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Quasilinear Dirichlet Problems with Degenerated p-Laplacian and Convection Term

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Abstract: The paper develops a sub-supersolution approach for quasilinear elliptic equations driven by degenerated p-Laplacian and containing a convection term. The presence of the degenerated operator forces a substantial change to the functional setting of previous works. The existence and location of solutions through a sub-supersolution is established. The abstract result is applied to find nontrivial, nonnegative and bounded solutions.

Keywords: quasilinear elliptic problem; degenereted p-Laplacian; convection term; sub-supersolution; nonnegative solution

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1. Introduction

In this paper, we study the following quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u) = f(x,u,\nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $p \in (1, N)$. We assume that the boundary $\partial\Omega$ of Ω is locally Lipschitzian, i.e., each point of $\partial\Omega$ has a neighborhood whose intersection with $\partial\Omega$ is the graph of a Lipschitz continuous function. Throughout the text we denote by $|\cdot|$ and $\cdot$ the standard Euclidean norm and scalar product on $\mathbb{R}^N$, respectively. A main feature of the present work is that the leading part of the equation in (P) is the differential operator in divergence form $\mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$ known as the degenerated p-Laplacian with the weight $a \in L^1_{\mathrm{loc}}(\Omega)$. It is supposed that the function a be positive almost everywhere in Ω and that the following condition holds

$$a^{-s} \in L^1(\Omega) \text{ for some } s \in \left(\frac{N}{p}, +\infty\right) \cap \left[\frac{1}{p-1}, +\infty\right). \tag{1}$$

In the case where $a(x) \equiv 1$ we recover the ordinary p-Laplacian. Various examples of useful weights meeting the requirement (1) are given in [1]. For instance, it is obvious that defining $a(x) = \mathrm{dist}(x, S)$ for $x \in \Omega$, with a nonempty closed subset S of $\partial\Omega$, one obtains a function a on Ω for which (1) holds true with any listed s.

The natural space associated with problem (P) is $W_0^{1,p}(a,\Omega)$ that is the closure of $C_0^\infty(\Omega)$ in the weighted Sobolev space $W^{1,p}(a,\Omega)$. In Section 2 we briefly survey the spaces $W^{1,p}(a,\Omega)$ and $W_0^{1,p}(a,\Omega)$. The (negative) degenerated p-Laplacian with the weight $a \in L^1_{\mathrm{loc}}(\Omega)$ under condition (1) is defined on $W_0^{1,p}(a,\Omega)$ and takes values in the dual space $(W_0^{1,p}(a,\Omega))^*$.

Corresponding to the constant s in (1) we set

$$p_s = \frac{ps}{s+1}$$

and the Sobolev critical exponent $p_s^* = \frac{Np_s}{N-p_s}$ (we note that $1 \le p_s < N$). There is a continuous embedding $W^{1,p}(a,\Omega) \hookrightarrow L^{p_s^*}(\Omega)$, so a continuous embedding $L^{(p_s^*)'}(\Omega) \hookrightarrow (W_0^{1,p}(a,\Omega))^*$, where $(p_s^*)'$ stands for the Hölder conjugate of p_s^*, i.e., $(p_s^*)' = \frac{p_s^*}{p_s^*-1}$. In order to handle problem (P) the idea is to arrange that the right-hand side $f(x,u,\nabla u)$ become an element of $L^{(p_s^*)'}(\Omega)$, which basically will be achieved through an adequate growth condition (see Hypothesis 1). We emphasize that the nonlinearity $f(x,u,\nabla u)$ depends on the solution u and on its gradient ∇u, which generally makes the variational methods be ineffective. Such a term $f(x,u,\nabla u)$ is often called convection. It is expressed by means of a function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ that is Carathéodory, i.e., $f(\cdot,t,\xi)$ is measurable for every $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x,\cdot,\cdot)$ is continuous for a.e. $x \in \Omega$.

The goal of our work is to build a systematical approach to problem (P) via the method of sub-supersolution. It is for the first time when the method of sub-supersolution is implemented for problem (P) involving the degenerated p-Laplacian and related convection. In this respect, the functional setting is adapted to the novel situation of degenerated operators relying in an essential way on the associated exponent p_s. For results on the method of sub-supersolution applied to problems exhibiting convection terms but not driven by degenerated differential operators we refer to [2–6].

By a (weak) solution to problem (P) we mean a function $u \in W_0^{1,p}(a,\Omega)$ such that $f(x,u,\nabla u) \in L^{(p_s^*)'}(\Omega)$ and

$$\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx = \int_\Omega f(x,u(x),\nabla u(x))v(x)dx, \ \forall v \in W_0^{1,p}(a,\Omega). \quad (2)$$

A function $\underline{u} \in W^{1,p}(a,\Omega)$ is called a subsolution for problem (P) if $\underline{u} \le 0$ on $\partial\Omega$ (in the sense of traces), $f(\cdot,\underline{u}(\cdot),\nabla\underline{u}(\cdot)) \in L^{(p_s^*)'}(\Omega)$ and

$$\int_\Omega a(x)|\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x) \cdot \nabla v(x)dx \le \int_\Omega f(x,\underline{u}(x),\nabla \underline{u}(x))v(x)dx \quad (3)$$

for all $v \in W_0^{1,p}(a,\Omega)$, $v \ge 0$ a.e. in Ω. Symmetrically, a function $\overline{u} \in W^{1,p}(a,\Omega)$ is called a supersolution for problem (P) if $\overline{u} \ge 0$ on $\partial\Omega$ (in the sense of traces), $f(\cdot,\overline{u}(\cdot),\nabla\overline{u}(\cdot)) \in L^{(p_s^*)'}(\Omega)$ and

$$\int_\Omega a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx \ge \int_\Omega f(x,\overline{u}(x),\nabla \overline{u}(x))v(x)dx \quad (4)$$

for all $v \in W_0^{1,p}(a,\Omega)$, $v \ge 0$ a.e. in Ω. Corresponding to a subsolution $\underline{u}$ and a supersolution $\overline{u}$ with $\underline{u} \le \overline{u}$ a.e. in Ω we can consider the ordered interval

$$[\underline{u},\overline{u}] = \{w \in W^{1,p}(a,\Omega) : \underline{u} \le w \le \overline{u}\}.$$

