Supplementary Material

Part A: Graph Families

1. mod 4

   Class 1: $G_8[0,1,2]$: Cube $\text{fix}(G)=3$

2. mod 5

   Class 1: $G_{10}[0,1,2]$: Circulant Graph $C_{10}(1,5)$, $\text{fix}(G)=2$

3. mod 6

   Class 1: $G_{12}[0,1,2]$: $P_2 \times C_6$, $\text{fix}(G)=2$

   Class 2: $G_{12}[0,1,3]$: Half-step graph (Theorem 4.5), $\text{fix}(G)=3$

   Class 3: $G_{12}[0,2,4]$: $2K_{3,3}$: $\text{fix}(G)=10$

4. mod 7

   Class 1: $G_{14}[0,1,2]$: Circulant Graph $C_{14}(1,7)$, $\text{fix}(G)=2$

   Class 2: $G_{14}[0,1,3]$: $\text{fix}(G) \leq 3$

5. mod 8

   Class 1: $G_{16}[0,1,2]$: $P_2 \times C_8$, $\text{fix}(G)=2$

   Class 2: $G_{16}[0,1,3]$: $\text{fix}(G) \leq 3$

   Class 3: $G_{16}[0,1,4]$: Half-step graph, (Theorem 4.5) $\text{fix}(G)=4$

   Class 4: $G_{16}[0,2,4]$: $2\,(P_2 \times C_4)$, $\text{fix}(G)=4$

6. mod 9

   Class 1: ML_{18}, $\text{fix}(G)=2$, $G_{18}[0,1,2]$, $G_{18}[0,2,4]$, and $G_{18}[0,1,5]$.

   Class 2: $G_{18}[0,1,3]$, $\text{fix}(G)=1$, $G_{18}[0,1,3]$, $G_{18}[0,1,7]$, $G_{18}[0,1,4]$,
   $G_{18}[0,1,6]$, $G_{18}[0,2,5]$, and $G_{18}[0,2,6]$.

   Class 3: $3K_{3,3}$, $\text{fix}(G)=12$, $G_{18}[0,3,6]$

7. mod 10

   Class 1: $P_2 \times C_{10}$, $\text{fix}(G)=2$: $G_{20}[0,1,2]$, and $G_{20}[0,3,6]$

   Class 2: $2ML_{10}$, $\text{fix}(G)=4$: $G_{20}[0,2,4]$, and $G_{20}[0,4,8]$

   Class 3: $\text{fix}(G)=1$ (any vertex can be fixed): $G_{20}[0,1,3]$, $G_{20}[0,1,4]$, $G_{20}[0,1,7]$, and $G_{20}[0,1,8]$

   Class 4: Half-step (Theorem 4.5), $\text{fix}(G)=5$: $G_{20}[0,1,5]$, $G_{20}[0,1,6]$, $G_{20}[0,2,5]$, and $G_{20}[0,2,7]$

8. mod 11

   Class 1: ML_{22}, $\text{fix}(G)=2$: $G_{22}[0,1,2]$, $G_{22}[0,2,4]$, $G_{22}[0,3,6]$, $G_{22}[0,4,8]$, and $G_{22}[0,5,10]$.

   Class 2: $\text{fix}(G)=1$ (any vertex can be fixed): $G_{22}[0,1,3]$, $G_{22}[0,1,4]$, $G_{22}[0,1,5]$, $G_{22}[0,1,7]$,
   $G_{22}[0,1,8]$, $G_{22}[0,1,9]$, $G_{22}[0,2,5]$, $G_{22}[0,2,6]$, $G_{22}[0,2,7]$, and $G_{22}[0,2,8]$.

9. mod 12

   Class 1: $P_2 \times C_{12}$, $\text{fix}(G)=2$: $G_{24}[0,1,2]$, $G_{24}[0,5,10]$

   Class 2: $\text{fix}(G)=1$ (any vertex can be fixed): $G_{24}[0,1,3]$, $G_{24}[0,1,10]$, $G_{24}[0,2,5]$, and $G_{24}[0,2,9]$.

