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Modified Mean-Variance Risk Measures for Long-Term Portfolios

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Abstract: This paper proposes modified mean-variance risk measures for long-term investment portfolios. Two types of portfolios are considered: constant proportion portfolios and increasing amount portfolios. They are widely used in finance for investing assets and developing derivative securities. We compare the long-term behavior of a conventional mean-variance risk measure and a modified one of the two types of portfolios, and we discuss the benefits of the modified measure. Subsequently, an optimal long-term investment strategy is derived. We show that the modified risk measure reflects the investor's risk aversion on the optimal long-term investment strategy; however, the conventional one does not. Several factor models are discussed as concrete examples: the Black–Scholes model, Kim–Omberg model, Heston model, and 3/2 stochastic volatility model.

Keywords: continuous-time factor model; modified risk measures; mean-variance analysis; long-term investment; optimal strategy

1. Introduction

Risk measure is an important topic in modern portfolio theory. There are numerous risk measures for portfolios, one of them being the mean-variance risk measure. A conventional formulation of the mean-variance risk measure is as follows:

$$\mathcal{R}_t := \gamma \sqrt{\text{var}[\Pi_t] - \mathbb{E}[\Pi_t]}, \quad (1)$$

where $\gamma > 0$ is the investor's risk aversion parameter, and Π_t is the portfolio value at time t . A conventional mean-variance risk measure has two flaws when dealing with long-term investment portfolios. One is that the growth rate of the conventional mean-variance risk measure depends on only either the mean or variance, and the other is that the growth rate does not depend on parameter γ . Therefore, we propose a modified risk measure to overcome these flaws and discuss its benefits in comparison with the conventional measure.

Two types of portfolios are considered in this study: a constant proportion portfolio (CPP) and an increasing amount portfolio (IAP). A CPP is a portfolio in which the ratio of investments in safe assets and risky assets is fixed. The CPP is worthwhile to study because they are widely used in finance. Many financial institutions and companies use this type of strategy for investing assets and developing derivative securities. A Leveraged Exchange Traded Fund (LETF) is a typical example of commercialized products based on the CPP strategy. Indeed, a LETF is mathematically same with the CPP because their portfolio structures are identical. It is natural to ask the following questions.

- What kind of risk measures should we use for long-term investment in CPPs?
- Given a risk measure, what is the optimal strategy for long-term investment in CPPs?

This paper answers these questions.

A similar work is conducted for IAPs. An IAP is a portfolio in which the amount invested in the risky assets increases over time, and the increasing rate is equal to the short rate. This type of portfolio might be less interesting than the CPPs; however, it is still useful for investors with a restrictive short position amount. In financial markets, the amount in the short position in the risky asset is occasionally restricted. When this amount is a



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constant multiplied by the money market account, the risk measure for the IAPs becomes practically useful.

This paper mainly discusses modified risk measures for long-term CPPs and IAPs. A conventional risk measure is $\mathcal{R}_t$, given in Equation (1) for the portfolio value Π_t at time $t \geq 0$. In fact, $-\mathcal{R}_t$ is a more conventional form than $\mathcal{R}_t$ (for example, the Markowitz portfolio theory); however, $\mathcal{R}_t$ is considered in this study for computational convenience. As modified risk measures, we propose

$$\Lambda_t := \frac{(\text{var}[\Pi_t])^{\gamma/2}}{\mathbb{E}[\Pi_t]}$$

and

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}]$$

for CPPs and IAPs, respectively, where $\gamma > 0$ is the investor's risk aversion parameter, and $r \geq 0$ is the short rate. For long-term investments, we focus on the large-time behavior of these risk measures. More precisely, for CPPs, we compute

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t$$

and compare these two values. These limits give the growth rates of the conventional and modified risk measures as $t \rightarrow \infty$. Similarly, for IAPs, two limit values

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t}$$

are compared.

The conventional risk measure has two flaws when working with long-term portfolios. The growth rate of the conventional risk measure depends on only either the mean or variance. It depends on only the variance for CPPs and only on the mean for IAPs. In other words, the conventional risk measure cannot offer a balance between the mean and variance of long-term portfolios. On the contrary, the growth rate of the modified risk measure reflects both the mean and variance. Another limitation is that the growth rate of the conventional risk measure does not depend on parameter γ . Thus, the conventional risk of long-term portfolios cannot reflect the investor's risk aversion. We show that the modified risk measure can reflect the investor's risk aversion. More details are discussed in Sections 3 and 4.

Optimal investment strategies are also investigated. Investors construct portfolios of financial assets depending on their levels of acceptable risk. In the mean-variance analysis, the return and risk of a portfolio are expressed as the mean and variance, respectively. We aim to identify a portfolio that minimizes the growth rate of the mean-variance risk measures. More precisely, we will calculate a constant proportion for CPPs and a constant amount for IAPs, to minimize the growth rate of the modified risk measure. Several factor models are analyzed as concrete examples: the Black–Scholes model, Kim–Omberg model, Heston model, and 3/2 stochastic volatility model.

As closely related topics, many authors have studied long-term CPPs. Leung and Park [1] investigated the long-term growth rate of expected utility from holding a CPP. For a given value process $(\Pi_t)_{t \geq 0}$ of CPP and a power utility function of the form $U(x) = x^p, 0 < p < 1$, the limit value

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathbb{E}[U(\Pi_t)]$$

was computed for several Markovian market models. In addition, a constant ratio that maximized the long-term growth rate was determined. Yao [2] analyzed the deviation probability estimate for a CPP. The logarithmic limit of the tail probability was computed using the large deviation principle. Moreover, the author presented optimal constant ratios for long-term CPPs. Zhu [3] investigated optimal strategies for a long-term static investor.

The author derived the optimal allocation of capital to maximize the long-term growth rate of the expected utility of wealth. Three models for the underlying stock price processes were covered: the Heston model, 3/2 model, and jump diffusion model.

Various studies have proposed better risk measures. Chen et al. [4] derived a class of time-consistent multi-period risk measures under regime switching. They analyzed a multi-stage portfolio selection model by using the time-consistent multi-period risk measure. Emmer et al. [5] considered three risk measures: VaR, ES, and expectiles. They checked whether these measures satisfied properties such as coherence, comonotonic additivity, robustness, and elicibility. They concluded that the ES can be considered a good risk measure, and there is no sufficient evidence to justify an all-inclusive replacement of ES by expectiles in applications. Rachev et al. [6] investigated the properties that a risk measure should satisfy to characterize the investor's risk preferences. They analyzed the relationship between distributional modeling and risk measures and described desirable features of an ideal risk measure for a portfolio selection problem. Ruttiens [7] proposed the "accrued returns variability," which was measured from the actual dispersion of successive cumulated returns relative to the corresponding successive cumulated returns produced by an accrued performance of null volatility. This risk measure outperformed the traditional risk measure, which was computed from the standard deviation of a series of past returns. Zakamouline and Koekebakker [8] presented a risk measure that takes into account higher moments of distribution. This measure is motivated by the investor's preferences represented by utility functions. They introduced the notion of relative preferences over absolute preferences and explained the several advantages.

The remainder of this paper is structured as follows. Section 2 describes the underlying market model considered in this study. Section 3 presents a modified risk measure for CPPs and investigates an optimal constant proportion for long-term investments. Several specific market models such as the Black–Scholes model, Kim–Omberg model, Heston model, and 3/2 stochastic volatility model are analyzed. A similar work is conducted in Section 4 for IAPs. Section 5 summarizes the paper. Technical details are presented in the Appendices A–C.

2. Factor Models

The underlying market model considered in this study is a factor model, which is defined as follows. A state process $(X_t)_{t \geq 0}$ is a solution of the stochastic differential equation (SDE)

$$dX_t = k(X_t) dt + a(X_t) dW_t$$

for continuous functions $k, \sigma : \mathbb{R} \rightarrow \mathbb{R}$. Assume that this SDE has a unique strong solution. A money market account G and a risky asset S are modeled as

$$G_t = e^{\int_0^t r(X_s) ds}, \quad t \geq 0$$

and

$$S_t = S_0 e^{\int_0^t (r(X_s) + \mu(X_s) - \frac{1}{2}\sigma^2(X_s)) ds + \int_0^t \sigma(X_s) dB_s}, \quad t \geq 0$$

for $S_0 > 0$ and continuous functions $r, \mu, \sigma : \mathbb{R} \rightarrow \mathbb{R}$. Here, the short rate is $(r(X_t))_{t \geq 0}$, and $(W_t, B_t)_{t \geq 0}$ is a correlated Brownian motion with correlation $-1 \leq \rho \leq 1$. In the SDE form,

$$\begin{aligned} dG_t/G_t &= r(X_t) dt, \quad G_0 = 1, \\ dS_t/S_t &= (r(X_t) + \mu(X_t)) dt + \sigma(X_t) dB_t, \quad S_0 > 0. \end{aligned} \quad (2)$$

This type of factor models covers a broad range of market models used in finance. Two kinds of portfolios are studied under this factor model: CPP (Section 3) and IAP (Section 4). Details are examined in the following sections.

3. Constant Proportion Portfolio

This section proposes a modified risk measure for CPPs and finds an optimal constant ratio for long-term investments. First, we explain the concept of CPPs with its mathematical formulation under the factor model. Second, both the conventional and modified mean-variance risk measures are introduced. We also discuss why the modified method is better than the conventional one. Finally, the analytic expressions of the two risk measures are computed, and an optimal constant ratio is provided for several market models: Black–Scholes model, Kim–Omberg model, Heston model, and 3/2 stochastic volatility model.

