Re(\varkappa) > 1 \text{ when } |z| = 1).$$

A generalization of (2) was proposed by Goyal and Laddha [15] in 1997, in the following formula:

$$\psi_{\mu, \varkappa}^{\wp}(z) = \sum_{\kappa=0}^{\infty} \frac{(\wp)_{\kappa}}{\kappa!} \frac{z^{\kappa}}{(\kappa + \mu)^{\varkappa}}, \quad \text{for } \Re(\varkappa) > 1, \Re(\mu) > 1. \tag{3}$$

$$(\mu \in \wp \in \mathbb{C} \setminus \mathbb{Z}_0^-, \varkappa \in \mathbb{C} \text{ when } |z| < 1; \Re(\varkappa - \wp) > 1 \text{ when } |z| = 1).$$

Along with these, there are more remarkable diverse extensions and generalizations that contributed to the rise of new classes of the Hurwitz–Lerch Zeta function in ([16–26]).

In this effort, by utilizing analytic techniques, a new linear (convolution) operator of morphometric functions is investigated and introduced in terms of the generalized Hurwitz–Lerch Zeta functions and Kummer functions. Moreover, sufficient stipulations are determined and examined in order for some formulas of this new operator to achieve subordination. Therefore, these outcomes are an extension for some well known outcomes of starlikeness, convexity, and close to convexity.

### 2. Preliminaries

Consider the class  $\mathcal{H}$  of regular functions in  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . The function  $f_1$  is named subordinate to  $f_2$  (or  $f_2$  is named superordinate to  $f_1$ ) and denotes  $f_1 \prec f_2$ , if there is a regular function  $\omega$  in  $\mathbb{D}$ , with  $\omega(0) = 0$  and  $|\omega(z)| < 1$  and  $f_1(z) = f_2(\omega(z))$ . If the function  $f_2$  is univalent in  $\mathbb{D}$ , then

$$f_1 \prec f_2 \Leftrightarrow f_1(0) = f_2(0) \quad \text{and} \quad f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

Let  $\Sigma$  represent the class of normalized meromorphic functions  $f(z)$  by

$$f(z) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \eta_{\kappa} z^{\kappa},$$

that are regular in the punctured unit disk

$$\mathbb{D}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}.$$

Furthermore, it indicates the classes of meromorphic starlike functions of order  $\zeta$  and meromorphic convex of order  $\zeta$  by  $\Sigma_{\mathcal{S}^*(\zeta)}$  and  $\Sigma_{\mathcal{K}(\zeta)}$ , ( $\zeta \geq 0$ ), respectively (see [22,23,27,28]).

The convolution product of two meromorphic functions  $f_{\ell}(z)$  ( $\ell = 1, 2$ ) in the following formula:

$$f_{\ell}(z) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \eta_{\kappa, \ell} z^{\kappa} \quad (\ell = 1, 2),$$

is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{\kappa=1}^{\infty} \eta_{\kappa,1} \eta_{\kappa,2} z^{\kappa}. \tag{4}$$

The meromorphic Kummer function  $\tilde{K}(\varrho; \omega, z)$  is formulated as:

$$\tilde{K}(\varrho; \omega, z) = \frac{1}{z} + \sum_{\kappa=0}^{\infty} \frac{(\varrho)_{\kappa+1}}{(\omega)_{\kappa+1}} \frac{z^{\kappa}}{(\kappa+1)!}, \tag{5}$$

$$(\varrho \in \mathbb{C}, \omega \in \mathbb{C} \setminus \{0, -1, \dots\}, z \in \mathbb{D}^*).$$