The following hypothesis for $f(x,s,\xi)$ is adapted to an ordered sub-supersolution $\underline{u} \le \overline{u}$.

Hypothesis 1. *Given an ordered sub-supersolution $\underline{u} \le \overline{u}$ for problem (P), the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the growth condition*

$$|f(x,t,\xi)| \le \sigma(x) + b|\xi|^r \quad \text{for a.e. } x \in \Omega, \ \text{for all } t \in [\underline{u}(x),\overline{u}(x)], \quad \xi \in \mathbb{R}^N,$$

with a function $\sigma \in L^{\frac{p_s}{r}}(\Omega)$ and constants $b > 0$ and $r \in (0, \frac{p_s}{(p_s^*)'})$.

According to Hypothesis 1 we have

$$f(x,u,\nabla u) \in L^{(p_s^*)'}(\Omega),\ \forall u \in [\underline{u}, \overline{u}],$$

thus the integrals in the definitions above exist since

$$f(x,u,\nabla u)v \in L^1(\Omega),\ \forall u \in [\underline{u}, \overline{u}],\ v \in W_0^{1,p}(a,\Omega).$$

Under Hypothesis 1, our main result establishes the existence of a weak solution to problem (P) with the additional location property $u \in [\underline{u}, \overline{u}]$. We stress that this location property represents a significant qualitative information for the solution giving actually a priori estimates for it. As an application we prove the existence of a nontrivial nonnegative solution for a class of problems of type (P). The applicability of the stated result is demonstrated by an example.

2. Preliminary Material

The notation $|\Omega|$ stands for the Lebesgue measure of the bounded domain Ω in $\mathbb{R}^N$. In this section we discuss a few facts about the degenerated p-Laplacian entering problem (P). More details can be found in [1].

We note that (1) implies

$$a^{-\frac{1}{p-1}} \in L^1(\Omega).$$

Indeed, it is seen that

$$\begin{aligned} \int_\Omega a(x)^{-\frac{1}{p-1}} dx &= \int_{\{a(x)<1\}} a(x)^{-\frac{1}{p-1}} dx + \int_{\{a(x)\geq 1\}} a(x)^{-\frac{1}{p-1}} dx \\ &\leq \int_{\{a(x)<1\}} a(x)^{-s} dx + |\Omega| < \infty \end{aligned}$$

since according to (1) one has $s \geq \frac{1}{p-1}$ and $a^{-s} \in L^1(\Omega)$.

The weighted Sobolev space $W^{1,p}(a,\Omega)$ consists of all the functions $u \in L^p(\Omega)$ for which $a^{\frac{1}{p}}|\nabla u| \in L^p(\Omega)$. It is endowed with the norm

$$\|u\|_{1,p,a} = \left(\int_\Omega |u|^p dx + \int_\Omega a(x)|\nabla u|^p dx \right)^{\frac{1}{p}}$$

becoming a uniformly convex Banach space (due to the preceding property of the weight $a(x)$, see ([1], [Theorem 1.3])), thus reflexive, that contains $C_0^\infty(\Omega)$. The space $W_0^{1,p}(a,\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p,a}$.

There is an extensive literature devoted to the weighted Sobolev spaces including embeddings and traces related to different boundary value problems (see, e.g., [1,7,8]). The results depend strongly on what type of weight is used, generally attempting reduction to nonweighted spaces. As described below, under assumption (1), we can embed the space $W^{1,p}(a,\Omega)$ into the ordinary Sobolev space $W^{1,p_s}(\Omega)$, hence automatically having the trace (note the boundary $\partial\Omega$ is Lipschitz). This fact is needed in the definition of the sub-supersolution.

From (1) it is known that $s \geq \frac{1}{p-1}$, so one has $p_s \geq 1$ and the continuous embedding

$$W^{1,p}(a,\Omega) \hookrightarrow W^{1,p_s}(\Omega), \tag{5}$$

which is relation (1.22) in [1]. More precisely, observing that $p > p_s$, through Holder's inequality and (1) we get

$$\int_\Omega |\nabla u|^{p_s} dx = \int_\Omega a^{-\frac{p_s}{p}} a^{\frac{p_s}{p}} |\nabla u|^{p_s} dx \leq \left(\int_\Omega a^{-s} dx \right)^{\frac{1}{s+1}} \left(\int_\Omega a|\nabla u|^p dx \right)^{\frac{p_s}{p}}$$

for all $u \in W^{1,p}(a,\Omega)$. As a consequence of the above inequality, we can endow $W_0^{1,p}(a,\Omega)$ with an equivalent norm

$$\|u\| = \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}}$$

for which it holds

$$\|u\|_{W_0^{1,p_s}(\Omega)} \leq \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{p_s}} \|u\|. \tag{6}$$