   Class 3: $\text{fix}(G)=1$ (any vertex can be fixed): $G_{24}[0,1,4]$, $G_{24}[0,1,9]$, $G_{24}[0,3,7]$, and $G_{24}[0,3,8]$.

Class 4: $\mathrm{fix}(G) = 2$: $G_{24}[0,1,5]$, $G_{24}[0,1,8]$

Class 5: Half-step graph, $\mathrm{fix}(G) = 6$, $G_{24}[0,1,6]$, $G_{24}[0,1,7]$

Class 6: $G_{24}[0,2,4]$: $2\,(P_2 \times C_6)$, $\mathrm{fix}(G) = 4$

Class 7: $G_{24}[0,2,6]$, $G_{24}[0,2,8]$: Two half-step graphs, $\mathrm{fix}(G) = 6$

Class 8: $G_{24}[0,3,6]$: $3\,(P_2 \times C_4)$, $\mathrm{fix}(G) = 6$

Class 9: $G_{24}[0,4,8]$: $4K_{3,3}$, $\mathrm{fix}(G) = 16$

10. mod 13

    Class 1: ML_{26},$\mathrm{fix}(G) = 2$:$G_{26}[0,1,2]$, $G_{26}[0,2,4]$,$G_{26}[0,3,6]$,$G_{26}[0,4,8]$, $G_{26}[0,5,10]$, and $G_{26}[0,6,12]$

    Class 2: $\mathrm{fix}(G) = 1$ (any vertex can be fixed): $G_{26}[0,1,3]$, $G_{26}[0,1,5]$, $G_{26}[0,1,6]$, $G_{26}[0,1,8]$,

    $G_{26}[0,1,9]$, $G_{26}[0,1,11]$, $G_{26}[0,2,5]$, $G_{26}[0,2,6]$, $G_{26}[0,2,9]$, $G_{26}[0,2,10]$, $G_{26}[0,3,7]$, and $G_{26}[0,3,9]$

    Class 3: $\mathrm{fix}(G) = 1$ (any vertex can be fixed): $G_{26}[0,1,4]$, $G_{26}[0,1,10]$, $G_{26}[0,2,7]$, and $G_{26}[0,2,8]$.

11. mod 14

    Class 1: $P_2 \times C_{14}$, $\mathrm{fix}(G) = 2$: $G_{28}[0,1,2]$, $G_{28}[0,3,6]$, and $G_{28}[0,5,10]$

    Class 2: $2ML_{14}$, $\mathrm{fix}(G) = 4$: $G_{28}[0,2,4]$, $G_{28}[0,4,8]$, and $G_{28}[0,6,12]$

    Class 3: $\mathrm{fix}(G) = 1$, any vertex: $G_{28}[0,1,3]$, $G_{28}[0,1,12]$, $G_{28}[0,1,5]$, $G_{28}[0,1,10]$, $G_{28}[0,3,8]$, and $G_{28}[0,3,9]$.

    Class 4: $\mathrm{fix}(G) = 1$, any vertex: $G_{28}[0,1,4]$, $G_{28}[0,1,11]$, $G_{28}[0,1,6]$,

    $G_{28}[0,1,9]$, $G_{28}[0,2,5]$, and $G_{28}[0,2,11]$.

    Class 5: half-step, $\mathrm{fix}(G) = 7$, $G_{28}[0,1,7]$, $G_{28}[0,1,8]$, $G_{28}[0,2,7]$,

    $G_{28}[0,2,9]$, $G_{28}[0,3,7]$, and $G_{28}[0,3,10]$.

    Class 6. Two disjoint graphs, $\mathrm{fix}(G) = 6$: $G_{28}[0,2,6]$, and $G_{28}[0,2,10]$.