A CPP is a portfolio in which the ratio of investments in the money market account and a risky asset is fixed. We denote the ratio invested in the risky asset as $\alpha \in \mathbb{R}$. The wealth process $(\Pi_t)_{t \geq 0}$ of the CPP with ratio α is constructed as follows: At any time $t \geq 0$, the cash amount of $\alpha \Pi_t$ (α times the CPP value) is invested in the risky asset and the amount $(1 - \alpha) \Pi_t$ is invested at the risk-free rate. In the factor model, with the notations described in Section 2, the wealth process $(\Pi_t)_{t \geq 0}$ with ratio α is described as

$$d\Pi_t / \Pi_t = (r(X_t) + \alpha \mu(X_t)) dt + \alpha \sigma(X_t) dB_t.$$

This implies the following:

$$\Pi_t = \Pi_0 e^{\int_0^t (r(X_s) + \alpha \mu(X_s) - \frac{1}{2} \alpha^2 \sigma^2(X_s)) ds + \int_0^t \alpha \sigma(X_s) dB_s}, \quad t \geq 0.$$

We always assume that the initial wealth is positive, in other words, $\Pi_0 > 0$, and this implies that $\Pi_t > 0$ for $t \geq 0$ almost surely. If $\alpha > 0$ (respectively, $\alpha < 0$), then the CPP takes a long position (respectively, a short position) in the risky asset, and if $\alpha = 0$, then the investment is only done in the money market account. Only the case with $\alpha > 0$ is analyzed in this study. This is because the case with $\alpha < 0$ can be analyzed by analogy, and the case with $\alpha = 0$ is trivial.

A conventional mean-variance risk measure is

$$\mathcal{R}_t := \gamma \sqrt{\text{var}[\Pi_t] - \mathbb{E}[\Pi_t]}$$

for risk-averse parameter $\gamma > 0$. We compute the limit value

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t$$

if it exists and find α that minimizes this limit value. This minimizing value α gives the lowest growth rate of $\mathcal{R}_t$ as $t \rightarrow \infty$, and this gives the optimal constant ratio for long-term investments.

This limit value can be calculated in a simpler way. We define

$$\hat{\mathcal{R}}_t := \gamma \sqrt{\mathbb{E}[\Pi_t^2]}.$$

If

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[\Pi_t]}{\sqrt{\mathbb{E}[\Pi_t^2]}} = 0 \quad (3)$$

then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\mathcal{R}}_t. \quad (4)$$

It can be verified that Equation (3) holds for all factor models below. Because the right-hand side of Equation (4) is simpler to compute than the left-hand side, $\hat{\mathcal{R}}_t$ is considered instead of $\mathcal{R}_t$ in the following sections.

The conventional mean-variance risk measure has two flaws when dealing with long-term investments. The principle of the mean-variance risk measure is to let an investor choose a balance between maximizing the mean and minimizing the variance based on the

investor's risk aversion. However, as Equation (4) indicates, the long-term behavior of the conventional risk measure is only determined by the variance and is therefore unaffected by the mean value. Thus, the long-term limit of the conventional risk measure does not reflect the underlying principle of the mean-variance risk measure.

Another flaw is that the growth rate of the conventional risk measure cannot capture the investor's risk aversion. From Equation (4), it is clear that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \sqrt{\mathbb{E}[\Pi_t^2]}.$$

This implies that the limit value is independent of the risk aversion parameter γ . Therefore, the conventional risk measure cannot reflect the investor's risk aversion for long-term investment portfolios.

We now propose a modified mean-variance risk measure. Define

$$\Lambda_t := \frac{(\text{var}[\Pi_t])^{\gamma/2}}{\mathbb{E}[\Pi_t]}, \quad \gamma > 0,$$

which concerns the fraction between the variance powered by $\gamma/2$ and the mean. For long-term investments, we calculate

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t.$$

The limit value of this modified risk measure overcomes the two flaws of the conventional one. This limit value can be calculated in a simpler way as follows. We define

$$\hat{\Lambda}_t := \frac{(\mathbb{E}[\Pi_t^2])^{\gamma/2}}{\mathbb{E}[\Pi_t]}.$$

If (3) holds, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\Lambda}_t. \quad (5)$$

Because the right-hand side is simpler to compute, we consider $\hat{\Lambda}_t$ instead of Λ_t in the following sections.

3.1. Black–Scholes Model

As a warm-up, we consider the CPP for the Black–Scholes model:

$$dS_t/S_t = (r + \mu) dt + \sigma dB_t \quad (6)$$

with $r, \mu \in \mathbb{R}$ and $\sigma > 0$. The value of the CPP with ratio α is

$$\Pi_t = \Pi_0 e^{(r+\alpha\mu - \frac{1}{2}\alpha^2\sigma^2)t + \alpha\sigma B_t}, \quad t \geq 0.$$

Proposition 1. Under the Black–Scholes model, for the CPP with ratio $\alpha > 0$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = r + \alpha\mu + \frac{1}{2}\alpha^2\sigma^2.$$

Proof. By direct calculation,

$$\mathbb{E}[\Pi_t] = \Pi_0 e^{(r+\alpha\mu)t}$$

and

$$\mathbb{E}[\Pi_t^2] = \Pi_0^2 e^{(2r+2\alpha\mu+\alpha^2\sigma^2)t}.$$

It is clear that Equation (3) is satisfied. Then,

$$\hat{\mathcal{R}}_t = \gamma \sqrt{\mathbb{E}[\Pi_t^2]} = \gamma \Pi_0 e^{(r+\alpha\mu + \frac{1}{2}\alpha^2\sigma^2)t}, \quad (7)$$

and it follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\mathcal{R}}_t = r + \alpha\mu + \frac{1}{2}\alpha^2\sigma^2.$$

This gives the desired result. $\square$

Proposition 2. Under the Black–Scholes model, for the CPP with ratio $\alpha > 0$, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = (\gamma - 1)r + (\gamma - 1)\alpha\mu + \frac{1}{2}\gamma\alpha^2\sigma^2.$$

Proof. From the proof of Proposition 1, we know

$$\hat{\Lambda}_t = \frac{(\mathbb{E}[\Pi_t^2])^{\gamma/2}}{\mathbb{E}[\Pi_t]} = \Pi_0^{\gamma-1} e^{((\gamma-1)r + (\gamma-1)\alpha\mu + \frac{1}{2}\gamma\alpha^2\sigma^2)t}. \quad (8)$$

Thus,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\Lambda}_t = (\gamma - 1)r + (\gamma - 1)\alpha\mu + \frac{1}{2}\gamma\alpha^2\sigma^2. \quad (9)$$

This completes the proof. $\square$

We find an optimal ratio α^* that minimizes the growth rate of Λ_t as $t \rightarrow \infty$. Assume that ratio α is allowed in a compact interval $[L, R]$ for $0 < L < R$. As a mapping of α , the function

$$\bar{\Lambda}(\alpha) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = (\gamma - 1)r + (\gamma - 1)\alpha\mu + \frac{1}{2}\gamma\alpha^2\sigma^2$$

is continuous on the compact interval $[L, R]$. Thus, the function achieves its minimum value. Because $\bar{\Lambda}$ is a quadratic function of α , and $\alpha = \frac{(1-\gamma)\mu}{\gamma\sigma^2}$ is a critical point, we obtain the following cases.

(i) If $L \leq \frac{(1-\gamma)\mu}{\gamma\sigma^2} \leq R$, then the optimal ratio is $\alpha^* = \frac{(1-\gamma)\mu}{\gamma\sigma^2}$ and

$$\bar{\Lambda}(\alpha^*) = -\frac{(\gamma - 1)^2\mu^2}{2\gamma\sigma^2}.$$

(ii) If $\frac{(1-\gamma)\mu}{\gamma\sigma^2} < L$, then the optimal ratio is $\alpha^* = L$ and

$$\bar{\Lambda}(\alpha^*) = (\gamma - 1)r + (\gamma - 1)\mu L + \frac{1}{2}\gamma\sigma^2 L^2.$$

(iii) If $\frac{(1-\gamma)\mu}{\gamma\sigma^2} > R$, then the optimal ratio is $\alpha^* = R$ and

$$\bar{\Lambda}(\alpha^*) = (\gamma - 1)r + (\gamma - 1)\mu R + \frac{1}{2}\gamma\sigma^2 R^2.$$

Let us compare two risk measures $\mathcal{R}_t$ and Λ_t . We can also find an optimal ratio α^* that minimizes the growth rate of $\mathcal{R}_t$ as $t \rightarrow \infty$ using Proposition 1. Because $\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t$ is independent of γ , the optimal α^* is also independent of the investor's risk aversion. The computation is similar to the one above; thus, we omit it. In this sense, the modified risk measure is better for long-term portfolios. It gives the optimal ratio α^* depending on the risk-averse parameter γ , but the conventional risk measure does not.

3.2. Kim–Omberg Model

We consider CPPs under the Kim–Omberg model [9]. Assume that the state process X satisfies

$$dX_t = (\theta - kX_t) dt + a dW_t, \quad X_0 \in \mathbb{R}$$

for $k, a > 0$ and $\theta \in \mathbb{R}$. This SDE has an explicit solution:

$$X_t = X_0 e^{-kt} + \frac{\theta}{k}(1 - e^{-kt}) + a e^{-kt} \int_0^t e^{ks} dW_s, \quad t \geq 0. \quad (10)$$

The short rate is a constant $r \in \mathbb{R}$, and the risky asset is

$$dS_t/S_t = (r + \mu X_t) dt + \sigma dB_t$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. This implies that the state process X is the risk premium divided by μ . The wealth process of the CPP with ratio α is

$$\Pi_t = \Pi_0 e^{(r - \frac{1}{2}\alpha^2\sigma^2)t + \alpha\mu \int_0^t X_s ds + \alpha\sigma B_t}, \quad t \geq 0.$$

Proposition 3. Under the Kim–Omberg model, we define

$$C_1 = \frac{a^2\mu^2}{k^2} - \frac{2\rho a\sigma\mu}{k} + \frac{\sigma^2}{2}, \quad C_2 = \frac{\theta\mu}{2k}, \quad C_3 = r.$$

Then, for the CPP with ratio $\alpha > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = C_1\alpha^2 - 2C_2\alpha + C_3.$$

Proof. We define a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{1}{2}\alpha^2\sigma^2 t + \alpha\sigma B_t}.$$

Then, $(\tilde{W}_s, \tilde{B}_s)_{0 \leq s \leq t} = (W_s - \alpha\rho\sigma s, B_s - \alpha\sigma s)_{0 \leq s \leq t}$ is a Brownian motion with correlation ρ under measure $\tilde{\mathbb{P}}$.