The Sobolev embedding theorem ensures the continuous embedding $W_0^{1,p_s}(\Omega) \hookrightarrow L^{p_s^*}(\Omega)$, with the critical exponent $p_s^* = \frac{Np_s}{N-p_s}$ (note that $1 \leq p_s < N$). Hence there exists a constant $T_0 > 0$ such that

$$\|u\|_{L^{p_s^*}(\Omega)} \leq T_0 \|u\|_{W_0^{1,p_s}(\Omega)}, \ \forall u \in W_0^{1,p_s}(\Omega). \tag{7}$$

The best embedding constant T_0 has been estimated by Talenti [9] as follows

$$T_0 \leq \pi^{-\frac{1}{2}} N^{-\frac{1}{p_s}} \left(\frac{p_s - 1}{N - p_s}\right)^{1-\frac{1}{p_s}} \left(\frac{\Gamma\left(1+\frac{N}{2}\right)\Gamma(N)}{\Gamma\left(\frac{N}{p_s}\right)\Gamma\left(1+N-\frac{N}{p_s}\right)}\right)^{\frac{1}{N}},$$

where Γ is the Euler function

$$\Gamma(t) = \int_0^{+\infty} z^{t-1}e^{-z}dz, \ \forall t > 0.$$

Moreover, by the Rellich–Kondrachov compact embedding theorem, if $1 \leq r < p_s^*$ then the embedding $W_0^{1,p_s}(\Omega) \hookrightarrow L^r(\Omega)$ is compact.

By (7) and Hölder's inequality we infer that

$$\|u\|_{L^r(\Omega)} \leq T_0 |\Omega|^{\frac{p_s^*-r}{p_s^*}} \|u\|_{W_0^{1,p_s}(\Omega)} \tag{8}$$

for every $u \in W_0^{1,p_s}(\Omega)$ and $r \in [1, p_s^*]$. Combining (6) and (8) we arrive at

$$\|u\|_{L^r(\Omega)} \leq \kappa_r \|u\| \tag{9}$$

for all $u \in W_0^{1,p}(a,\Omega)$ and $r \in [1, p_s^*]$, with the constant

$$\kappa_r = T_0 |\Omega|^{\frac{p_s^*-r}{p_s^*}} \|a^{-s}\|_{L^1(\Omega)}^{\frac{1}{p_s}}.$$

The (negative) degenerated p-Laplacian with the weight $a \in L^1_{\mathrm{loc}}(\Omega)$ satisfying condition (1) is the operator $A : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by

$$\langle A(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx, \ \forall u, v \in W_0^{1,p}(a,\Omega). \tag{10}$$

We readily check that the operator A in (10) is well defined noticing by means of Hölder's inequality that for all $u, v \in W_0^{1,p}(a,\Omega)$ it holds

$$\left|\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx\right| \leq \int_\Omega a(x)^{\frac{p-1}{p}} |\nabla u|^{p-1} a(x)^{\frac{1}{p}} |\nabla v| dx \tag{11}$$

$$\leq \left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega a(x)|\nabla v|^p dx\right)^{\frac{1}{p}} < \infty.$$

Important properties of the operator A introduced in (10) are listed in the statement below.

Proposition 1. *Assume that the measurable function $a : \Omega \to \mathbb{R}$ satisfies condition (1). Then the (negative) degenerated p-Laplacian $A : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by (10) has the following properties:*

(i) *A is a bounded operator in the sense that it maps bounded sets to bounded sets;*

(ii) *A is a coercive operator, i.e.,*

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty;$$

(iii) *A is a strictly monotone operator, i.e.,*

$$\langle Au - Av, u - v\rangle > 0, \quad u \neq v;$$

(iv) *A has the S_+ property meaning that any sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ that satisfies $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0 \tag{12}$$

is strongly convergent.

Proof. (i) From (10) and (11) we infer that

$$|\langle Au, v\rangle| = \left| \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx \right| \leq \|u\|^{p-1}\|v\|, \ \forall u, v \in W_0^{1,p}(a,\Omega).$$

We obtain

$$\|Au\|_{(W_0^{1,p}(a,\Omega))^*} = \sup_{v\in W_0^{1,p}(a,\Omega),\ \|v\|\leq 1} |\langle Au, v\rangle| \leq \|u\|^{p-1}, \ \forall u \in W_0^p(a,\Omega),$$

whence A is bounded.

(ii) By (10) we have that

$$\langle Au, u\rangle = \int_\Omega a(x)|\nabla u|^p dx = \|u\|^p, \ \forall u \in W_0^{1,p}(a,\Omega).$$

Taking into account that $p > 1$, it follows that the operator A is coercive.

(iii) In view of the strict monotonicity of the mapping $\xi \mapsto |\xi|^{p-2}\xi$ on $\mathbb{R}^N$, it turns out

$$\langle Au - Av, u - v\rangle = \int_\Omega a(x)\Big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\Big) \cdot (\nabla u - \nabla v)dx > 0, \quad u \neq v,$$

so A is a strictly monotone operator.