12. mod 15

    Class 1: ML_{15}, $\mathrm{fix}(G) = 2$: $G_{30}[0,1,2]$, $G_{30}[0,2,4]$, $G_{30}[0,4,8]$, and

    $G_{30}[0,7,14]$.

    Class 2: $3ML_5$, $\mathrm{fix}(G) = 6$: $G_{30}[0,3,6]$, and $G_{30}[0,6,12]$.

    Class 3: $5ML_3$, $\mathrm{fix}(G) = 10$: $G_{30}[0,5,10]$.

    Class 4: $\mathrm{fix}(G) = 1$:any vertex: $G_{30}[0,1,3]$, $G_{30}[0,1,13]$,$G_{30}[0,1,7]$,

    $G_{30}[0,1,9]$,$G_{30}[0,2,6]$,$G_{30}[0,2,11]$,$G_{30}[0,3,7]$, and $G_{30}[0,3,11]$.

    Class 5: $\mathrm{fix}(G) = 2$: $G_{30}[0,1,4]$, $G_{30}[0,1,12]$, $G_{30}[0,2,8]$, and $G_{30}[0,2,9]$.

    Class 6: $\mathrm{fix}(G) = 2$: $G_{30}[0,1,5]$, $G_{30}[0,1,11]$, $G_{30}[0,2,7]$, and $G_{30}[0,2,10]$.

    Class 7: $\mathrm{fix}(G) = 1$: any vertex: $G_{30}[0,1,6]$, $G_{30}[0,1,10]$, $G_{30}[0,2,5]$, $G_{30}[0,2,12]$,

    $G_{30}[0,3,8]$, $G_{30}[0,3,10]$, $G_{30}[0,4,9]$, and $G_{30}[0,4,10]$.

13. mod 16

    Class 1: $P_2 \times C_{16}$: $\mathrm{fix}(G) = 2$: $G_{32}[0,1,2]$, $G_{32}[0,3,6]$, $G_{32}[0,5,10]$, and $G_{32}[0,7,14]$.

    Class 2: $2\,(P_2 \times C_8)$, $\mathrm{fix}(G) = 4$: $G_{32}[0,2,4]$, and $G_{32}[0,6,12]$.

    Class 3: $4\,(P_2 \times C_4)$: $G_{32}[0,4,8]$.

    Class 4: $\mathrm{fix}(G) = 1$: any vertex: $G_{32}[0,1,3]$, $G_{32}[0,1,14]$, $G_{32}[0,1,6]$, $G_{32}[0,1,11]$,

    $G_{32}[0,2,7]$, $G_{32}[0,2,11]$, $G_{32}[0,3,9]$, and $G_{32}[0,3,10]$.

Class 5: $\text{fix}(G) = 1$: any vertex: $G_{32}[0, 1, 4]$, $G_{32}[0, 1, 13]$, $G_{32}[0, 1, 5]$, $G_{32}[0, 1, 12]$, $G_{32}[0, 3, 7]$, $G_{32}[0, 3, 12]$, $G_{32}[0, 4, 9]$, and $G_{32}[0, 4, 11]$.

Class 6: $\text{fix}(G) = 2$: $G_{32}[0, 1, 7]$, $G_{32}[0, 1, 10]$, $G_{32}[0, 2, 5]$, and $G_{32}[0, 2, 13]$.

Class 7: half-step: $\text{fix}(G) = 8$: $G_{32}[0, 1, 8]$, $G_{32}[0, 1, 9]$, $G_{32}[0, 3, 8]$, and $G_{32}[0, 3, 11]$.

Class 8: $\text{fix}(G) = 4$: $G_{32}[0, 2, 6]$, and $G_{32}[0, 2, 12]$.

Class 9: $\text{fix}(G) = 4$: $G_{32}[0, 2, 8]$, and $G_{32}[0, 2, 10]$.

We note the cases become increasingly complex for cases when n is highly composite.