The state process satisfies

$$dX_s = (\theta + \alpha\rho a\sigma - kX_s) ds + a d\tilde{W}_s, \quad 0 \leq s \leq t.$$

Observe that

$$\begin{aligned} \mathbb{E}^\mathbb{P}[\Pi_t] &= \Pi_0 e^{rt} \mathbb{E}^\mathbb{P}[e^{\alpha\mu \int_0^t X_s ds - \frac{1}{2}\alpha^2\sigma^2 t + \alpha\sigma B_t}] \\ &= \Pi_0 e^{rt} \mathbb{E}^{\tilde{\mathbb{P}}}[e^{\alpha\mu \int_0^t X_s ds}] \\ &= \Pi_0 e^{(r + \frac{\alpha^2 a^2 \mu^2}{2k^2} - \frac{(\theta + \alpha\rho a\sigma)\alpha\mu}{k})t} f_1(t) \end{aligned} \quad (11)$$

where $f_1(t)$ converges to a positive constant as $t \rightarrow \infty$. For the last equality, we have used Lemma A1. By a similar computation, we obtain

$$\mathbb{E}^\mathbb{P}[\Pi_t^2] = \Pi_0^2 e^{(2r + \frac{2\alpha^2 a^2 \mu^2}{k^2} - \frac{2(\theta + 2\alpha\rho a\sigma)\alpha\mu}{k} + \alpha^2\sigma^2)t} f_2(t), \quad (12)$$

where $f_2(t)$ converges to a positive constant as $t \rightarrow \infty$. It is clear that Equation (3) is satisfied. Then,

$$\hat{\mathcal{R}}_t = \gamma \sqrt{\mathbb{E}[\Pi_t^2]} = \gamma \Pi_0 e^{(r + \frac{\alpha^2 a^2 \mu^2}{k^2} - \frac{(\theta + 2\alpha\rho a\sigma)\alpha\mu}{k} + \frac{\alpha^2\sigma^2}{2})t} \sqrt{f_2(t)}. \quad (13)$$

It follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\mathcal{R}}_t = r + \frac{\alpha^2 a^2 \mu^2}{k^2} - \frac{(\theta + 2\alpha \rho a \sigma) \alpha \mu}{k} + \frac{\alpha^2 \sigma^2}{2} = C_1 \alpha^2 - 2C_2 \alpha + C_3.$$

This gives the desired result. $\square$

Proposition 4. Under the Kim–Omberg model, we define

$$C_1 = \frac{(\gamma - \frac{1}{2}) a^2 \mu^2}{k^2} - \frac{(2\gamma - 1) \rho a \sigma \mu}{k} + \frac{\gamma \sigma^2}{2}, \quad C_2 = \frac{(\gamma - 1) \theta \mu}{2k}, \quad C_3 = \gamma(r - 1).$$

Then, for the CPP with ratio $\alpha > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = C_1 \alpha^2 - 2C_2 \alpha + C_3.$$

Proof. From the proof of Proposition 3, we obtain

$$\hat{\Lambda}_t = \frac{(\mathbb{E}[\Pi_t^2])^{\gamma/2}}{\mathbb{E}[\Pi_t]} = \Pi_0^{\gamma-1} e^{(\gamma(r-1) + \frac{(\gamma-\frac{1}{2})a^2\mu^2}{k^2} - \frac{(\gamma-1)\theta\alpha\mu + (2\gamma-1)a^2\rho a\sigma\mu}{k} + \frac{\gamma\alpha^2\sigma^2}{2})t} \frac{f_2^{\frac{\gamma}{2}}(t)}{f_1(t)}. \quad (14)$$

It follows that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\Lambda}_t \\ &= \gamma(r - 1) + \frac{(\gamma - \frac{1}{2}) a^2 \mu^2}{k^2} - \frac{(\gamma - 1) \theta \alpha \mu + (2\gamma - 1) a^2 \rho a \sigma \mu}{k} + \frac{\gamma \alpha^2 \sigma^2}{2} \\ &= C_1 \alpha^2 - 2C_2 \alpha + C_3. \end{aligned} \quad (15)$$

This completes the proof. $\square$

We find an optimal ratio α^* that minimizes the growth rate of Λ_t as $t \rightarrow \infty$. Assume that ratio α is allowed in a compact interval $[L, R]$ for $0 < L < R$. As a mapping of α , the function

$$\bar{\Lambda}(\alpha) := \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = C_1 \alpha^2 - 2C_2 \alpha + C_3$$

is continuous on the compact interval $[L, R]$. Thus, the function achieves its minimum value.

- (i) If $C_1 > 0$, then $\bar{\Lambda}$ is a convex quadratic function in α , and $\alpha = \frac{C_2}{C_1}$ is a critical point. If $L \leq \frac{C_2}{C_1} \leq R$, then the optimal ratio is $\alpha^* = \frac{C_2}{C_1}$, and $\bar{\Lambda}(\alpha^*) = -\frac{C_2^2}{C_1} + C_3$. If $\frac{C_2}{C_1} < L$, then the optimal ratio is $\alpha^* = L$, and $\bar{\Lambda}(\alpha^*) = C_1 L^2 - 2C_2 L + C_3$. If $\frac{C_2}{C_1} > R$, then the optimal ratio is $\alpha^* = R$, and $\bar{\Lambda}(\alpha^*) = C_1 R^2 - 2C_2 R + C_3$.
- (ii) If $C_1 < 0$, then $\bar{\Lambda}$ is a concave quadratic function of α , and $\alpha = \frac{C_2}{C_1}$ is a critical point. If $\frac{C_2}{C_1} < \frac{1}{2}(L + R)$, then the optimal ratio is $\alpha^* = R$. If $\frac{C_2}{C_1} > \frac{1}{2}(L + R)$, then the optimal ratio is $\alpha^* = L$. If $\frac{C_2}{C_1} = \frac{1}{2}(L + R)$ then both $\alpha^* = R$ and $\alpha^* = L$ are optimal.
- (iii) If $C_1 = 0, C_2 > 0$, then the optimal ratio is $\alpha^* = R$, and $\bar{\Lambda}(\alpha^*) = -2C_2 R + C_3$. If $C_1 = 0, C_2 < 0$, then the optimal ratio is $\alpha^* = L$, and $\bar{\Lambda}(\alpha^*) = -2C_2 L + C_3$. If $C_1 = 0, C_2 = 0$, then $\bar{\Lambda}(\alpha) = C_3$ for all α .

We can also find an optimal ratio α^* that minimizes the growth rate of $\mathcal{R}_t$ as $t \rightarrow \infty$. Because the computation is similar to the one above, we omit it. We can observe that the modified risk measure gives the optimal ratio α^* depending on the risk-averse parameter γ , but the conventional one does not.

3.3. Heston Model

We consider CPPs under the Heston model [10]. Assume that the state process X satisfies

$$dX_t = (\theta - kX_t) dt + a\sqrt{X_t} dW_t, \quad X_0 > 0$$

for $k, a > 0$ and $\theta \geq a^2/2$. The short rate is a constant $r \in \mathbb{R}$, and the risky asset is

$$dS_t/S_t = (r + \mu) dt + \sigma X_t^{1/2} dB_t$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. This implies that the state process X is the squared volatility divided by σ^2 . We assume that ratio α is allowed between L and R for $0 < L < R$. The wealth process of the CPP with ratio

$$\alpha \in [L, R] \quad (16)$$

is

$$\Pi_t = \Pi_0 e^{(r+\alpha\mu)t - \frac{1}{2}\alpha^2\sigma^2 \int_0^t X_s ds + \alpha\sigma \int_0^t \sqrt{X_s} dB_s}, \quad t \geq 0.$$

We assume that

$$k \geq (2|\rho| + \sqrt{2})a\sigma R. \quad (17)$$

Proposition 5. Under the Heston model, we define

$$C_1 = \mu - \frac{\rho\theta\sigma}{a}, \quad C_2 = r + \frac{\theta k}{2a^2}, \quad C_3 = \frac{\theta^2\sigma^2}{2a^2}(2\rho^2 - 1), \quad C_4 = \frac{\rho k\theta^2\sigma}{2a^3}, \quad C_5 = \frac{\theta^2 k^2}{4a^4}. \quad (18)$$

Then, for the CPP with ratio $\alpha \in [L, R]$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

Proof. It is clear that $\mathbb{E}^\mathbb{P}[\Pi_t] = \Pi_0 e^{(r+\alpha\mu)t}$. We estimate $\mathbb{E}^\mathbb{P}[\Pi_t^2]$ using a method similar to that presented in the proof of Proposition 3. Let us define a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-2\alpha^2\sigma^2 \int_0^t X_s ds + 2\alpha\sigma \int_0^t \sqrt{X_s} dB_s}.$$