(iv) Let a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and (12). Using the monotonicity of the operator A and (12) we have

$$\lim_{n\to\infty} \langle A(u_n) - A(u), u_n - u\rangle = 0.$$

Through Hölder's inequality we obtain

$$
\begin{aligned}
&\langle A(u_n)-A(u),u_n-u\rangle=\int_\Omega a(x)\Big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\Big)\cdot(\nabla u_n-\nabla u)dx\\
= \;&\int_\Omega a(x)|\nabla u_n|^p dx-\int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla u dx-\int_\Omega a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla u_n dx+\int_\Omega a(x)|\nabla u|^p dx\\
\geq \;&\int_\Omega a(x)|\nabla u_n|^p dx-\left(\int_\Omega a(x)|\nabla u_n|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{1}{p}}\\
&-\left(\int_\Omega a(x)|\nabla u|^p dx\right)^{\frac{p-1}{p}}\left(\int_\Omega a(x)|\nabla u_n|^p dx\right)^{\frac{1}{p}}+\int_\Omega a(x)|\nabla u|^p dx\\
= \;&(\|u_n\|-\|u\|)(\|u_n\|^{p-1}-\|u\|^{p-1})\geq 0,
\end{aligned}
$$

from which we find that $\lim_{n\to+\infty}\|u_n\|=\|u\|$. Due to the uniform convexity of $W_0^{1,p}(a,\Omega)$ it follows that $u_n\to u$ in $W_0^{1,p}(a,\Omega)$, thus completing the proof. □

We also need the first eigenvalue λ_1 of the operator $A: W_0^{1,p}(a,\Omega)\to(W_0^{1,p}(a,\Omega))^*$ in (10). Precisely, $\lambda_1>0$ is the least (positive) number for which the equation

$$
\begin{cases}
-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)=\lambda_1|u|^{p-2}u & \text{in } \Omega\\
\\
u=0 & \text{on } \partial\Omega
\end{cases}
\tag{13}
$$

admits a nontrivial solution called eigenfunction corresponding to the first eigenvalue λ_1. A solution to (13) is understood in the weak sense, i.e., $u\in W_0^{1,p}(a,\Omega)$ satisfying

$$
\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx=\lambda_1\int_\Omega |u(x)|^{p-2}u(x)v(x)dx,\ \forall v\in W_0^{1,p}(a,\Omega).
$$

It is known that there exists an eigenfunction $u_1\in W_0^{1,p}(a,\Omega)$ corresponding to the first eigenvalue λ_1 such that $u_1(x)\geq 0$ for a.e. $x\in\Omega$, $u_1\not\equiv 0$, and $u_1\in L^\infty(\Omega)$. For the proofs of these properties we refer to ([1], Chapter 3).

3. Main Results

Our main abstract result provides the existence of a solution to problem (P) and its location within the ordered interval determined by a sub-supersolution.

Theorem 1. *Let the weight $a\in L^1_{\text{loc}}(\Omega)$ fulfill the requirement (1) and assume that Hypothesis 1 for a subsolution $\underline{u}$ and a supersolution $\overline{u}$ with $\underline{u}\leq\overline{u}$ a.e. is satisfied. Then problem (P) possesses at least a solution $u\in W_0^{1,p}(a,\Omega)$ with the location property $\underline{u}\leq u\leq\overline{u}$ for a.e. $x\in\Omega$.*

Proof. By means of the given sub-supersolution $\underline{u}\leq\overline{u}$ for problem (P), we introduce some related mappings. The cut-off function $\pi:\Omega\times\mathbb{R}\to\mathbb{R}$ is defined by

$$
\pi(x,t)=\begin{cases}
-(\underline{u}(x)-t)^{\frac{r}{p_s-r}} & \text{if}\quad t<\underline{u}(x)\\
0 & \text{if}\quad \underline{u}(x)\leq t\leq\overline{u}(x)\\
(t-\overline{u}(x))^{\frac{r}{p_s-r}} & \text{if}\quad t>\overline{u}(x),
\end{cases}
\tag{14}
$$

where s and r are the constants given in (1) and Hypothesis 1. Using (14) in conjunction with $\underline{u},\overline{u}\in L^{p_s^*}(\Omega)$ enables us to find that

$$
|\pi(x,t)|\leq c|t|^{\frac{r}{p_s-r}}+\varrho(x)\ \text{for a.e. } x\in\Omega,\ \text{all } t\in\mathbb{R},
\tag{15}
$$

with a constant $c > 0$ and a function $\varrho \in L^{\frac{p_s^*(p_s-r)}{r}}(\Omega)$. Moreover, proceeding as in [4], we can establish that

$$\int_\Omega \pi(x,u(x))u(x)\,dx \geq b_1\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)} - b_2 \text{ for all } u \in W_0^{1,p}(a,\Omega), \tag{16}$$

with positive constants b_1 and b_2.

In view of (15), the Nemytskij operator $u \mapsto \pi(\cdot,u(\cdot))$ generated by π maps continuously $L^{p_s^*}(\Omega)$ to $L^{\frac{p_s^*(p_s-r)}{r}}(\Omega)$. Therefore, the mapping $\Pi : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by

$$\langle \Pi(u),v\rangle = \int_\Omega \pi(x,u)v dx,\ \forall u,v \in W_0^{1,p}(a,\Omega)$$

is completely continuous. This is true because the inclusion $L^{\frac{p_s^*(p_s-r)}{r}}(\Omega) \subset (W_0^{1,p}(a,\Omega))^*$ is compact being the adjoint of the compact inclusion $W_0^{1,p}(a,\Omega) \subset L^{\frac{p_s^*(p_s-r)}{p_s^*(p_s-r)-r}}(\Omega)$ (note that $\frac{p_s^*(p_s-r)}{p_s^*(p_s-r)-r} < p_s^*$ owing to the assumption $r \in (0, \frac{p_s}{(p_s^*)'})$ in Hypothesis 1).