Corresponding to (5) and (3), based on a convolution tool, we imposed the following new convolution complex operator for  $f(z) \in \Sigma$  as:

$$\begin{aligned} L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z) &= \tilde{K}(\varrho; \omega, z) * \mathfrak{A}_{\varkappa, \mu}(z) * f(z) \\ &= \frac{1}{z} + \sum_{\kappa=1}^{\infty} \frac{(\varrho)_{\kappa+1} (\wp)_{\kappa+1}}{(\omega)_{\kappa+1} (\kappa+1)! (\kappa+1)!} \left(\frac{\mu+1}{\mu+\kappa+1}\right)^{\varkappa} \eta_{\kappa} z^{\kappa}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathfrak{A}_{\varkappa, \mu}(z) &= (\mu+1)^{\varkappa} \left[ \psi_{\mu, \varkappa}^{\wp}(z) - \frac{1}{\mu^{\varkappa}} + \frac{1}{z(\mu+1)^{\varkappa}} \right] \\ &= \frac{1}{z} + \sum_{\kappa=1}^{\infty} \frac{(\wp)_{\kappa+1}}{(\kappa+1)!} \left(\frac{\mu+1}{\mu+\kappa+1}\right)^{\varkappa} z^{\kappa} \quad (z \in \mathbb{U}^*) \end{aligned} \tag{7}$$

The major goal of this paper is to study the following subordinations:

$$\frac{L_{\mu}^{\varkappa}(\varrho+1, \omega, \wp)f(z)}{L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z)} \prec \frac{h(1-z)}{h-z}, \quad (h > 1)$$

$$\frac{L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z)}{z} \prec \frac{1+Ez}{1-z}, \quad (-1 \leq E < 1)$$

and

$$\frac{L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z)}{z} \prec \frac{h(1-z)}{h-z}, \quad (h > 1). \tag{8}$$

In particular, we obtain sufficient conditions for which the function  $f \in \Sigma$  satisfies such subordination, which extends certain outcomes in this direction concerning starlikeness, convexity, and close to convexity.

The following lemma will be needed to accomplish our proofs. We refer the reader to [29], Theorem 3.4, p. 132, for the proof of this lemma.

**Lemma 1.** Let  $q(z)$  be univalent in  $\mathbb{D}$  and let  $\Theta$  and  $\Phi$  be regular in a domain  $\mathfrak{D} \supset q(\mathbb{D})$ , with  $\Phi(\varpi) \neq 0$  when  $\varpi \in q(\mathbb{D})$ . Set

$$Y(z) = zq'(z)\Phi(q(z)), \quad \Lambda(z) = \Theta(q(z)) + Y(z)$$

Suppose that

- (1)  $Y(z)$  is starlike in  $\mathbb{D}$ , and

(2)  $\Re \frac{z\Lambda'(z)}{Y(z)} > 0$  for  $z \in \mathbb{D}$ .

If  $p(z)$  is regular in  $\mathbb{D}$  with  $p(0) = q(0)$ ,  $p(\mathbb{D}) \subset \mathfrak{D}$  and

$$\Theta(p(z)) + zp'(z)\Phi(p(z)) \prec \Theta(q(z)) + zq'(z)\Phi(q(z)), \tag{9}$$

then  $p(z) \prec q(z)$  and  $q(z)$  is the best dominant.

### 3. Main Outcomes

First, we treat the first subordinate in (8).

**Theorem 1.** Let  $\varrho > 0$ ,  $h > 1$ ,  $\zeta \in \mathbb{R}$  and  $f \in \Sigma$ . Then, if  $|\zeta| \leq 1$ ,  $L_\mu^\zeta(\varrho, \omega, \wp)f(z)/z \neq 0$  in  $\mathbb{D}^*$  and

$$\left( \frac{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho, \omega, \wp)f(z)} \right)^\zeta \left( (\varrho + 1) \frac{L_\mu^\zeta(\varrho + 2, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)} - 1 \right) \prec \Lambda(z), \tag{10}$$

where

$$\Lambda(z) = \left( \frac{h(1-z)}{h-z} \right)^{\zeta+1} \left( \varrho - \frac{(h-1)z}{h(1-z)^2} \right),$$

we have

$$\frac{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho, \omega, \wp)f(z)} \prec \frac{h(1-z)}{h-z}.$$