Then, $(\tilde{W}_s, \tilde{B}_s)_{0 \leq s \leq t} = (W_s - 2\alpha\rho\sigma \int_0^s \sqrt{X_u} du, B_s - 2\alpha\sigma \int_0^s \sqrt{X_u} du)_{0 \leq s \leq t}$ is a Brownian motion with correlation ρ under measure $\tilde{\mathbb{P}}$. The state process satisfies

$$dX_s = (\theta - \ell X_s) ds + a\sqrt{X_s} d\tilde{W}_s, \quad 0 \leq s \leq t,$$

where $\ell := k - 2\alpha\rho a\sigma > 0$. Subsequently, we define

$$\eta := \frac{\ell}{a^2} - \sqrt{\frac{\ell^2}{a^4} - \frac{2\alpha^2\sigma^2}{a^2}}.$$

Observe that the inside of the square root is non-negative and $\ell, \eta > 0$ by Equation (17). It follows that

$$\begin{aligned} \mathbb{E}^\mathbb{P}[\Pi_t^2] &= \Pi_0^2 e^{2(r+\alpha\mu)t} \mathbb{E}^\mathbb{P}[e^{-\alpha^2\sigma^2 \int_0^t X_s ds + 2\alpha\sigma \int_0^t \sqrt{X_s} dB_s}] \\ &= \Pi_0^2 e^{2(r+\alpha\mu)t} \mathbb{E}^{\tilde{\mathbb{P}}}[e^{\alpha^2\sigma^2 \int_0^t X_s ds}] \\ &= \Pi_0^2 e^{(2r+2\alpha\mu+\theta\eta)t} f(t), \end{aligned} \quad (19)$$

where $f(t)$ converges to a positive constant as $t \rightarrow \infty$. For the last equality, we have used Lemma A2. It is clear that Equation (3) is satisfied. Then,

$$\hat{\mathcal{R}}_t = \gamma \sqrt{\mathbb{E}[\Pi_t^2]} = \gamma \Pi_0 e^{(r+\alpha\mu+\frac{\theta\eta}{2})t} \sqrt{f(t)}. \quad (20)$$

It follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\mathcal{R}}_t = r + \alpha\mu + \frac{\theta\eta}{2} = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

This gives the desired result. $\square$

Proposition 6. Under the Heston model, we define

$$\begin{aligned} C_1 &= (\gamma - 1)\mu - \frac{\gamma\rho\theta\sigma}{a}, \quad C_2 = (\gamma - 1)r + \frac{\gamma\theta k}{2a^2}, \\ C_3 &= \frac{\gamma^2\theta^2\sigma^2}{2a^2}(2\rho^2 - 1), \quad C_4 = \frac{\gamma^2\rho k\theta^2\sigma}{2a^3}, \quad C_5 = \frac{\gamma^2\theta^2k^2}{4a^4}. \end{aligned} \quad (21)$$

Then, for the CPP with ratio $\alpha \in [L, R]$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

Proof. From the proof of Proposition 5, we obtain

$$\hat{\Lambda}_t = \frac{(\mathbb{E}[\Pi_t^2])^{\gamma/2}}{\mathbb{E}[\Pi_t]} = \Pi_0^{\gamma-1} e^{((\gamma-1)r + (\gamma-1)\alpha\mu + \frac{1}{2}\gamma\theta\eta)t} f^{\frac{\gamma}{2}}(t). \quad (22)$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\Lambda}_t = (\gamma - 1)r + (\gamma - 1)\alpha\mu + \frac{1}{2}\gamma\theta\eta \\ &= C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}. \end{aligned} \quad (23)$$

This completes the proof. $\square$

We now consider an optimal ratio α^* for the long-term CPPs. To emphasize the dependence of the long-term limit on α , we define

$$\bar{\Lambda}(\alpha) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}. \quad (24)$$

By direct calculation,

$$\bar{\Lambda}'(\alpha) = C_1 - \frac{C_3\alpha - C_4}{\sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}}, \quad \bar{\Lambda}''(\alpha) = \frac{C_4^2 - C_3C_5}{(C_3\alpha^2 - 2C_4\alpha + C_5)^{3/2}}.$$

- (i) If $C_4^2 - C_3C_5 > 0$ and $C_3 < C_1^2$, then function $\bar{\Lambda}$ is strictly convex, and $\bar{\Lambda}'(\alpha) = 0$ has a unique solution

$$\bar{\alpha} = -\frac{C_4}{C_3} + \frac{|C_1|}{C_3} \sqrt{\frac{C_3C_5 - C_4^2}{C_3 - C_1^2}}.$$

If $\bar{\alpha}$ lies in $[L, R]$, then $\alpha^* = \bar{\alpha}$ is the optimal ratio. If $\bar{\alpha} > R$, then $\alpha^* = R$ is optimal, and if $\bar{\alpha} < L$, then $\alpha^* = L$ is optimal.

- (ii) If $C_4^2 - C_3C_5 > 0$ and $C_3 \geq C_1^2$, then function $\bar{\Lambda}$ is strictly convex, and $\bar{\Lambda}'(\alpha) = 0$ has no solutions. Furthermore, if $C_1 > 0$, $\bar{\Lambda}(\alpha)$ is an increasing function; thus, the optimal ratio is $\alpha^* = L$. If $C_1 \leq 0$, $\bar{\Lambda}(\alpha)$ is a decreasing function; thus, the optimal ratio is $\alpha^* = R$.
- (iii) If $C_4^2 - C_3C_5 < 0$ and $C_3 > C_1^2$, function $\bar{\Lambda}$ is strictly concave, and $\bar{\Lambda}'(\alpha) = 0$ has a unique solution $\bar{\alpha}$ defined above. If $\bar{\alpha}$ lies in $[L, R]$ and $\bar{\Lambda}(L) \geq \bar{\Lambda}(R)$, then $\alpha^* = R$ is

optimal, and if $\bar{\alpha}$ lies in $[L, R]$ and $\bar{\Lambda}(L) \leq \bar{\Lambda}(R)$, then $\alpha^* = L$ is optimal. If $\bar{\alpha} > R$, then $\alpha^* = L$ is optimal, and if $\bar{\alpha} < L$, then $\alpha^* = R$ is optimal.

- (iv) If $C_4^2 - C_3C_5 < 0$ and $C_3 \leq C_1^2$, then function $\bar{\Lambda}$ is strictly concave, and $\bar{\Lambda}'(\alpha) = 0$ has no solutions. Furthermore, if $C_1 > 0$, $\bar{\Lambda}(\alpha)$ is an increasing function; thus, the optimal ratio is $\alpha^* = L$. If $C_1 < 0$, $\bar{\Lambda}(\alpha)$ is a decreasing function; thus, the optimal ratio is $\alpha^* = R$.
- (v) If $C_4^2 - C_3C_5 = 0$, then $C_3 \geq 0$ (because $C_3 = C_4^2/C_5$ and $C_5 > 0$), and the function $\bar{\Lambda}$ is equal to

$$\bar{\Lambda}(\alpha) = C_1\alpha + C_2 - \sqrt{C_3}\left|\alpha - \frac{C_4}{C_3}\right|.$$

If $C_1 - \sqrt{C_3} \geq 0$, then $\bar{\Lambda}$ is monotonically increasing; thus, $\alpha^* = L$ is optimal. If $C_1 - \sqrt{C_3} < 0$ and $C_1 + \sqrt{C_3} \leq 0$, then $\bar{\Lambda}$ is monotone decreasing; thus, $\alpha^* = R$ is optimal. If $C_1 - \sqrt{C_3} < 0$, $C_1 + \sqrt{C_3} > 0$ and $C_4/C_3 \geq (L + R)/2$, then $\alpha^* = L$ is optimal. If $C_1 - \sqrt{C_3} < 0$, $C_1 + \sqrt{C_3} > 0$ and $C_4/C_3 \leq (L + R)/2$, then $\alpha^* = R$ is optimal.

3.4. 3/2 Stochastic Volatility Model

We consider CPPs under the 3/2 stochastic volatility model [11]. Assume that the state process X satisfies

$$dX_t = (\theta - kX_t)X_t dt + aX_t^{3/2} dW_t, \quad X_0 > 0$$

for $\theta, k, a > 0$. The short rate is a constant $r \in \mathbb{R}$, and the risky asset is

$$dS_t/S_t = (r + \mu) dt + \sigma X_t^{1/2} dB_t$$

for $\mu \in \mathbb{R}$ and $\sigma > 0$. This implies that the state process X is the squared volatility divided by σ^2 . We assume that ratio α is allowed between L and R for $0 < L < R$. The wealth process of the CPP with ratio

$$\alpha \in [L, R] \quad (25)$$

is

$$\Pi_t = \Pi_0 e^{(r+\alpha\mu)t - \frac{1}{2}\alpha^2\sigma^2 \int_0^t X_s ds + \alpha\sigma \int_0^t \sqrt{X_s} dB_s}, \quad t \geq 0.$$

Assume that

$$k \geq \frac{a^2}{2} + (2|\rho| + \sqrt{2})a\sigma R. \quad (26)$$

Proposition 7. Under the 3/2 stochastic volatility model, we define

$$\begin{aligned} C_1 &= \mu - \frac{\rho\theta\sigma}{a}, \quad C_2 = r + \frac{\theta}{2} \left(\frac{k}{a^2} + \frac{1}{2} \right), \\ C_3 &= \frac{\theta^2\sigma^2}{2a^2} (2\rho^2 - 1), \quad C_4 = \frac{\rho\theta^2\sigma}{4} \left(\frac{2k}{a^3} + \frac{1}{a} \right), \quad C_5 = \frac{\theta^2}{4} \left(\frac{k}{a^2} + \frac{1}{2} \right)^2. \end{aligned} \quad (27)$$

Then, for the CPP with ratio $\alpha \in [L, R]$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