Hypothesis 1 and (5) imply that the Nemytskij operator $u \mapsto f(\cdot,u(\cdot),\nabla u(\cdot))$ maps continuously $[\underline{u},\overline{u}] \subset W^{1,p}(a,\Omega)$ to $L^{\frac{p_s}{r}}(\Omega)$ with $r \in (0, \frac{p_s}{(p_s^*)'})$. Composing the preceding Nemytskij operator with the inclusion $L^{\frac{p_s}{r}}(\Omega) \subset (W_0^{1,p}(a,\Omega))^*$, which is compact because it is the adjoint operator of the compact inclusion $W_0^{1,p}(a,\Omega) \subset L^{\frac{p_s}{p_s-r}}(\Omega)$ (note that $\frac{p_s}{p_s-r} < p_s^*$ since $r \in (0, \frac{p_s}{(p_s^*)'})$ in Hypothesis 1), we obtain a completely continuous mapping $N_f : [\underline{u},\overline{u}] \to (W_0^{1,p}(a,\Omega))^*$ given by

$$\langle N_f(u),v\rangle = \int_\Omega f(x,u(x),\nabla u(x))v(x)\,dx$$

for all $u \in [\underline{u},\overline{u}]$ and $v \in W_0^{1,p}(a,\Omega)$.

We also make use of the truncation operator $T : W_0^{1,p}(a,\Omega) \to W^{1,p}(a,\Omega)$ given by

$$(Tu)(x) = \begin{cases} \underline{u}(x) & \text{if} \quad u(x) < \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \overline{u}(x) & \text{if} \quad u(x) > \overline{u}(x) \end{cases} \tag{17}$$

for all $u \in W_0^{1,p}(a,\Omega)$ and a.e. $x \in \Omega$. It is a continuous and bounded mapping (in the sense that it maps bounded sets to bounded sets). Notice that its range lies in $[\underline{u},\overline{u}]$, so T can be composed with the operator N_f.

Now we consider for every $\lambda > 0$ the operator $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ defined by

$$A_\lambda = A + \lambda\Pi - N_f \circ T. \tag{18}$$

Explicitly, it reads as

$$\begin{aligned} \langle A_\lambda(u),v\rangle &= \int_\Omega a(x)|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \lambda\int_\Omega \pi(x,u)v\,dx \\ &- \int_\Omega f(x,Tu,\nabla(Tu))v\,dx \text{ for all } u,\, v \in W_0^{1,p}(a,\Omega). \end{aligned} \tag{19}$$

From Proposition 1(i) it is known that the operator $A : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is bounded, while the above comments demonstrate that the operators Π, N_f and T are all of them bounded. Therefore from (18) we infer that the operator $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is bounded.

We claim that $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is a pseudomonotone operator. In this respect, let a sequence $\{u_n\} \subset W_0^{1,p}(a,\Omega)$ satisfy $u_n \rightharpoonup u$ in $W_0^{1,p}(a,\Omega)$ and

$$\limsup_{n\to\infty} \langle A_\lambda(u_n), u_n - u\rangle \le 0. \tag{20}$$

The sequence $\{\Pi(u_n)\}$ is bounded in $L^{\frac{p_s^*(p_s-r)}{r}}(\Omega)$, while $u_n \to u$ in $L^{\frac{p_s^*(p_s-r)}{p_s^*(p_s-r)-r}}(\Omega)$ by the compact embedding $W_0^{1,p}(a,\Omega) \subset L^{\frac{p_s^*(p_s-r)}{p_s^*(p_s-r)-r}}(\Omega)$, thus

$$\lim_{n\to\infty} \langle \Pi(u_n), u_n - u\rangle = 0.$$

The sequence $\{N_f \circ T(u_n)\}$ is bounded in $L^{\frac{p_s}{r}}(\Omega)$, while $u_n \to u$ in $L^{\frac{p_s}{p_s-r}}(\Omega)$ by the compact embedding $W_0^{1,p}(a,\Omega) \subset L^{\frac{p_s}{p_s-r}}(\Omega)$, producing

$$\lim_{n\to\infty} \langle N_f \circ T(u_n), u_n - u\rangle = 0.$$

Consequently, complying with (18), we see that (20) reduces to (12). This, in conjunction with the weak convergence $u_n \rightharpoonup u$, enables us to apply Proposition 1(iv) ensuring that the strong convergence $u_n \to u$ in $W_0^{1,p}(a,\Omega)$ holds.

From the strong convergence $a(\cdot)^{\frac{1}{p}}\nabla u_n(\cdot) \to a(\cdot)^{\frac{1}{p}}\nabla u(\cdot)$ in $(L^p(\Omega))^N$ it follows the strong convergence $a(\cdot)^{\frac{p-1}{p}}|\nabla u_n(\cdot)|^{p-2}\nabla u_n(\cdot) \to a(\cdot)^{\frac{p-1}{p}}|\nabla u(\cdot)|^{p-2}\nabla u(\cdot)$ in $(L^{\frac{p}{p-1}}(\Omega))^N$. This amounts to saying that $Au_n \rightharpoonup Au$ in $(W_0^{1,p}(a,\Omega))^*$ since

$$\langle Au_n, v\rangle = \int_\Omega a(x)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v dx \to \int_\Omega a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla v dx = \langle Au, v\rangle,\ \forall v \in W_0^{1,p}(a,\Omega).$$

Again, from the strong convergence $a(\cdot)^{\frac{1}{p}}\nabla u_n(\cdot) \to a(\cdot)^{\frac{1}{p}}\nabla u(\cdot)$ in $(L^p(\Omega))^N$ we infer that

$$\langle Au_n, u_n\rangle = \int_\Omega a(x)|\nabla u_n|^p dx \to \int_\Omega a(x)|\nabla u|^p dx = \langle Au, u\rangle$$

as $n \to \infty$. Taking into account the continuity of the mappings Π and $N_f \circ T$, we have

$$\langle A_\lambda u_n, v\rangle \to \langle A_\lambda u, v\rangle,\ \forall v \in W_0^{1,p}(a,\Omega),$$

and

$$\langle A_\lambda u_n, u_n\rangle \to \langle A_\lambda u, u\rangle$$

as $n \to \infty$, for every $\lambda > 0$. We can conclude that $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is a pseudomonotone operator (see, e.g., ([2], Definition 2.97)).