**Proof.** From (10) and the assumption

$$L_\mu^\zeta(\varrho, \omega, \wp)f(z)/z \neq 0$$

in  $\mathbb{D}^*$ , we infer that  $L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)/z \neq 0$  in  $\mathbb{D}^*$ . Define

$$p(z) = \frac{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho, \omega, \wp)f(z)}.$$

Then,  $p(z)$  is regular in  $\mathbb{D}^*$  and

$$\frac{zp'(z)}{p(z)} = \frac{z\left(L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)\right)'}{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)} - \frac{z\left(L_\mu^\zeta(\varrho, \omega, \wp)f(z)\right)'}{L_\mu^\zeta(\varrho, \omega, \wp)f(z)}. \tag{11}$$

By virtue of the identity

$$z\left(L_\mu^\zeta(\varrho, \omega, \wp)f(z)\right)' = \varrho\left(L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)\right) - (\varrho + 1)L_\mu^\zeta(\varrho, \omega, \wp)f(z)$$

and (11), we get

$$(\varrho + 1) \frac{L_\mu^\zeta(\varrho + 2, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)} = 1 + \varrho p(z) + \frac{zp'(z)}{p(z)}. \tag{12}$$

Now, (12) together with (10) imply

$$\varrho(p(z))^{\zeta+1} + zp'(z)(p(z))^{\zeta-1} \prec \Lambda(z). \tag{13}$$

Let

$$q(z) := \frac{h(1-z)}{h-z}.$$

Then,  $q$  is, clearly, convex in  $\mathbb{D}^*$  and

$$\Lambda(z) = \varrho(q(z))^{\zeta+1} + zq'(z)(q(z))^{\zeta-1}.$$

Let

$$\Theta(\omega) = \varrho\omega^{\zeta+1} \text{ and } \Phi(\omega) = \omega^{\zeta-1}.$$

Then, (13) may be written in the form of (9). Denoting  $zq'(z)\Phi(q(z))$  by  $Y(z)$  yields

$$Y(z) = \frac{(1-h)zh^{\zeta}(1-z)^{\zeta-1}}{(h-z)^{1+\zeta}},$$

and

$$\Lambda(z) = \Theta(q(z)) + Y(z) = \left(\frac{h(1-z)}{h-z}\right)^{1+\zeta} \left(\varrho - \frac{(h-1)z}{h(1-z)^2}\right).$$

But,  $h > 1$  and  $|\zeta| \leq 1$ . Hence,

$$\begin{aligned} \Re \frac{zY'(z)}{Y(z)} &= \Re \left(1 + \frac{z(1-\zeta)}{1-z} + (1-\zeta)\frac{z}{h-z}\right) \\ &> -1 + \frac{1}{2}(1-\zeta) + \frac{(1+\zeta)h}{1+h} \\ &= \frac{(1+\zeta)(h-1)}{2(1+h)} > 0. \end{aligned}$$

Consequently,  $Y(z)$  is starlike. Moreover,

$$\Re \frac{z\Lambda'(z)}{Y(z)} = \varrho(1+\zeta)\Re \frac{h(1-z)}{h-z} + \Re \frac{zY'(z)}{Y(z)} \geq 0.$$

By employing Lemma 1, we gain  $p(z) \prec q(z)$  that is

$$\frac{L_{\mu}^{\varkappa}(\varrho+1, \omega, \wp)f(z)}{L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z)} \prec \frac{h(1-z)}{h-z}.$$

The proof is completed.  $\square$

A special case of Theorem 1 is when  $\varkappa = \zeta = 0$  and  $\varrho = \omega = \wp = 1$ , where we get

**Corollary 1.** *If  $h > 1$  and  $f \in \Sigma$  attains  $f(z)/z \neq 0$  in  $\mathbb{D}^*$  and*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{h(1-z)}{h-z} - \frac{(h-1)z}{(h-z)(1-z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{h(1-z)}{h-z}.$$

**Remark 1.** *When  $z \in \mathbb{R}$ ,*

$$\Lambda(z) = \frac{h(1-z)}{h-z} - \frac{(h-1)z}{(h-z)(1-z)} = \frac{z}{h-z} + \frac{1}{1-z} \in \mathbb{R}.$$

Moreover,  $\Lambda(0) = 1$  and  $\Lambda(\mathbb{D}) = \Re\Lambda(z) < \frac{(h+1)}{2(h-1)}$  for  $1 < h \leq 2$  and  $\Re\Lambda(z) < \frac{(5h-1)}{2(h+1)}$  for  $2 < h$ . Hence, this outcome is a generalization of the outcome obtained in [30].