Proof. It is clear that $\mathbb{E}^\mathbb{P}[\Pi_t] = \Pi_0 e^{(r+\alpha\mu)t}$. We estimate $\mathbb{E}^\mathbb{P}[\Pi_t^2]$ by analogy with the proof of Proposition 3. Let us define a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-2\alpha^2\sigma^2 \int_0^t X_s ds + 2\alpha\sigma \int_0^t \sqrt{X_s} ds}.$$

Then, $(\tilde{W}_s, \tilde{B}_s)_{0 \leq s \leq t} = (W_s - 2\alpha\rho\sigma \int_0^s \sqrt{X_u} du, B_s - 2\alpha\sigma \int_0^s \sqrt{X_u} du)_{0 \leq s \leq t}$ is a Brownian motion with correlation ρ under measure $\tilde{\mathbb{P}}$. The state process satisfies

$$dX_s = (\theta - \ell X_s)X_s ds + aX_s^{3/2} d\tilde{W}_s, \quad 0 \leq s \leq t,$$

where $\ell := k - 2\alpha\rho a\sigma$. Furthermore, we define

$$\eta := \frac{\ell}{a^2} + \frac{1}{2} - \sqrt{\left(\frac{\ell}{a^2} + \frac{1}{2}\right)^2 - \frac{2\alpha^2\sigma^2}{a^2}}.$$

Observe that the inside of the square root is non-negative and $\ell, \eta > 0$ by Equation (26). It follows that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\Pi_t^2] &= \Pi_0^2 e^{2(r+\alpha\mu)t} \mathbb{E}^{\mathbb{P}}[e^{-\alpha^2\sigma^2 \int_0^t X_s ds + 2\alpha\sigma \int_0^t \sqrt{X_s} dB_s}] \\ &= \Pi_0^2 e^{2(r+\alpha\mu)t} \mathbb{E}^{\tilde{\mathbb{P}}}[e^{\alpha^2\sigma^2 \int_0^t X_s ds}] \\ &= \Pi_0^2 e^{(2r+2\alpha\mu+\theta\eta)t} f(t), \end{aligned} \quad (28)$$

where $f(t)$ converges to a positive constant as $t \rightarrow \infty$. For the last equality, we have used Lemma A3. It is clear that Equation (3) is satisfied. Then,

$$\hat{\mathcal{R}}_t = \gamma \sqrt{\mathbb{E}[\Pi_t^2]} = \gamma \Pi_0 e^{(r+\alpha\mu+\frac{\theta\eta}{2})t} \sqrt{f(t)}. \quad (29)$$

It follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\mathcal{R}}_t = r + \alpha\mu + \frac{\theta\eta}{2} = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

This gives the desired result. $\square$

Proposition 8. Under the 3/2 stochastic volatility model, we define

$$\begin{aligned} C_1 &= (\gamma - 1)\mu - \frac{\gamma\rho\theta\sigma}{a}, \quad C_2 = (\gamma - 1)r + \frac{\gamma\theta}{2} \left(\frac{k}{a^2} + \frac{1}{2} \right), \\ C_3 &= \frac{\gamma^2\theta^2\sigma^2}{2a^2} (2\rho^2 - 1), \quad C_4 = \frac{\gamma^2\theta^2\rho\sigma}{4} \left(\frac{2k}{a^3} + \frac{1}{a} \right), \quad C_5 = \frac{\gamma^2\theta^2}{4} \left(\frac{k}{a^2} + \frac{1}{2} \right)^2. \end{aligned} \quad (30)$$

Then, for the CPP with ratio $\alpha \in [L, R]$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t = C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}.$$

Proof. From the proof of Proposition 7, we obtain

$$\hat{\Lambda}_t = \frac{(\mathbb{E}[\Pi_t^2])^{\gamma/2}}{\mathbb{E}[\Pi_t]} = \Pi_0^{\gamma-1} e^{((\gamma-1)r + (\gamma-1)\alpha\mu + \frac{1}{2}\gamma\theta\eta)t} f^{\frac{\gamma}{2}}(t). \quad (31)$$

Thus,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t &= \lim_{t \rightarrow \infty} \frac{1}{t} \ln \hat{\Lambda}_t = (\gamma - 1)r + (\gamma - 1)\alpha\mu + \frac{1}{2}\gamma\theta\eta \\ &= C_1\alpha + C_2 - \sqrt{C_3\alpha^2 - 2C_4\alpha + C_5}. \end{aligned} \quad (32)$$

This completes the proof. $\square$

The optimal ratio α^* for the long-term CPPs can be computed in the same way as the analysis of Equation (24). Thus, we omit the details here.

4. Increasing Amount Portfolio

We consider an increasing amount portfolio (IAP) in which the amount invested in the risky asset is increasing with time. More precisely, the amount at time t is αe^{rt} for a constant $\alpha \in \mathbb{R}$, where $r \geq 0$ is the constant short rate. When $\alpha < 0$, the amount is decreasing; however, we simply say that this portfolio is an IAP for all $\alpha \in \mathbb{R}$ for convenience. The value of the IAP is

$$d\Pi_t = (r\Pi_t + \alpha\mu(X_t)e^{rt})dt + \alpha\sigma(X_t)e^{rt}dB_t,$$

and this gives

$$\Pi_t = \Pi_0 e^{rt} + \alpha e^{rt} \int_0^t \mu(X_s) ds + \alpha e^{rt} \int_0^t \sigma(X_s) dB_s, \quad t \geq 0. \quad (33)$$

This section proposes a modified mean-variance risk measure for IAPs and investigates an optimal amount for long-term investments.

We propose a modified mean-variance risk measure:

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}].$$

As is well known, Π_t / e^{rt} is the discounted portfolio value at time t . For long-term investments, we focus on large-time behaviors of $\mathcal{R}_t$ and Δ_t . Specifically, two limit values

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t}$$

are computed for several factor models.

The conventional mean-variance risk measure $\mathcal{R}_t$ has two flaws when working with long-term IAPs. We will see that

$$\mathbb{E}[\Pi_t] \simeq te^{rt} \quad \text{and} \quad \sqrt{\text{var}[\Pi_t]} \simeq \sqrt{t}e^{rt}.$$

Here, for two functions f and g , the notation $f(t) \simeq g(t)$ means that $\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)}$ exists and is finite. It follows that

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} = - \lim_{t \rightarrow \infty} \frac{\mathbb{E}[\Pi_t]}{te^{rt}}. \quad (34)$$

Hence, the growth rate of the conventional risk measure is only determined by the mean and cannot capture the variance. On the contrary, the growth rate of the modified risk measure reflects both the mean and variance.

Another flaw is that the growth rate of the conventional risk measure cannot capture the investor's risk aversion. This flaw is common for both CPPs and IAPs. From Equation (34), it is clear that $\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}}$ is independent of parameter γ . On the contrary, we will see that the growth rate of the modified risk measure depends on parameter γ . This implies that the modified risk measure captures the risk aversion parameter better than the conventional risk measure for long-term IAPs.

4.1. Black–Scholes Model

First, we consider the Black–Scholes model presented in Equation (6). The value of the increasing amount portfolio with $\alpha \in \mathbb{R}$ is

$$\Pi_t = \Pi_0 e^{rt} + \alpha \mu t e^{rt} + \alpha \sigma e^{rt} B_t, \quad t \geq 0,$$

by Equation (33).

Proposition 9. *Under the Black–Scholes model, for the IAP with $\alpha \in \mathbb{R}$, we have*

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} = -\alpha\mu, \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = \gamma\alpha^2\sigma^2 - \alpha\mu.$$

Proof. By direct calculation,

$$\mathbb{E}[\Pi_t] = \Pi_0 e^{rt} + \alpha \mu t e^{rt}$$

and

$$\text{Var}[\Pi_t] = \alpha^2 \sigma^2 t e^{2rt}.$$

Thus,

$$\mathcal{R}_t = \gamma \sqrt{\text{Var}[\Pi_t]} - \mathbb{E}[\Pi_t] = \gamma |\alpha| \sigma \sqrt{t} e^{rt} - (\Pi_0 e^{rt} + \alpha \mu t e^{rt})$$

and

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}] = \gamma \alpha^2 \sigma^2 t - (\Pi_0 + \alpha \mu t).$$

This gives us the desired result. $\square$

Let us consider the optimal amount α^* for long-term IAPs. To emphasize the dependence of the growth rate on α , we define

$$\bar{\Delta}(\alpha) = \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = C_1 \alpha^2 - 2C_2 \alpha, \quad (35)$$

where

$$C_1 = \gamma \sigma^2, \quad C_2 = \frac{\mu}{2}.$$

Because $\bar{\Delta}$ is a quadratic function of α and $\bar{\Delta}'' = 2C_1 > 0$, the function $\bar{\Delta}$ achieves its minimum at $\alpha^* = \frac{C_2}{C_1}$, and the minimum value is $\bar{\Delta}(\alpha^*) = -\frac{C_2^2}{C_1}$.

4.2. Kim–Omberg Model

We recall the Kim–Omberg model presented in Section 3.2. The IAP value is

$$\Pi_t = \Pi_0 e^{rt} + \alpha \mu e^{rt} \int_0^t X_s ds + \alpha \sigma e^{rt} B_t, \quad t \geq 0,$$

by Equation (33).