The next step in the proof is to show that the operator $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is coercive provided $\lambda > 0$ is large enough. Taking advantage of the fact that $Tu \in [\underline{u}, \overline{u}]$ whenever $u \in W_0^{1,p}(a,\Omega)$, let us note by (16), (19) and Hypothesis 1 that

$$\begin{aligned} \langle A_\lambda(u), u\rangle &= \langle A(u), u\rangle + \lambda \int_\Omega \pi(x,u)u\,dx - \int_\Omega f(x,Tu,\nabla(Tu))u\,dx \qquad (21) \\ \ge\quad & \|u\|^p + \lambda\Big(b_1\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)} - b_2\Big) - \|\sigma\|_{L^{\frac{p_s}{r}}(\Omega)}\|u\|_{L^{\frac{p_s}{p_s-r}}(\Omega)} - b\int_\Omega |\nabla(Tu)|^r|u|dx \end{aligned}$$

for all $u \in W_0^{1,p}(a,\Omega)$. Now we estimate the last term in (21) based on the fact that by (5) we know that $\nabla u \in (L^{p_s}(\Omega))^N$, and so $\nabla(Tu) \in (L^{p_s}(\Omega))^N$. Using the definition of Tu in (17), Hölder's inequality and the continuous embedding in (9) it turns out that

$$\begin{aligned}&\int_\Omega |\nabla(Tu)|^r|u|dx = \int_{\{\underline{u}\le u\le \overline{u}\}} |\nabla u|^r|u|dx + \int_{\{u<\underline{u}\}} |\nabla \underline{u}|^r|u|dx + \int_{\{u>\overline{u}\}} |\nabla \overline{u}|^r|u|dx \\ \le \quad &\int_\Omega |\nabla u|^r|u|dx + c_1\|u\|, \ \forall u \in W_0^{1,p}(a,\Omega),\end{aligned}$$

with a constant $c_1 > 0$. We can insert the preceding inequality in (21) to derive

$$\langle A_\lambda(u),u\rangle \ge \|u\|^p + \lambda(b_1\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)} - b_2) - c_2\|u\| - b\int_\Omega |\nabla u|^r|u|dx, \tag{22}$$

with a constant $c_2 > 0$. The Hölder's and Young's inequalities in conjunction with embedding (5) imply

$$\int_\Omega |\nabla u|^r|u|dx \le \|\nabla u\|^r_{L^{p_s}(\Omega)}\|u\|_{L^{\frac{p_s}{p_s-r}}(\Omega)} \le c_3\|u\|^{p_s} + c_4\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)},$$

with constants $c_3 > 0$ and $c_4 > 0$. Then (22) entails

$$\langle A_\lambda(u),u\rangle \ge \|u\|^p + \lambda(b_1\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)} - b_2) - c_2\|u\| - b(c_3\|u\|^{p_s} + c_4\|u\|^{\frac{p_s}{p_s-r}}_{L^{\frac{p_s}{p_s-r}}(\Omega)}) \tag{23}$$

for all $u \in W_0^{1,p}(a,\Omega)$. Recalling from (16) that $b_1 > 0$, we can choose $\lambda > 0$ so large to have $\lambda b_1 > bc_4$. Hence due to $p > p_s \ge 1$ (see (1)), (23) yields the coercivity of A_λ, i.e.,

$$\lim_{\|u\|\to+\infty} \frac{\langle A_\lambda(u),u\rangle}{\|u\|} = +\infty.$$

We have shown that the nonlinear operator $A_\lambda : W_0^{1,p}(a,\Omega) \to (W_0^{1,p}(a,\Omega))^*$ is bounded, pseudomonotone and coercive provided $\lambda > 0$ is sufficiently large. Therefore, for such an A_λ we can apply the main theorem of pseudomonotone operators (see, e.g., ([2], Theorem 2.99)) ensuring that there exists a solution $u \in W_0^{1,p}(a,\Omega)$ to the equation

$$A_\lambda(u) = 0. \tag{24}$$

Fix an admissible $\lambda > 0$ as pointed out above. We are going to prove that $u \in W_0^{1,p}(a,\Omega)$ resolving (24) is a weak solution of the original problem (P), which means that (2) is satisfied. To this end, notice that (19) and (24) yield

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx + \lambda\int_\Omega \pi(x,u)vdx \\ =&\int_\Omega f(x,Tu,\nabla(Tu))vdx \text{ for all } v \in W_0^{1,p}(a,\Omega).\end{aligned} \tag{25}$$

We proceed by comparing u with the subsolution $\underline{u}$ and supersolution $\overline{u}$ postulated in Hypothesis 1. We claim that $u \le \overline{u}$ a.e. in Ω. Towards this, it can be readily checked that $(u-\overline{u})^+ = \max\{u-\overline{u},0\} \in W_0^{1,p}(a,\Omega)$, where the condition $\overline{u} \ge 0$ on $\partial\Omega$ in the sense of traces is essentially used. Thus, we can insert $v = (u-\overline{u})^+$ in (25) and (4) which gives

$$\begin{aligned}&\int_\Omega a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla(u-\overline{u})^+(x)dx + \lambda\int_\Omega \pi(x,u(x))(u-\overline{u})^+(x)dx \\ =&\int_\Omega f(x,Tu(x),\nabla(Tu)(x))(u-\overline{u})^+(x)dx\end{aligned} \tag{26}$$

and

$$\int_{\Omega} a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x)\cdot\nabla(u-\overline{u})^{+}(x)dx \geq \int_{\Omega} f(x,\overline{u}(x),\nabla\overline{u}(x))(u-\overline{u})^{+}(x)dx. \quad (27)$$