Part B: Remaining cases from Theorem 4.2

1. 1 fixed block - second possibility $[k-1, 0, 2] \to [k-1, 0, 2]$, $[0, 1, 3] \to [k-3, k-2, 0]$, $[k-3, k-2, 0] \to [0, 1, 3]$

This implies that $k-1 \to \{k-1, 2\}$, $2 \to \{k-1, 2\}$, $1 \to \{k-3, k-2\}$, $3 \to \{k-3, k-2\}$, $k-3 \to \{1, 3\}$ and $k-2 \to \{1, 3\}$.

We then consider the block $[2, 3, 5]$. It has to be mapped to a block with a $k-1$ or 2 and either a $k-3$ or $k-2$. We consider the four following cases:

- $2 \to k-1$, $3 \to k-3$

  Then the block $[2, 3, 5]$ would be mapped to a block with a $k-1$ and a $k-3$. This would be the block $[k-4, k-3, k-1]$, implying that $5 \to k-4$, and also that $k-1 \to 2$ and $1 \to k-2$. Then the block $[1, 2, 4]$ has to be mapped to a block with a $k-2$ and a $k-1$. This would be the block $[k-2, k-1, 1]$, implying that $4 \to 1$. This is a contradiction because only $k-3$ or $k-2$ can be mapped to 1.

- $2 \to k-1$, $3 \to k-2$

  In this case, $[2, 3, 5]$ is going to be mapped to a block with a $k-1$ and a $k-2$. This is the block $[k-2, k-3, 1]$, further implying that $5 \to 1$. This is a contradiction because only $k-3$ or $k-2$ can be mapped to 1.

- $2 \to 2$, $3 \to k-2$

  In this case, the block $[2, 3, 5]$ must be mapped to a block with a 2 and a $k-2$. Such block does not exist.

- $2 \to 2$, $3 \to k-3$

  In this case, $[2, 3, 5]$ must be mapped to a block with a 2 and a $k-3$. Such block does not exist.

1. 1 Fixed block - third possibility

$[k-3, k-2, 0] \to [k-3, k-2, 0]$, $[0, 1, 3] \to [k-1, 0, 2]$, $[k-1, 0, 2] \to [0, 1, 3]$

This implies that $k-3 \to \{k-3, k-2\}$, $k-2 \to \{k-3, k-2\}$, $1 \to \{k-1, 2\}$, $3 \to \{k-1, 2\}$, $k-1 \to \{1, 3\}$ and $2 \to \{1, 3\}$

We first consider the block $[2, 3, 5]$. It must be mapped to a block with a 1 or 3 and either a $k-1$ or 2. We consider the four following cases:

- $2 \to 1$, $3 \to 2$

  In this case, $[2, 3, 5]$ would have to be mapped to a block with a 1 and a 2, which would be the block $[1, 2, 4]$, implying $5 \to 4$, $k-1 \to 3$ and $1 \to k-1$. Then the block $[1, 2, 4]$ would be mapped to a block with a $k-1$ and a 1, which would be the block $[k-2, k-1, 1]$, implying that $4 \to k-2$. This is a contradiction because only $k-3$ or $k-2$ can be mapped to $k-2$.

- $2 \to 1$, $3 \to k-1$

  In this case, $[2, 3, 5]$ would have to be mapped to a block with a 1 and a $k-1$. This would be the block $[k-2, k-1, 1]$, implying that $5 \to k-2$. This is a contradiction because only $k-3$ or $k-2$ can be mapped to $k-2$.

- $2 \to 3$, $3 \to 2$

  In this case, $[2, 3, 5]$ has to be mapped to a block with a 3 and a 2. This would mean $[2, 3, 5] \to [2, 3, 5]$, implying $5 \to 5$, $k-1 \to 1$ and $1 \to k-1$. In this case, the block $[1, 2, 4]$ would have to be mapped to a block with a 3 and a $k-1$. Such block does not exist.

- $2 \to 3$, $3 \to k-1$

  In this case, the block $[2, 3, 5]$ would have to be mapped to a block with a 3 and a $k-1$. Such block does not exist.