Proposition 10. Under the Kim–Omberg model, for the IAP with $\alpha \in \mathbb{R}$, we have

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{t e^{rt}} = -\frac{\alpha \mu \theta}{k}, \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = \gamma \alpha^2 \left(\frac{\mu^2 a^2}{k^2} + \frac{2\rho \mu a \sigma}{k} + \sigma^2 \right) - \frac{\alpha \mu \theta}{k}.$$

Proof. We compute $\mathbb{E}[\Pi_t]$ and $\text{Var}[\Pi_t]$. From Equation (10),

$$\mathbb{E}[X_s] = X_0 e^{-kt} + \frac{\theta}{k} (1 - e^{-kt}),$$

and this gives

$$\mathbb{E}[\Pi_t] = \Pi_0 e^{rt} + \alpha \mu e^{rt} \int_0^t \mathbb{E}[X_s] ds = \Pi_0 e^{rt} + \frac{\alpha \mu}{k} e^{rt} \left(X_0 (1 - e^{-kt}) + \theta \left(t - \frac{1}{k} (1 - e^{-kt}) \right) \right). \quad (36)$$

We define $Y_t = X_t - \mathbb{E}[X_t] = a e^{-kt} \int_0^t e^{ks} dW_s$. Then,

$$\begin{aligned} \text{Var}[\Pi_t] &= \alpha^2 e^{2rt} \text{Var} \left[\mu \int_0^t X_s ds + \sigma B_t \right] = \alpha^2 e^{2rt} \mathbb{E} \left[\left(\mu \int_0^t Y_s ds + \sigma B_t \right)^2 \right] \\ &= \alpha^2 e^{2rt} \left(\mu^2 \mathbb{E} \left[\left(\int_0^t Y_s ds \right)^2 \right] + 2\mu \sigma \mathbb{E} \left[B_t \int_0^t Y_s ds \right] + \sigma^2 \mathbb{E}[B_t^2] \right). \end{aligned} \quad (37)$$

We calculate the two expectations

$$\mathbb{E} \left[\left(\int_0^t Y_s ds \right)^2 \right] \quad \text{and} \quad \mathbb{E} \left[B_t \int_0^t Y_s ds \right].$$

For $0 \leq s \leq u$,

$$\begin{aligned}\mathbb{E}[Y_s Y_u] &= a^2 e^{-k(s+u)} \mathbb{E}\left[\int_0^s e^{kv} dW_v \int_0^u e^{kv} dW_v\right] \\ &= a^2 e^{-k(s+u)} \int_0^s e^{2kv} dv \\ &= \frac{a^2}{2k} (e^{-k(u-s)} - e^{-k(u+s)}).\end{aligned}\quad (38)$$

Then,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t Y_s ds\right)^2\right] &= \mathbb{E}\left[\int_0^t \int_0^t Y_s Y_u ds du\right] \\ &= 2 \int_0^t \int_0^u \mathbb{E}[Y_s Y_u] ds du \\ &= \frac{a^2}{k} \int_0^t \int_0^u (e^{-k(u-s)} - e^{-k(u+s)}) ds du \\ &= \frac{a^2}{k^2} \left(t - \frac{2}{k}(1 - e^{-kt}) + \frac{1}{2k}(1 - e^{-2kt})\right).\end{aligned}\quad (39)$$

Given that (W, B) is a correlated Brownian motion with correlation ρ , we know that

$$\begin{aligned}\mathbb{E}[B_s X_s] &= a e^{-ks} \mathbb{E}\left[B_s \int_0^s e^{ku} dW_u\right] \\ &= a \rho e^{-ks} \mathbb{E}\left[W_s \int_0^s e^{ku} dW_u\right] \\ &= \frac{a \rho}{k} (1 - e^{-ks}).\end{aligned}\quad (40)$$

Thus,

$$\begin{aligned}\mathbb{E}\left[B_t \int_0^t X_s ds\right] &= \mathbb{E}\left[\int_0^t B_t X_s ds\right] = \int_0^t \mathbb{E}[B_t X_s] ds \\ &= \int_0^t \mathbb{E}[B_s X_s] ds = \frac{a \rho}{k} \int_0^t (1 - e^{-ks}) ds \\ &= \frac{a \rho}{k} \left(t - \frac{1}{k}(1 - e^{-kt})\right).\end{aligned}\quad (41)$$

Therefore, Equation (37) becomes

$$\text{Var}[\Pi_t] = \alpha^2 e^{2rt} \left(\frac{\mu^2 a^2}{k^2} \left(t - \frac{2}{k}(1 - e^{-kt}) + \frac{1}{2k}(1 - e^{-2kt})\right) + \frac{2\rho\mu a\sigma}{k} \left(t - \frac{1}{k}(1 - e^{-kt})\right) + \sigma^2 t \right). \quad (42)$$

Moreover, direct calculation gives us

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{t e^{rt}} = -\frac{\alpha \mu \theta}{k}, \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = \gamma \alpha^2 \left(\frac{\mu^2 a^2}{k^2} + \frac{2\rho\mu a\sigma}{k} + \sigma^2 \right) - \frac{\alpha \mu \theta}{k}.$$

This completes the proof. $\square$

We now consider the optimal amount α^* for long-term IAPs. Let

$$\bar{\Delta}(\alpha) = \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = C_1 \alpha^2 - 2C_2 \alpha, \quad (43)$$

where

$$C_1 = \gamma \left(\frac{\mu^2 a^2}{k^2} + \frac{2\rho\mu a\sigma}{k} + \sigma^2 \right), \quad C_2 = \frac{\mu \theta}{2k}.$$

Because $\bar{\Delta}$ is a quadratic function of α and $\bar{\Delta}'' = 2C_1 > 0$, the function $\bar{\Delta}$ achieves its minimum at $\alpha^* = \frac{C_2}{C_1}$, and the minimum value is $\bar{\Delta}(\alpha^*) = -\frac{C_2^2}{C_1}$.

4.3. Heston Model

We recall the Heston model presented in Section 3.3. The IAP value is

$$\Pi_t = \Pi_0 e^{rt} + \alpha \mu t e^{rt} + \alpha \sigma e^{rt} \int_0^t \sqrt{X_s} dB_s, \quad t \geq 0.$$

Proposition 11. Under the Heston model, for the IAP with $\alpha \in \mathbb{R}$, we have

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} = -\alpha\mu, \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = \frac{\gamma\alpha^2\theta\sigma^2}{k} - \alpha\mu.$$

Proof. We first estimate the behavior of $\mathbb{E}[X_s]$ as $s \rightarrow \infty$. The process X is a Cox-Ingersoll-Ross (CIR) process, and it has an invariant density function

$$d\pi(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx,$$

where $\alpha = \frac{2\theta}{a^2}$ and $\beta = \frac{2k}{a^2}$. As $s \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}[X_s] &\rightarrow \int_0^\infty x d\pi(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\beta x} dx \\ &= \frac{1}{\Gamma(\alpha)\beta} \int_0^\infty x^\alpha e^{-x} dx \\ &= \frac{1}{\Gamma(\alpha)\beta} \Gamma(\alpha+1) = \frac{\alpha}{\beta} = \frac{\theta}{k}. \end{aligned} \quad (44)$$

Thus, we have

$$\int_0^t \mathbb{E}[X_s] ds = tf(t),$$

where $f(t)$ converges to $\frac{\theta}{k}$ as $t \rightarrow \infty$.

By direct calculation,

$$\mathbb{E}[\Pi_t] = \Pi_0 e^{rt} + \alpha \mu t e^{rt}$$

and

$$\begin{aligned} \text{Var}[\Pi_t] &= \alpha^2 \sigma^2 e^{2rt} \mathbb{E} \left[\left(\int_0^t \sqrt{X_s} dB_s \right)^2 \right] \\ &= \alpha^2 \sigma^2 e^{2rt} \mathbb{E} \left[\int_0^t X_s ds \right] \\ &= \alpha^2 \sigma^2 e^{2rt} \int_0^t \mathbb{E}[X_s] ds = \alpha^2 \sigma^2 t e^{2rt} f(t). \end{aligned} \quad (45)$$

Therefore,

$$\mathcal{R}_t = \gamma \sqrt{\text{Var}[\Pi_t]} - \mathbb{E}[\Pi_t] = \gamma |\alpha| \sigma \sqrt{te^{rt}} \sqrt{f(t)} - (\Pi_0 e^{rt} + \alpha \mu t e^{rt})$$

and

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}] = \gamma \alpha^2 \sigma^2 t f(t) - (\Pi_0 + \alpha \mu t).$$

This implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} &= -\alpha\mu, \\ \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} &= \lim_{t \rightarrow \infty} (\gamma \alpha^2 \sigma^2 f(t) - \alpha\mu) = \frac{\gamma \alpha^2 \theta \sigma^2}{k} - \alpha\mu. \end{aligned} \quad (46)$$

This gives the desired result. $\square$

We now consider the optimal amount α^* for long-term IAPs. Let

$$\bar{\Delta}(\alpha) = \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = C_1 \alpha^2 - 2C_2 \alpha, \quad (47)$$

where

$$C_1 = \frac{\gamma \theta \sigma^2}{k}, \quad C_2 = \frac{\mu}{2}.$$

Because $\bar{\Delta}$ is a quadratic function of α and $\bar{\Delta}'' = 2C_1 > 0$, the function $\bar{\Delta}$ achieves its minimum at $\alpha^* = \frac{C_2}{C_1}$, and the minimum value is $\bar{\Delta}(\alpha^*) = -\frac{C_2^2}{C_1}$.