From (26) and (27), by subtraction we are led to

$$\begin{aligned}&\int_{\Omega} a(x)\Big(|\nabla u(x)|^{p-2}\nabla u(x)-|\nabla\overline{u}(x)|^{p-2}\nabla\overline{u}(x)\Big)\cdot\nabla(u-\overline{u})^{+}(x)dx+\lambda\int_{\Omega}\pi(x,u(x))(u-\overline{u})^{+}(x)dx\\ &\leq\int_{\Omega}(f(x,Tu(x),\nabla(Tu)(x))-f(x,\overline{u}(x),\nabla\overline{u}(x)))(u-\overline{u})^{+}(x)dx.\end{aligned}$$

By (14), (17), and the preceding inequality we get

$$\begin{aligned}&\int_{\{u>\overline{u}\}} a(x)\Big(|\nabla u(x)|^{p-2}\nabla u(x)-|\nabla\overline{u}(x)|^{p-2}\nabla\overline{u}(x)\Big)\cdot\nabla(u-\overline{u})dx+\lambda\int_{\{u>\overline{u}\}}(u(x)-\overline{u}(x))^{\frac{ps}{ps-r}}dx\\ &\leq\int_{\{u>\overline{u}\}}(f(x,Tu,\nabla(Tu))-f(x,\overline{u},\nabla\overline{u}))(u-\overline{u})dx=0.\end{aligned}$$

Since the function $a(x)$ is positive almost everywhere in Ω and the mapping $\xi\mapsto|\xi|^{p-2}\xi$ on $\mathbb{R}^N$ is monotone, we arrive at

$$\int_{\{u>\overline{u}\}}(u(x)-\overline{u}(x))^{\frac{ps}{ps-r}}dx\leq 0.$$

Therefore, the Lebesgue measure of the set $\{u>\overline{u}\}$ is zero, i.e., $u\leq\overline{u}$ a.e. in Ω.

Similarly, we can prove that $\underline{u}\leq u$ a.e. in Ω. Specifically, relying on the condition $\underline{u}\leq 0$ on $\partial\Omega$ (in the sense of traces), it holds $(\underline{u}-u)^{+}=\max\{\underline{u}-u,0\}\in W_0^{1,p}(a,\Omega)$, which allows us to test (25) and (3) with $v=(\underline{u}-u)^{+}\in W_0^{1,p}(a,\Omega)$. This results in

$$\begin{aligned}&\int_{\Omega} a(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla(\underline{u}-u)^{+}(x)dx+\lambda\int_{\Omega}\pi(x,u(x))(\underline{u}-u)^{+}(x)dx\\ &=\int_{\Omega} f(x,Tu(x),\nabla(Tu)(x))(\underline{u}-u)^{+}(x)dx\end{aligned} \quad (28)$$

and

$$\int_{\Omega} a(x)|\nabla\underline{u}(x)|^{p-2}\nabla\underline{u}(x)\cdot\nabla(\underline{u}-u)^{+}(x)dx\leq\int_{\Omega} f(x,\underline{u}(x),\nabla\underline{u}(x))(\underline{u}-u)^{+}(x)dx. \quad (29)$$

Arguing as before, we deduce from (28), (29), (14), and (17) the following estimate

$$\begin{aligned}&\int_{\{\underline{u}>u\}} a(x)\Big(|\nabla\underline{u}(x)|^{p-2}\nabla\underline{u}(x)-|\nabla u(x)|^{p-2}\nabla u(x)\Big)\cdot\nabla(\underline{u}-u)dx+\lambda\int_{\{\underline{u}>u\}}(\underline{u}(x)-u(x))^{\frac{ps}{ps-r}}dx\\ &\leq\int_{\Omega}(f(x,\underline{u},\nabla\underline{u})-f(x,Tu,\nabla(Tu))(\underline{u}-u)^{+}dx\\ &=\int_{\{\underline{u}>u\}}(f(x,\underline{u},\nabla\underline{u})-f(x,Tu,\nabla(Tu)))(\underline{u}-u)^{+}dx=0.\end{aligned}$$

At this point, the positivity of the function $a(x)$ on Ω and the monotonicity of the mapping $\xi\mapsto|\xi|^{p-2}\xi$ on $\mathbb{R}^N$ confirm that

$$\int_{\{\underline{u}>u\}}(\underline{u}(x)-u(x))^{\frac{ps}{ps-r}}dx\leq 0,$$

from which we can readily derive that $\underline{u}\leq u$ a.e in Ω.

Based on the enclosure property $\underline{u}\leq u\leq\overline{u}$ a.e. in Ω, it follows through (17) that $T(u)=u$ and through (14) that $\Pi(u)=0$. As a result, (25) takes the form of (2), thus the proof is complete. □

Now we present an application of Theorem 1 describing how the existence of a nontrivial nonnegative solution can be established by effectively determining a sub-supersolution. In the sequel, by λ_1 we denote the first eigenvalue of problem (13) (see Section 2).