Note that, when

$$\Lambda(z) = 1 - \frac{(h-1)z}{h(1-z)^2},$$

$$\Lambda(\mathbb{D}) = \mathbb{C} - \left[ \frac{5h-1}{4h}, \infty \right).$$

Thus, setting  $\varkappa = 0$ ,  $\zeta = -1$  and  $\varrho = \omega = \wp = 1$  in the Theorem 1 implies the following outcome:

**Corollary 2.** Let  $h > 1$  and  $f \in \Sigma$  satisfy  $f(z)/z \neq 0$  in  $\mathbb{D}^*$  and

$$\Re \left( \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \right) < \frac{5h-1}{4h}.$$

Then,

$$\frac{zf'(z)}{f(z)} \prec \frac{h(1-z)}{h-z}.$$

**Theorem 2.** Let  $\varrho > 0$ ,  $-1 \leq \zeta < 0$ ,  $-1 \leq E < 1$  and  $f \in \Sigma$ . If  $L_\mu^\varkappa(\varrho, \omega, \wp)f(z)/z \neq 0$  in  $\mathbb{D}^*$  and

$$\left( \frac{L_\mu^\varkappa(\varrho, \omega, \wp)f(z)}{z} \right)^\zeta \left( \varrho \frac{L_\mu^\varkappa(\varrho+1, \omega, \wp)f(z)}{z} \right) \prec \Lambda(z), \tag{14}$$

for

$$\Lambda(z) = \left( \frac{1+ Ez}{1-z} \right)^\zeta \left( \varrho \frac{1+ Ez}{1-z} + \frac{(1+E)z}{(1-z)^2} \right),$$

then

$$\frac{L_\mu^\varkappa(\varrho, \omega, \wp)f(z)}{z} \prec \frac{1+ Ez}{1-z}.$$

**Proof.** Let

$$p(z) = \frac{L_\mu^\varkappa(\varrho, \omega, \wp)f(z)}{z}. \tag{15}$$

Then, clearly  $p$  is regular in  $\mathbb{D}^*$ . Then, it follows by (9), that

$$\varrho \left( L_\mu^\varkappa(\varrho+1, \omega, \wp)f(z) \right)' = zp'(z) - (\varrho-1)p(z). \tag{16}$$

Thus, (14) becomes

$$\varrho p(z)^{1+\zeta} + p(z)^\lambda zp'(z) \prec \Lambda(z).$$

Now, we define  $q(z)$  by

$$q(z) = \frac{1 + Ez}{1 - z}.$$

Then,  $q(z)$  is univalent in  $\mathbb{D}$  and  $q(\mathbb{D}) = \{z : \Re q(z) > (1 - E)/2\}$ . Let  $\Theta$  and  $\Phi$  be

$$\Theta(\omega) = \varrho \omega^{\zeta+1} \quad \text{and} \quad \Phi(\omega) = \omega^\zeta.$$

After that,  $\Phi$  and  $\Theta$  are regular in  $\mathbb{C} \setminus \{0\}$ , and (14) has the form of (9). Moreover, by letting

$$Y(z) = zq'(z)\Phi(q(z)) = \frac{(1 + E)z(1 + Ez)^\zeta}{(1 - z)^{2+\zeta}},$$

we have

$$\Lambda(z) = f(q(z)) + Y(z).$$

Next, by the assumptions of the theorem,

$$\begin{aligned} \Re \frac{z\Lambda'(z)}{Y(z)} &= \Re \left[ 1 + \zeta \frac{Ez}{1 + Ez} + (2 + \zeta) \frac{z}{1 - z} \right] \\ &> 1 - \frac{\zeta|E|}{1 + |E|} - \frac{2 + \lambda}{2} = \frac{-\zeta(1 - |E|)}{2(1 + |E|)} > 0, \end{aligned}$$