4.4. 3/2 Stochastic Volatility Model

We recall the 3/2 stochastic volatility model presented in Section 3.4. The IAP value is

$$\Pi_t = \Pi_0 e^{rt} + \alpha \mu t e^{rt} + \alpha \sigma e^{rt} \int_0^t \sqrt{X_s} dB_s, \quad t \geq 0.$$

Proposition 12. Under the 3/2 stochastic volatility model, for the IAP with $\alpha \in \mathbb{R}$, we have

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{t e^{rt}} = -\alpha \mu, \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = \frac{2\gamma \alpha^2 \theta \sigma^2}{2k + a^2} - \alpha \mu.$$

Proof. We first estimate the behavior of $\mathbb{E}[X_s]$ as $s \rightarrow \infty$. Let $Y = 1/X$. Then, Y is a CIR process, and it has an invariant density function

$$d\mu_\infty(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y} dy,$$

where $\alpha = \frac{2k}{a^2} + 2$ and $\beta = \frac{2\theta}{a^2}$. As $s \rightarrow \infty$,

$$\begin{aligned} \mathbb{E}[X_s] &= \mathbb{E}[1/Y_s] \rightarrow \int_0^\infty \frac{1}{y} d\mu_\infty(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-2} e^{-\beta y} dy \\ &= \frac{\beta}{\Gamma(\alpha)} \int_0^\infty y^{\alpha-2} e^{-y} dy \\ &= \frac{\beta}{\Gamma(\alpha)} \Gamma(\alpha-1) = \frac{\beta}{\alpha-1} = \frac{2\theta}{2k+a^2}. \end{aligned} \quad (48)$$

Thus, we have

$$\int_0^t \mathbb{E}[X_s] ds = t f(t),$$

where $f(t)$ converges to $\frac{2\theta}{2k+a^2}$ as $t \rightarrow \infty$.

By direct calculation,

$$\mathbb{E}[\Pi_t] = \Pi_0 e^{rt} + \alpha \mu t e^{rt}$$

and

$$\begin{aligned} \text{Var}[\Pi_t] &= \alpha^2 \sigma^2 e^{2rt} \mathbb{E} \left[\left(\int_0^t \sqrt{X_s} dB_s \right)^2 \right] \\ &= \alpha^2 \sigma^2 e^{2rt} \mathbb{E} \left[\int_0^t X_s ds \right] \\ &= \alpha^2 \sigma^2 e^{2rt} \int_0^t \mathbb{E}[X_s] ds = \alpha^2 \sigma^2 t e^{2rt} f(t). \end{aligned} \quad (49)$$

Therefore,

$$\mathcal{R}_t = \gamma \sqrt{\text{Var}[\Pi_t]} - \mathbb{E}[\Pi_t] = \gamma |\alpha| \sigma \sqrt{t e^{rt}} \sqrt{f(t)} - (\Pi_0 e^{rt} + \alpha \mu t e^{rt})$$

and

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}] = \gamma \alpha^2 \sigma^2 t f(t) - (\Pi_0 + \alpha \mu t).$$

This implies that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} &= -\alpha \mu, \\ \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} &= \lim_{t \rightarrow \infty} (\gamma \alpha^2 \sigma^2 f(t) - \alpha \mu) = \frac{2\gamma \alpha^2 \theta \sigma^2}{2k + a^2} - \alpha \mu. \end{aligned} \quad (50)$$

This gives the desired result. $\square$

Finally, we consider the optimal amount α^* for long-term IAPs. Let

$$\bar{\Delta}(\alpha) = \lim_{t \rightarrow \infty} \frac{\Delta_t}{t} = C_1 \alpha^2 - 2C_2 \alpha, \quad (51)$$

where

$$C_1 = \frac{2\gamma \theta \sigma^2}{2k + a^2}, \quad C_2 = \frac{\mu}{2}.$$

Because $\bar{\Delta}$ is a quadratic function of α and $\bar{\Delta}'' = 2C_1 > 0$, the function $\bar{\Delta}$ achieves its minimum at $\alpha^* = \frac{C_2}{C_1}$, and the minimum value is $\bar{\Delta}(\alpha^*) = -\frac{C_2^2}{C_1}$.

5. Conclusions

This study proposed modified mean-variance risk measures for long-term investment portfolios. Two types of portfolios were considered: CPP and IAP. In contrast to the conventional mean-variance risk measure

$$\mathcal{R}_t := \gamma \sqrt{\text{var}[\Pi_t]} - \mathbb{E}[\Pi_t],$$

we provided modified risk measures

$$\Lambda_t := \frac{(\text{var}[\Pi_t])^{\gamma/2}}{\mathbb{E}[\Pi_t]}$$

for CPPs and

$$\Delta_t = \gamma \text{Var}[\Pi_t / e^{rt}] - \mathbb{E}[\Pi_t / e^{rt}]$$

for IAPs, where $\gamma > 0$ is the risk aversion parameter.

The long-term growth rates of the conventional and modified risk measures of the two types of portfolios were calculated. For CPPs, two values

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln \mathcal{R}_t \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \ln \Lambda_t$$

were computed. For IAPs,

$$\lim_{t \rightarrow \infty} \frac{\mathcal{R}_t}{te^{rt}} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{\Delta_t}{t}$$

were computed. Several benefits of the modified risk measures were discussed. The growth rate of the modified risk measure depends on both the mean and variance of the portfolio; however, that of the conventional risk measure does not. In addition, the growth rate of the modified risk measure reflects the investor's risk aversion, whereas the conventional risk measure does not.

Our analysis was used for finding optimal long-term investment strategies. Based on the long-term growth rate, we calculated a constant proportion for CPPs and a constant amount for IAPs, which minimizes the growth rate of the modified risk measure. Several

factor models were covered as concrete examples: the Black–Scholes model, Kim–Omberg model, Heston model, and 3/2 stochastic volatility model.

The author would suggest the following topic for future research. In Economics and Finance, a risk measure usually comes from the preferences of an investor, which can be represented by a utility function. The relevant moments and parameters that affect the investor portfolio choice can be obtained from the utility function. It would be meaningful to argue which utility function gives rise to the modified risk measure and how it differs from other commonly used utility functions that gave rise to the conventional risk measures.

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Appendix A. Ornstein–Uhlenbeck (OU) Process

Appendices A–C describe several lemmas used in the proof of Propositions 3, 5, and 7, respectively. One of the main ideas of these lemmas is the Hansen–Scheinkman decomposition, and similar computations have been conducted at [12–14]. However, the formulations used in this study differ from the above literature. Thus, we present full computations in the proofs of these lemmas for the completeness of this paper.

The following lemma is used in the proof of Proposition 3. Here, we denote an underlying measure and a Brownian motion by $\tilde{\mathbb{P}}$ and $\tilde{W}$, respectively, instead of $\mathbb{P}$ and W . This is to ensure notational consistency with the proof of Proposition 3.

Several studies (for example, Proposition 2.6.2.1 in [15]) have been conducted regarding the expectation in Equation (A1) for $p < 0$. However, for $p \in \mathbb{R}$, no useful expressions could be found for our analysis. We therefore derive an asymptotic expression of expectation as $t \rightarrow \infty$.

Lemma A1. *Let X be a solution of the SDE*

$$dX_t = (\delta - kX_t) dt + a d\tilde{W}_t, \quad X_0 \in \mathbb{R},$$

where $k, a > 0$, $\delta \in \mathbb{R}$, and $\tilde{W}$ is a Brownian motion under a measure $\tilde{\mathbb{P}}$. Then, for $p \in \mathbb{R}$,

$$\mathbb{E}^{\tilde{\mathbb{P}}} [e^{-p \int_0^t X_s ds}] = f_1(t) e^{-\frac{p}{k} X_0} e^{-\lambda t}, \quad (\text{A1})$$

where $f_1(t)$ converges to a positive constant as $t \rightarrow \infty$.

Proof. Let $\mathcal{L}$ be the infinitesimal generator of the process X with killing rate $p \cdot$. Then,

$$(\mathcal{L}\phi)(x) = \frac{1}{2} a^2 \phi''(x) + (\delta - kx) \phi'(x) - px \phi(x).$$

By direct calculation, we obtain that

$$(\lambda, \phi(x)) = \left(-\frac{a^2 p^2}{2k^2} + \frac{\delta p}{k}, e^{-\frac{p}{k} x} \right)$$

satisfies $\mathcal{L}\phi = -\lambda\phi$. Therefore,

$$M_s := \frac{\phi(X_s)}{\phi(X_0)} e^{-p \int_0^s X_u du + \lambda s} = e^{-\frac{p}{k}(X_s - X_0) - p \int_0^s X_u du + \lambda s}, \quad 0 \leq s \leq t,$$

is a $\tilde{\mathbb{P}}$ -local martingale. It can be easily verified that this process is a $\tilde{\mathbb{P}}$ -martingale by using Theorem 5.1.8 of [16]. We define a measure $\hat{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}} = M_t. \quad (\text{A2})$$

Then, the process

$$\hat{W}_s = \tilde{W}_s + \frac{pa}{k}s, \quad 0 \leq s \leq t$$

is a $\hat{\mathbb{P}}$ -Brownian motion, and X satisfies

$$dX_t = \left(\delta - \frac{pa^2}{k} - kX_t \right) dt + a d\hat{W}_t.$$

It follows that

$$\mathbb{E}^{\hat{\mathbb{P}}}[e^{-p \int_0^t X_s ds}] = \mathbb{E}^{\tilde{\mathbb{P}}}[M_t e^{\frac{p}{k} X_t}] e^{-\frac{p}{k} X_0} e^{-\lambda t} = \mathbb{E}^{\hat{\mathbb{P}}}[e^{\frac{p}{k} X_t}] e^{-\frac{p}{k} X_0} e^{-\lambda t} = f_1(t) e^{-\frac{p}{k} X_0} e^{-\lambda t}, \quad (\text{A3})$$

where

$$f_1(t) = \mathbb{E}^{\hat{\mathbb{P}}}[e^{\frac{p}{k} X_t}]. \quad (\text{A4})$$

Now, we show that $f_1(t)$ converges to a positive constant as $t \rightarrow \infty$. Observe that the density function of X_t can be expressed as

$$z(x; t) := \frac{1}{\Sigma_t \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(x - m_t)^2}{\Sigma_t^2}},$$

where $m_t := X_0 e^{-kt} + \frac{1}{k}(\delta - \frac{pa^2}{k})(1 - e^{-kt})$ is the mean and $\Sigma_t^2 := \frac{a^2}{2k}(1 - e^{-2kt})$ is the variance. Then,

$$f_1(t) = \mathbb{E}^{\hat{\mathbb{P}}}[e^{\frac{p}{k} X_t}] = \int_{\mathbb{R}} e^{\frac{p}{k} x} z(x; t) dx \rightarrow \int_{\mathbb{R}} e^{\frac{p}{k} x} \pi(x) dx$$

as $t \rightarrow \infty$, where

$$\pi(x) := \frac{1}{\sqrt{\pi a^2/k}} e^{-\frac{(x - (\delta - pa^2/k)/k)^2}{a^2/k}}.$$

This completes the proof. $\square$

Appendix B. CIR Process

The following lemma is used in the proof of Proposition 5. Here, we denote an underlying measure and a Brownian motion by $\tilde{\mathbb{P}}$ and $\tilde{W}$, respectively, instead of $\mathbb{P}$ and W . This is for the notational consistency with the proof of Proposition 5.