Theorem 2. *Let the weight $a \in L^1_{\mathrm{loc}}(\Omega)$ fulfill the requirement (1). Assume that the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the conditions:*

(j) *there is a constant $\mu > 0$ such that*

$$\lambda_1 t^{p-1} \leq f(x,t,\xi) \text{ for a.e. } x \in \Omega, \text{ all } t \in [0,\mu],\, \xi \in \mathbb{R}^N;$$

(jj) *there is a constant $C > 0$ such that*

$$f(x,C,0) \leq 0 \text{ for a.e. } x \in \Omega;$$

(jjj) *there are a function $\sigma \in L^{\frac{p_s}{r}}(\Omega)$ and constants $b > 0$ and $r \in (0, \frac{p_s}{(p_s^*)'})$ such that*

$$|f(x,t,\xi)| \leq \sigma(x) + b|\xi|^r \quad \text{for a.e. } x \in \Omega, \quad \text{all } t \in [0,C], \quad \xi \in \mathbb{R}^N.$$

Then problem (P) has a nondegenerate, nonnegative and bounded weak solution $u \in W_0^{1,p}(a,\Omega)$ satisfying the estimate $u \leq C$.

Proof. Our goal is to apply Theorem 1 by constructing an appropriate sub-supersolution. In order to determine a subsolution, we use an eigenfunction $u_1 \in W_0^{1,p}(a,\Omega)$ corresponding to the first eigenvalue λ_1 of problem (13) with the properties $u_1(x) \geq 0$ for a.e. $x \in \Omega$, $u_1 \not\equiv 0$, and $u_1 \in L^\infty(\Omega)$ as mentioned in Section 2. Then we choose an $\varepsilon > 0$ sufficiently small to verify

$$\varepsilon u_1(x) \leq \mu \text{ for a.e. } x \in \Omega, \tag{30}$$

where μ is the positive constant postulated in assumption (j). Then assumption (j) implies

$$\lambda_1(\varepsilon u_1)^{p-1} \leq f(x, \varepsilon u_1, \nabla(\varepsilon u_1)) \text{ for a.e. } \ x \in \Omega. \tag{31}$$

For a possibly smaller $\varepsilon > 0$ we can suppose

$$\varepsilon u_1(x) \leq C \text{ for a.e. } x \in \Omega, \tag{32}$$

with $C > 0$ in assumption (jj).

Let us fix an $\varepsilon > 0$ for which (30) and (32) are fulfilled. We claim that $\underline{u} = \varepsilon u_1$ is a subsolution to problem (P). Indeed, by (13) with u_1 in place of u and (31) we note that

$$\begin{aligned} &\int_\Omega a(x)|\nabla \underline{u}(x)|^{p-2}\nabla \underline{u}(x) \cdot \nabla v(x)dx = \varepsilon^{p-1}\lambda_1 \int_\Omega u_1(x)^{p-1}v(x)dx \\ &\leq \int_\Omega f(x, \varepsilon u_1(x), \nabla(\varepsilon u_1)(x))v(x)dx = \int_\Omega f(x, \underline{u}(x), \nabla \underline{u}(x))v(x)dx \end{aligned}$$

for all $v \in W_0^{1,p}(a,\Omega)$, $v \geq 0$ a.e. in Ω, thereby proving the claim.

Next we claim that the constant function $\overline{u} = C$, with $C > 0$ in assumption (jj), is a supersolution to problem (P). Accordingly, from assumption (jj) we find that

$$\int_\Omega a(x)|\nabla \overline{u}(x)|^{p-2}\nabla \overline{u}(x) \cdot \nabla v(x)dx = 0 \geq \int_\Omega f(x,C,0)v(x)dx = \int_\Omega f(x, \overline{u}(x), \nabla \overline{u}(x))v(x)dx$$

for all $v \in W_0^{1,p}(a,\Omega)$, $v \geq 0$ a.e. in Ω, which proves the claim.

It is clear from (32) that $\underline{u}(x) \leq \overline{u}(x)$ for a.e. in Ω. Assumption (jjj) ensures that the growth condition required in Hypothesis 1 of Theorem 1 holds true. Therefore, all the hypotheses of Theorem 1 are verified, which permits the conclusion that there exists a solution $u \in W_0^{1,p}(a,\Omega)$ of problem (P) within the ordered interval $[\underline{u}, \overline{u}]$. Since the function $\underline{u} = \varepsilon u_1$ is nontrivial and nonnegative, and $u \geq \underline{u}$, we have that u is nontrivial and nonnegative, whereas $u \in [\underline{u}, \overline{u}]$ renders the boundedness of u and the a priori estimate $u \leq C$. The proof is complete. $\square$

We end the paper with a simple example for which Theorem 2 applies.

Example 1. Fix a positive weight $a \in L^1_{\text{loc}}(\Omega)$ with the property (1). Let the function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$f(x,t,\xi) = \begin{cases} 0 & \text{if } t < 0 \\ t^{p-1}(\rho(x) + |\xi|^r) & \text{if } 0 \leq t \leq 1 \\ (2-t)(\rho(x) + |\xi|^r) & \text{if } t > 1, \end{cases}$$

with some $r \in [1, \frac{p_s}{(p_s^*)'})$ and $\rho \in L^\infty(\Omega)$ satisfying $\rho(x) \geq \lambda_1$ for a.e. $x \in \Omega$. It follows that f is a Carathéodory function for which conditions $(j) - (jjj)$ in Theorem 2 are verified. Precisely, condition (j) holds with $\mu = 1$ because $\rho(x) \geq \lambda_1$, condition (jj) holds with $C = 2$, and condition (jjj) is fulfilled with the given r. Hence Theorem 2 applies to problem (P) whose equation has the right-hand side expressed with the function $f(x,t,\xi)$ given above.

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