and

$$\Re \frac{z\Lambda'(z)}{Y(z)} = \Re \left[ \frac{f'(q(z))}{f(q(z))} + \frac{zY'(z)}{Y(z)} \right] = \varrho(1 + \zeta) + \Re \frac{zY'(z)}{Y(z)} \geq 0.$$

An application of Lemma 1 now yields the result.  $\square$

Since the function  $\Lambda(z) = \varrho + \frac{1 + Ez}{(1 - z)(1 + Ez)}$  maps real values to real values,  $\Lambda(0) = \varrho$ ,  $\Lambda(\mathbb{D})$  is symmetric with respect to the real axis and

$$\Re \Lambda(z) > \varrho + \frac{1}{2} - \frac{1}{1 - |E|}, \quad z \in \mathbb{D}^*,$$

we may apply Theorem 2 by letting  $\zeta = -1$  to get the following.

**Corollary 3.** *Let  $-1 < E < 1$ ,  $\varrho > 0$  and  $f \in \Sigma$ . If  $L_a^t(\nu, \tau)f(z)/z \neq 0$  in  $\mathbb{D}^*$  and*

$$\Re \left( \frac{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)}{L_\mu^\zeta(\varrho, \omega, \wp)f(z)} \right) > 1 + \frac{1}{2\varrho} - \frac{1}{\varrho(1 - |E|)},$$

then

$$\frac{L_\mu^\zeta(\varrho, \omega, \wp)f(z)}{z} \prec \frac{1 + Ez}{1 - z}.$$

**Theorem 3.** *Let  $\zeta \geq -1$ ,  $h > 1$  and  $f \in \Sigma$ . If  $L_\mu^\zeta(\varrho, \omega, \wp)f(z)/z \neq 0$  in  $\mathbb{D}^*$  and*

$$\left( \frac{L_\mu^\zeta(\varrho, \omega, \wp)f(z)}{z} \right)^\zeta \left( \frac{L_\mu^\zeta(\varrho + 1, \omega, \wp)f(z)}{z} \right) \prec \frac{h^{1+\zeta}(1 - z)^\zeta}{(h - z)^{1+\zeta}} \left( \varrho(1 - z) - \frac{h(1 - z)}{h - z} \right),$$

then

$$\frac{L_{\mu}^{\varkappa}(\varrho, \omega, \wp)f(z)}{z} \prec \frac{h(1-z)}{h-z}.$$

**Proof.** The outcome yields from Lemma 1 by defining the functions  $\Phi$  and  $\Theta$  by  $\Theta(\omega) = \varrho \omega^{-(1+\zeta)}$  and  $\Phi(\omega) = -\omega^{-(2+\zeta)}$ .  $\square$

Observe that  $\Re\left(1 - \frac{(h-1)z}{(h-z)(1-z)}\right) < \frac{3h-1}{2(h-1)}$  when  $z \in \mathbb{D}^*$ . Hence, when letting  $\varrho = \omega = \wp = 1$  and  $\varkappa = 0$  in the above Theorem, we get the following.

**Corollary 4.** Let  $h > 1$  and  $f \in \Sigma$ . If  $f'(z) \neq 0$  in  $\mathbb{D}$  and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3h-1}{2(h-1)},$$

then

$$f'(z) \prec \frac{h(1-z)}{h-z}.$$

#### 4. Conclusions

In this analytic investigation, based on convolution concept, we have defined and applied prosperously a complex linear operator which is associated with the meromorphic Hurwitz–Lerch Zeta type functions and Kummer functions. By utilizing this new linear operator, we have discussed several interesting merits of some new geometric subclasses of meromorphic univalent functions in the punctured unit disk  $\mathbb{D}^*$ .

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