Lemma A2. Let X be a solution of the SDE

$$dX_t = (\theta - \ell X_t) dt + a \sqrt{X_t} d\tilde{W}_t, \quad X_0 > 0,$$

where $\ell, a > 0$, $\theta \geq a^2/2$, and $\tilde{W}$ is a Brownian motion under a measure $\tilde{\mathbb{P}}$. Then, for $p \geq -\frac{\ell^2}{2a^2}$,

$$\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-p \int_0^t X_s ds}] = f(t) e^{\eta X_0} e^{-\lambda t},$$

where

$$\eta := \frac{\ell}{a^2} - \sqrt{\frac{\ell^2}{a^4} + \frac{2p}{a^2}}, \quad \lambda := -\theta\eta,$$

and $f(t)$ converges to a positive constant as $t \rightarrow \infty$.

Proof. Let $\mathcal{L}$ be the infinitesimal generator of the process X with killing rate $p \cdot$. Then,

$$(\mathcal{L}\phi)(x) = \frac{1}{2}a^2x\phi''(x) + (\theta - \ell x)\phi'(x) - px\phi(x).$$

By direct calculation, we obtain that

$$(\lambda, \phi(x)) := (-\theta\eta, e^{\eta x})$$

satisfies $\mathcal{L}\phi = -\lambda\phi$. Therefore,

$$M_s := \frac{\phi(X_s)}{\phi(X_0)} e^{-p \int_0^s X_u du + \lambda s} = e^{\eta(X_s - X_0) - p \int_0^s X_u du + \lambda s}, \quad 0 \leq s \leq t,$$

is a $\tilde{\mathbb{P}}$ -local martingale. Using Theorem 5.1.8 of [16], it can be easily verified that this process is a $\tilde{\mathbb{P}}$ -martingale. We define a measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = M_t. \quad (\text{A5})$$

Then, the process

$$\hat{W}_s = \tilde{W}_s - a\eta \int_0^s \sqrt{X_u} du, \quad 0 \leq s \leq t,$$

is a $\hat{\mathbb{P}}$ -Brownian motion, and X satisfies

$$dX_t = (\theta - mX_t) dt + a\sqrt{X_t} d\hat{W}_t, \quad X_0 > 0,$$

where $m = \sqrt{\ell^2 + 2pa^2}$. It follows that

$$\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-p \int_0^t X_s ds}] = \mathbb{E}^{\tilde{\mathbb{P}}}[M_t e^{-\eta X_t}] e^{\eta X_0} e^{-\lambda t} = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{-\eta X_t}] e^{\eta X_0} e^{-\lambda t} = f(t) e^{\eta X_0} e^{-\lambda t}, \quad (\text{A6})$$

where

$$f(t) = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{-\eta X_t}]. \quad (\text{A7})$$

Now, we show that $f(t)$ converges to a positive constant as $t \rightarrow \infty$. From Corollary 6.3.4.4 in [15], we have

$$f(t) = \mathbb{E}^{\tilde{\mathbb{P}}}[e^{-\eta X_t}] = \left(\frac{1}{1 + \eta c(t)} \right)^{2\theta/a^2} e^{-\frac{\eta}{\eta c(t) + 1} e^{-mt} x}, \quad (\text{A8})$$

where $c(t) := \frac{a^2}{2m}(1 - e^{-mt})$. The proof is given for $\eta > 0$, but the same proof holds for $\eta > -2m/a^2$. It is evident that

$$f(t) \rightarrow \left(\frac{1}{1 + \frac{a^2\eta}{2m}} \right)^{2\theta/a^2} \quad (\text{A9})$$

as $t \rightarrow \infty$. $\square$

Appendix C. 3/2 Model

The following lemma is used in the proof of Proposition 7. Here, we denote an underlying measure and a Brownian motion by $\tilde{\mathbb{P}}$ and $\tilde{W}$, respectively, for the notational consistency with the proof of Proposition 7.

Lemma A3. Let X be a solution of the SDE

$$dX_t = (\theta - \ell X_t)X_t dt + aX_t^{3/2} d\tilde{W}_t, \quad X_0 > 0,$$

where $\theta, \ell, a > 0$, and $\tilde{W}$ is a Brownian motion under a measure $\tilde{\mathbb{P}}$. Then, for $p \geq -\frac{\ell^2}{2a^2}$,

$$\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-p \int_0^t X_s ds}] = f(t)X_0^\eta e^{-\lambda t},$$

where

$$\eta := \frac{\ell}{a^2} + \frac{1}{2} - \sqrt{\left(\frac{\ell}{a^2} + \frac{1}{2}\right)^2 + \frac{2p}{a^2}}, \quad \lambda := -\theta\eta,$$

and $f(t)$ converges to a positive constant as $t \rightarrow \infty$.

Proof. Let $\mathcal{L}$ be the infinitesimal generator of the process X with killing rate p . Then,

$$(\mathcal{L}\phi)(x) = \frac{1}{2}a^2x^3\phi''(x) + (\theta - \ell x)x\phi'(x) - px\phi(x).$$

By direct calculation, we obtain that

$$(\lambda, \phi(x)) := (-\theta\eta, x^\eta)$$

satisfies $\mathcal{L}\phi = -\lambda\phi$. Therefore,

$$M_s := \frac{\phi(X_s)}{\phi(X_0)} e^{-p \int_0^s X_u du + \lambda s} = \left(\frac{X_s}{X_0}\right)^\eta e^{-p \int_0^s X_u du + \lambda s}, \quad 0 \leq s \leq t,$$

is a $\tilde{\mathbb{P}}$ -local martingale. It can be easily verified that this process is a $\tilde{\mathbb{P}}$ -martingale by using Theorem 5.1.8 of [16]. We define a measure $\hat{\mathbb{P}}$ on $\mathcal{F}_t$ as

$$\frac{d\hat{\mathbb{P}}}{d\tilde{\mathbb{P}}} = M_t. \quad (\text{A10})$$

Then, the process

$$\hat{W}_s = \tilde{W}_s - a\eta \int_0^s \sqrt{X_u} du, \quad 0 \leq s \leq t,$$

is a $\hat{\mathbb{P}}$ -Brownian motion, and X follows

$$dX_t = (\theta - mX_t)X_t dt + aX_t^{3/2} d\hat{W}_t, \quad X_0 > 0,$$

where $m = \ell - a^2\eta$. It is easy to check that $m > 0$ using Equation (26). It follows that

$$\mathbb{E}^{\tilde{\mathbb{P}}}[e^{-p \int_0^t X_s ds}] = \mathbb{E}^{\tilde{\mathbb{P}}}[M_t X_t^{-\eta}] X_0^\eta e^{-\lambda t} = \mathbb{E}^{\hat{\mathbb{P}}}[X_t^{-\eta}] X_0^\eta e^{-\lambda t} = f(t) X_0^\eta e^{-\lambda t}, \quad (\text{A11})$$

where

$$f(t) = \mathbb{E}^{\hat{\mathbb{P}}}[X_t^{-\eta}]. \quad (\text{A12})$$

Now, let us show that $f(t)$ converges to a positive constant as $t \rightarrow \infty$. To this end, let $Y = 1/X$. Then, Y is a CIR process satisfying

$$dY_t = (\mu - \theta Y_t) dt - a\sqrt{Y_t} d\hat{W}_t, \quad Y_0 = 1/X_0,$$

where $\mu := m + a^2$. Since there is a constant $c > 0$ such that $y^\eta \leq ce^{\frac{\theta}{a^2}y}$ for all $y > 0$, we have

$$\mathbb{E}^{\hat{\mathbb{P}}}[Y_t^\eta] \leq c\mathbb{E}^{\hat{\mathbb{P}}}[e^{\frac{\theta}{a^2}Y_t}]. \quad (\text{A13})$$

From Equations (A8) and (A9),

$$\mathbb{E}^{\hat{\mathbb{P}}}[e^{\frac{\theta}{a^2}Y_t}] \rightarrow 2^{\frac{2\mu}{a^2}}$$

as $t \rightarrow \infty$. This implies that

$$f(t) = \mathbb{E}^{\hat{\mathbb{P}}}[X_t^{-\eta}] = \mathbb{E}^{\hat{\mathbb{P}}}[Y_t^\eta] \rightarrow \int_0^\infty y^\eta \pi(y) dy$$

as $t \rightarrow \infty$, where

$$\pi(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}, \quad \alpha = \frac{2\mu}{a^2}, \quad \beta = \frac{2\theta}{a^2},$$

is the invariant density function of Y under $\hat{\mathbb{P}}$. This gives us the desired result. $\square$

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